Lie Groups: Fall, 2024
SOLUTIONS FOR FINAL EXAM

December 21, 2024

1. For $g \in SU(n)$ show that the dimension of the centralizer of g is at least $n-1$.

Every $g \in SU(n)$ is contained in a maximal torus T and clearly T centralizes g. Thus, for every $g \in SU(n)$ the centralizer of g has dimension at least the rank of $SU(n)$. The diagonal matrices with diagaonal entries $(z_1, \ldots, z_n)$ complex numbers of norm 1 whose product is 1 form a torus in $SU(n)$ of dimension $n-1$. This torus is contained in a maximal torus of $SU(n)$ which then has dimension at least $(n-1)$. Consequently, the centralizer of any $g \in SU(3)$ has diension at least $(n-1)$. [In fact, we know that these diagonal matrices form a maximal torus for $SU(n-1)$.][1]

2. Let $\mathbb{H}$ be the quaternions $\{x+yi+zj+wk\}$ with norm $\sqrt{x^2+y^2+z^2+w^2}$. Show that the tangent space at the identity of the unit sphere $S^3 \subset \mathbb{H}$ is naturally identified with the purely imaginary quaternions, i,e,, the $\mathbb{R}$-subspace generated by i, j, k. Show that quaternion multiplication determines a Lie group structure on S^3. Describe the Lie algebra of this group in terms of quaternion multiplication.

The unit 3-sphere is the zeros of the square of the distance function to the origin. This function has nontrivial gradient at each point of the 3-sphere, so by the implicit function theorem S^3 is a smooth manifold. Quaternion multiplication is associative and for quaternions a, b we have $|ab| = |a| \cdot |b|$. Thus, quaternion multiplication induces an associate multiplication map $S^3 \times S^3 \to S^3$ with unit 1. Quaternion multiplication is given by quadratic polynomial functions of the coordinates and is thus a smooth function. As a result in it a smooth associativemultiplication on S^3. The inverse of a

[1]Just saying that the rank of $SU(n)$ is $(n-1)$ instead of establishing it was what I expected you to do.

non-zero quaternion a is $\overline{a}/|a|^2$ where $\overline{x+iy+jz+kw} = x - iy - jz - kw$. This map is smooth when restricted to non-zero quaternions. This proves that quaternion multiplication is a group structure[2] on S^3.

The gradient of the square of the distance function at 1 is $2x$ and hence the tangent space to the S^3 at 1 is the perpendicular space to $2x$, which is the space of purely imaginary quaternions. If $a \in S^3$ and $b(t)$ is a smooth one-parameter family of quaternions with $b(0) = 1$, then the derivative at $t = 0$ of $ab(t)a^{-1}$ is $ab'(0)a^{-1}$. Furthermore, if a now depends on s with $a(0) = 1$, then the derivitive of $a(s)b'(0)a(s)^{-1}$ at $s = 0$ is $a'(0)b'(0) - b'(0)a'(0)$. This proves that the Lie bracket on the purely imaginary quaternions α, β is given by $[\alpha, \beta] = \alpha\beta - \beta\alpha$. (Easy check that this commutator leaves invariant the space purely imaginary quaternions.)

3. Let $C^\infty(\mathbb{R})$ be the space of smooth functions on $\mathbb{R}$. Consider the differential operators on $C^\infty(\mathbb{R})$ defined by

$$a_+ = \frac{d}{dx}; \qquad a_- = -x^2\frac{d}{dx}.$$

Show that these two elements generate a finite dimensional Lie subalgebra of the Lie algebra associated to the algebra of all differential operators on $C^\infty(\mathbb{R})$. Give a Lie group with an isomorphic Lie algebra.

We compute

$$H = [a_+, a_-] = -2x\frac{d}{dx}.$$

Then

$$[H, a_+] = 2\frac{d}{dx} = 2a_+ \quad \text{and} \quad [H, a_-] = 4x^2\frac{d}{dx} - 2x^2\frac{d}{dx} = -2a_-.$$

This is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Hence, a Lie group with this Lie algebra is $SL(2, \mathbb{R})$.

