Lie Groups: Fall, 2024
Problems VII

October 8, 2024

Problem 1. A endomorphism N of a finite dimensional vector space is *nilpotent* if $N^k = 0$ for some $k \geq 1$. Show that an endomorphism of a finite dimensional vector space is nilpotent if and only if there is a basis in which it is strictly upper triangular.

Problem 2. (Jordan Canonical Form) Show any endomorphism A of a finite dimensional complex vector space can be decomposed as $A_{ss}A_{nil}$ where A_{ss} is diagonal in some basis ad A_{nil} is nilpotent and commutes with A_{ss}. [Hint: The eigenspace with eigenvalue λ is defined by induction. Let $A\colon V \to V$ be an endomorphism with V finite dimensional. If λ is not an eigenvalue of A, then the λ-eigenspace is zero. If $v \in V$ is a eigenvector of eigenvalue λ then define $\overline{V} = V/\langle v\rangle$. The endomorphsms A leaves invariant the one-dimensional space $\langle v\rangle$ of V spaned by v and hence induces an endomorphism of $\overline{V}$. The generalized λ-eigenspace of A is the pre-image under $V \to \overline{V}$ of the generalized λ-eigenspace of $\overline{A}$. This leads to an inductive definition of the generalized λ-eignspace of A.] Show that V is a direct sum of its generalized eigenspaces, that A preserves this direct sum decomposition, and that on the generalized λ-eigespace $A - \lambda\mathrm{Id}$ is a nilpotent transformation.

Problem 3. Show that a central element z of a compact, connected Lie group is contained in every maximal torus.

Problem 4. Let T be a maximal torus in a compact Lie group G. Define the *co-weight lattice* in $\mathfrak{t}$ as the kernel of the exponential map $\mathfrak{t} \to T$. Show that the co-weight lattice is naturally identified as the fundamental group of T. Let $\mathfrak{W} \subset \mathfrak{t}$ be the subspace $\mathfrak{W} = \cap_\alpha \alpha^{-1}(\mathbb{Z}))$. Show that $\mathfrak{W}$ is a sub group of $\mathfrak{t}$ containing the co-weight lattice, Show that the center of G is naturally identified with the quotient of $\mathfrak{W}$ by the co-weight lattice.

Problem 5. Let G and T be as in Problem 4. Show that the center of G is finite if and only if the roots span $\mathfrak{t}^*$ over the reals. In this case the *root* *lattice* is the lattice spanned by the roots and the *weight lattice* as the dual lattice to the root lattice in $\mathfrak{t}$. in this case, show that the center of G is identified with the quotient of the weight lattice by the co-weight lattice.

Problem 6. Recall that Quaternions $\mathbb{H}$ is a division algebra (algebra in which every non-zero element has an inverse) on $\mathbb{R}^4$. The standard generators are $1, i, j, k$ with 1 as the unit and $i^2 = j^2 = k^2 = -1$ and $ij = k; jk = i; ki = j$ and $ij = -ji; jk = -kj; ik = -ki$. The usual inner product makes $1, i, , j, k$ an orthonormal basis so that the norm of a quaternion is given by

$$|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Show that $|\xi \cdot \zeta| = |\xi| \cdot |\zeta|$. Show that the unit sphere $S^3 \subset \mathbb{H}$ with quaternion multiplication is a Lie group. Show that the one parameter subgroups of S^3 are the great circles through 1. These are also the maximal tori. Show that every point except ± 1 is in a unique maximal torus, whereas ± 1 are the central elements and are contained in every maximal torus.

Problem 7. Give a diffeomorphism between the real projective space $\mathbb{R}P^3$ and $SO(3)$.