Problem Set IX for Lie Groups: Fall 2024

November 19, 2024

Problem 1. Compute the translation lattice Λ_0 for W_{aff}, the co-weight lattice Λ and the dual to the root lattice Λ_R^* for $SO(8)$ and $SO(5)$. Compute $\pi_1(SO(5))$ and $\pi_1(SO(8))$.

Problem 2. Let G be a compact connected Lie group and $T \subset G$ a maximal torus. Show that the center of G, denoted $Z(G)$, is contained in T. Show that $Z(G) = \cap_\alpha \ker(\alpha)$, where α ranges over all roots of T. Show that if the roots generate $\mathfrak{t}^*$ over the reals, then the center of G is identified with the quotient of the dual to the weight lattice by the co-weight lattice.

Problem 3. Suppose G is a compact, connected Lie group. Show that the image of G under the adjoint representation is $G/Z(G)$. Show that if G has finite center, then in the adjoint form, $\mathrm{ad}(G)$, the dual to the root lattice is equal to the co-weight lattice.

Problem 4. Suppose that G is a compact, connected Lie group. Show that the Lie algebra of the center of G is the subspace of $\mathfrak{t}$ on which the Weyl group acts trivially, which is also the orthogonal complement of the subspace generated by the roots of T.

Problem 5. Suppose that G is a compact simply connected Lie group. Suppose that the Dynkin diagram D for G has connected components $D_1, \ldots, D_k$. Show that there are compact, simply connected Lie groups $G_1, \ldots, G_k$ with D_i being the Dynkin diagram of G_i and $G \cong G_1 \times \cdots \times G_k$.

More generally, suppose instead of being simply connected, G is compact with finite center. What is the analogous result?

Problem 6. Fix a notion of positive roots. A root β is *a highest root* if $\alpha + \beta$ is not a root for every simple root α. Show that a highest root lies in the closure of the fundamental Weyl chamber

Problem 7. Show that for $SU(n)$ and $SO(2n)$, $n \geq 3$, there is a unique root contained in the closure of the fudamental Weyl chamber and hence a unique highest root. Find the highest root.

Problem 8. Show that for $SO(2n+1)$, $n \geq 2$, there are two roots contained in the closure of the fundamental Weyl chamber, one long and one short. Find them. Show that there is a unique highest root for $SO(2n+1)$.