4. Let G be a Lie group and $H \subset G$ a sub-Lie group. Let $N_G(H)$ be the normalizer of H in G. Give a description of the Lie algebra of $N_G(H)$ and prove that your description is correct. Same question for the centralizer of H in G.

Let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{n}$ be the Lie algebra of $N_G(H)$. Let $h(t)$ be a one-parameter family in H through the identity and $n(s)$ a one parameter family in $N_G(H)$ through the identity. Then $n(s)h(t)n(s)^{-1} \in H$

[2] It would have sufficed to say that quaternion multiplication preserves the norm an hence induces a group structure on S^3 which is smooth and hence is a Lie group structure.

for all s, t. Differentiating this expression at $t = 0$ yields $\mathrm{ad}(n(s))(h'(0)) \in \mathfrak{h}$. Differentiating again at $s = 0$ gives $[n'(0), h'(0)] \in \mathfrak{h}$. This proves that

$$\mathfrak{n} \subset \{X \in \mathfrak{g} \big| [X, \mathfrak{h}] \subset \mathfrak{h}\}.$$

The exact same argument shows that letting $\mathfrak{z}$ be the Lie algebra of the centralizer of H we have

$$\mathfrak{z} \subset \{X \in \mathfrak{g} \big| [X, \mathfrak{h}] = 0\}.$$

In general, the opposite inclusions do not hold. As an example let $H \subset SO(3)$ be a finite cylic group of order greater than 1. The subgroup H is contained in a unique maximal torus of $SO(3)$, and its normalizer normalizes this torus and its centralizer is this torus. Thus, the Lie algebra of the normalizer and centeralizer of H is the Lie algebra of this maximal torus. On the other hand $\mathfrak{h} = \{0\}$ and hence all of $\mathfrak{g}$ both normalizes and centralizes this Lie algebra.

In both cases the reverse inclusions hold for H connected, which we now assume. The easier case is the centralizer, so let's do that one first. Suppose that $X \in \mathfrak{g}$ satisfies $[X, Y] = 0$ for every $Y \in \mathfrak{h}$. Fix an inner product on $\mathfrak{g}$ and consider $Y \in \mathfrak{h}$ of norm 1. Then by the BCH formula $\exp(tX)\exp(sY) = \exp(tX + sY) = \exp(sY)\exp(tX)$ for s, t sufficiently close to zero (independent of Y as long as it has norm 1). This proves that there is $\epsilon > 0$ and a neighborhood U of $e \in H$ such $\exp(tX)$ commutes with every $u \in U$ for all $t \in (-\epsilon, \epsilon)$. Since H is connected, U generates H in the sense that every element of H is a finite product of elements in U. It follows that $\exp(tX)$ commutes with H for all $t \in (-\epsilon, \epsilon)$. Since $\exp(tX)$ is a one parameter group, for every $t \in \mathbb{R}$, the element $\exp(tX)$ is a product of a finite number of terms of the form $\exp(t_i X)$ with $-\epsilon < t_i < \epsilon$. Thus, $\exp(tX)$ commutes with H for all $t \in \mathbb{R}$.

Now suppose that $X \in \mathfrak{g}$ normalizes $\mathfrak{h}$. We shall show that the one-parameter subgroup $\exp(tX)$ is contained in $N_G(H)$. If $X \in \mathfrak{h}$, then $\exp(tX) \in H$ and hence normalizes H. We assume $X \notin \mathfrak{h}$. Then $\mathfrak{p} = \mathbb{R}(X) \oplus \mathfrak{h}$ is a sub-Lie algebra of $\mathfrak{g}$. Hence, there is a simply connected Lie group P with Lie algebra $\mathfrak{p}$ and a map of Lie groups $\psi\colon P \to G$ such that the induced map on the Lie algebras is the inclusion $\mathfrak{p} \subset \mathfrak{g}$.

The exact sequence of Lie algebras $\mathfrak{h} \to \mathfrak{p} \to \mathbb{R}$ leads to a surjective homomorphism $\varphi\colon P \to \mathbb{R}$. The kernel, K, of φ is a normal Lie subgroup of P with Lie algebra $\mathfrak{h}$. Thus, ψ maps a small neighborhood of the identity in K isomorphically onto a small neighborhood of the identity in H. Since

H is connected. it follows that $\psi(K) = H$. Since $X \in \mathfrak{p}$, the one-parameter subgroup $\exp(tX)$ is contained in $\psi(P) \subset G$

Since K is normal in P, it follows that every $g \in \psi(P)$ normalizes H. Thus, the one-parameter subgroup $\exp(tX)$ normalizes H; i.e.. $\exp(tX) \subset N_G(H)$.

5. Compute explicitly the linear and quadratic terms in the Baker-Campbell-Hausdorff formula for $\exp(X)\exp(Y)$

We write

$$\exp(X)\exp(Y) = \exp(H(X,Y)),$$

where $H(X,Y)$ is the BCH formula. Expanding gives

$$(1+X+X^2/2+\cdots)(1+Y+Y^2/2\cdots) = (1+H_1(X,Y)+H_2(X,Y)+H_1(X,Y)^2/2+\cdots),$$

where $H_i(X,Y)$ is the term of degree i in the BCH formula.

Equating the linear terms gives $H_1(X,Y) = X + Y$. Equating the quadratic terms gives:

$$X^2/2 + XY + Y^2/2 = H_2(X,Y) + (X^2 + XY + YX + Y^2)//2.$$

Hence $XY/2 - YX/2 = H_2(X,Y)$. That is to say

$$H_2(X,Y) = \frac{1}{2}[X,Y].$$

6. Suppose that G is a connected Lie group (not necessarily compact) with Lie algebra $\mathfrak{g}$ and suppose that $X \in \mathfrak{g}$ is in the center of $\mathfrak{g}$, meaning that it has trivial bracket with every element of $\mathfrak{g}$. Show that for every $t \in \mathbb{R}$, the element $\exp(tX)$ is in the center of G.

This is a special case of the second inclusion for the centralizer in Problem 4. This time X is centralizing $\mathfrak{g}$ and one must show that $\exp(tX)$ centralizes G.

7. Let G be a compact, connected Lie group and $T \subset G$ a maximal torus. Suppose that $\rho\colon G \times V \to V$ is a finite dimensional real representation of G. Show that the restriction of ρ to T decomposes V in the form $V = \oplus_\lambda V_\lambda$ where λ ranges over the weight lattice $\mathrm{Hom}(T, S^1)$, where T acts on V_λ by (possibly several copies of) the character λ. Show that the function $\mathrm{Hom}(T, S^1) \to \mathbb{Z}$ that assigns to λ the dimension of V_λ is a Weyl invariant function. Show that the dimension of V_λ is even for every $\lambda \neq 0$.

We showed in class that every real representation of T is a direct sum of a subspace on which T acts by the identity and two dimensional spaces each one acted on by a character $T \to S^1$ followed by the standard rotation action. The latter spaces are two-dimensional. Here, we amalgamate all the two dimensional subspaces with the same character $\lambda\colon T \to S^1$ together into a subspace we denote V_λ. For $\lambda \neq 0$ the space V_λ is a direct sum of two-dimensional spaces and hence is of even dimension.

If $\rho_T\colon T \times V \to V$ is a finite dimensional linear representation, we denote by $d_{rho_T}\colon \mathrm{Hom}(T, S^1) \to \mathbb{Z}$, then function that assigns to λ the dimension of the λ character space V_λ in the decompostion of V induced by ρ_T.

Suppose that two representations $\rho_T\colon T \times V \to V$ and $\rho'_T\colon T \times V' \to V'$ are conjugate by an isomorphism $I\colon V \to V'$, meaning that

$$\rho'_T(t)(I(v)) = I((\rho_T)(t)(v)) \tag{0.1}$$

Let $\{e_1, \ldots, e_n\}$ be a basis for V and let $\{e'_1, \ldots, e'_n\}$ basis for V' with $e'_i = I(e_i)$,. It follows from Equation 0.1 that, for all $t \in T$, the matrix for $\rho_T(t)$ in the first basis agrees with the matrix for $\rho'_T(t)$ in the second basis. In particular, it follows that I takes V_λ isomorphicalliy to V'_λ. Thus, if ρ_T and ρ'_T are conjugate representations, then $d_{\rho_T} = d_{\rho'_T}$.

The action of $W \times \mathrm{Hom}(T, S^1) \to \mathrm{Hom}(T, S^1)$ is given by $(w * \lambda)(t) = \lambda(w^{-1}tw)$.

Let $\rho\colon G \times V \to V$ be the representation. Denote by $\rho_T\colon T \times V \to V$ the restriction of ρ to T. Consider the representation

$$T \xrightarrow{c(w^{-1})} T \xrightarrow{\rho_T} \mathrm{Aut}(V).$$

The decomosition of V into characters spaces for this representation is the same set of spaces as for ρ_T, but the character labels have changed. The space V_λ for ρ_T is the space $V_{w*\lambda}$ for $\rho_T \circ c(w^{-1})$. This shows that $d_{\rho_T}(\lambda) = d_{\rho_T \circ c(w^{-1})}(w * \lambda)$.

Next we show that ρ_T and $\rho_T \circ c(w^{-1})$ are conjugate representations.. We prove this by considering the diagram

$$\begin{array}{ccc} T \times V & \xrightarrow{\rho_T} & V \\ {\scriptstyle \mathrm{Id} \times \rho(w^{-1})}\big\downarrow & & {\scriptstyle \rho(w^{-1})}\big\downarrow \\ T \times V & \xrightarrow{\rho_T \circ c(w^{-1})} & V \end{array}.$$

It is commutative since

$$\begin{array}{ccc} (t,v) & \longrightarrow & \rho_T(t)(v) \\ \big\downarrow & & \big\downarrow \\ (t,\rho(w^{-1})(v)) & \longrightarrow & \rho_T(w^{-1}tw)(\rho(w^{-1})v) = \rho(w^{-1})(\rho_T(t)(v)) \end{array} .$$

This shows that $\rho(w^{-1})$ conjugates ρ_T to $\rho_T \circ c(w^{-1})$. Thus, $d_{\rho_T}(w*\lambda) = d_{\rho_T \circ c(w^{-1})}(w*\lambda)$. We have already seen that $d_{\rho_T}(\lambda) = d_{\rho_T \circ c(w^{-1})}(w*\lambda)$. Putting these together gives $d_{\rho_T}(\lambda) = d_{\rho_T}(w*\lambda)$ as required.

8(a). Classify, up to isomorphism, all root systems of rank 2. (Here, by *rank* 2, I mean that the roots lie in a 2-dimensional real vector space and span that space over $\mathbb{R}$.) For each isomorphism class of such root systems give the Weyl group and the dimension of any compact, connected Lie group realizing the root system.

Since the root system is two dimensional there are exactly two simple roots α, β. One possibility is that the roots are orthogonal. In this case reflections in the two simple roots commute with each other and the Weyl group they generate is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The walls of these two roots are perpendicular and the Weyl hambers are quadrants in the plane. Since the rank is two and there are exactly two pairs of roots, the dimension of any Lie group with this roots system is 6.

If the simple roots are not orthogonal, then we label them so that $|\alpha| \leq |\beta|$. Since α and β are linearly independent, the Cauchy-Schwartz inequlaity tells us that

$$1 \leq \left(\frac{-2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\right)\left(\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\right) < 4.$$

Thus,

$$\frac{-2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} = 1$$

and

$$\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \{1,2,3\}.$$

When $\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = 1$, the simple roots α, β have equal length and $\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = 1$. In this case, the angle between the kernel of α the kernel of β is $\pi/3$ and there are 6 Weyl chambers. The Weyl group is the dihedral group of order 6 (or equivalently the pemutation group on 3 letters). There are 3 pairs of roots $\pm\alpha, \pm\beta, \pm(\alpha+\beta)$, so that any Lie group with this root. system has dimension 8.

When $\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = 2$, the angle between kernel α and kernel β is $\pi/4$. In this case there are four pairs of roots $\pm\alpha, \pm\beta, \pm(\beta+\alpha), \pm(\beta+2\alpha)$. The Weyl group is the dihedral group of order 8 and the fundamental Weyl chamber is given by $\{y \geq 0\} \cap \{x \leq y\}$. Since there are 4 pairs of roots, the dimension of any Lie group with this root system is 10.

When $\frac{-2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = 3$, the angle between kernel α and kernel β is $\pi/6$. The Weyl group is the dihedral group of order 12 and there are 6 pairs of roots. Any Lie group with this root system has dimension 14.

8(b). For each the root system R in 8(a) that does not have any triple bonds (i.e. does not have roots $\alpha, \beta \in R$ with $\langle\alpha,\beta\rangle \neq 0$ and $|\beta|/|\alpha| = \sqrt{3}$), give an example of a compact Lie group realizing the root system and describe (or draw) the affine walls and the affine Weyl chambers for that group.

Examples of Lie groups in order with the cases considered above (i) $SO(3) \times SO(3)$; (ii) $SU(3)$; (iii) $SO(5)$. The affine Weyl walls, and affine Wyel chambers are: (i) the horizontal and vertical lines with integer intercepts with the other axis, and the Weyl chambers are squares with unit side length and corners at the integral lattice;

(ii) horizontal lines and lines with slope $\pm\pi/6$; the affine Weyl chambers are equilateral triangles.

(iii) horizontal and vertial llines with integral intercepts and diagonal lines at angles $\pm\pi/4$ through integral lattice points; the affine Wey chambers the traingles cut out of a unit-size square with with integral lattice point corners and horiontal and vertidal sides by the two diagonals.

8(c). For each root system R in 8(b): (i) compute the quotient Λ_R^*/Λ_0 of the dual to the root lattice divided by the translation lattice of the affine Weyl group; and (ii) give all possibilities, up to isomorphism, for the pairs $(\mathrm{center}(G), \pi_1(G))$ as G ranges over compact, connected Lie groups whose root system is isomorphic to R.

Since $SO(3)\times SO(3)$ has trivial center and fundamental group $\mathbb{Z}/2\times\mathbb{Z}/2$, in this case $\Lambda_R^*/\Lambda_0 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For (center, fundamental group), one can have $(\{e\}, \mathbb{Z}/2 \times \mathbb{Z}/2), (\mathbb{Z}/2, \mathbb{Z}/2), (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \{e\})$. Examples are $SO(3) \times SO(3)$, $SO(3) \times S^3$, and $S^3 \times S^3$.

Since $SU(3)$ is simply connected with center $\mathbb{Z}/3\mathbb{Z}$ given by the diagonal complex matrices of determinant 1 with cube roots of 1 down the diagonal, in this case $\Lambda_R^*/\Lambda_0 = \mathbb{Z}/3\mathbb{Z}$ and the possibilities for (center, fundamental group) in this case are: $(\mathbb{Z}/3\mathbb{Z}, \{e\})$ or $(\{e\}, \mathbb{Z}/3\mathbb{Z})$. Examples are $SU(3)$ and the adjoint form of $SU(3)$ (also called $PU(3)$.

Since $SO(5)$ has trivial center and fundamental group $\mathbb{Z}/2\mathbb{Z}$, in this case $\Lambda_R^*/\Lambda_0 = \mathbb{Z}/2\mathbb{Z}$ and the possibilities for the (center, fundamental group) are $(\{e\}, \mathbb{Z}/2\mathbb{Z})$ and $(\mathbb{Z}/2\mathbb{Z}, \{e\})$. Examples are $SO(5)$ and its double covering which goes under then name $Spin(5)$.