Invariant Measures on Compact Lie Groups

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Throughout G is a compact Lie group. We fix an orientation for $\mathfrak{g}$ and extend this to a global orientation of the the tangent spaces of G which is both left- and right-invariant.

1 Invariant Measure and Integration on a Compact Lie Group

Recall that a Riemannian metric on a manifold is a smoothly varying family of positive definite, symmetric inner products on the tangent bundle. Fix a positive definite symmetric inner product B_e on $\mathfrak{g}$. There is a unique left-invariant Riemannian metric on G whose value at the identity is B_e. This and the orientation of $\mathfrak{g}$ produce a volume form $\omega \in \Omega^n(G)$. A frame $\{v_1, \ldots, v_n\}$ at a point $g \in G$ is a positively oriented frame if and only if $\omega(g)(v_1 \wedge \cdots \wedge v_n) > 0$. In this case the value of ω on this frame is the n-dimensional of the parallelpiped spanned by the frame as measured in the inner product $g_* B_e$. This volume form is left-invariant in the sense that $(g\cdot)^*\omega = \omega$. This leads to a notion of measure and integration. The measure of an open subset U is $\int_U \omega$ where U has the induced orientation. Integration is denoted

$$\int_G f d\mathrm{vol}(G) = \int_G f\omega,$$

for a continuous function f on G. Since G is compact every open subset has finite measure and every continuous function has finite integral. This notion is left invariant in the sense that $g^*(U)$ and U have the same volume and $\int_G g^* f d\mathrm{vol}_G = \int_G f \mathrm{vol}_G$.

Since the space of n-forms is the space of sections of a line bundle over G, if we fix an orientation on G, then two left-invariant n-forms ω and ω' that are positive on this orientation differ by a positive constant: $\omega' = \lambda\omega$

where

$$\lambda = \mathrm{vol}'(G)/\mathrm{vol}(G) > 0,$$

with $\mathrm{vol}(G)$, resp. $\mathrm{vol}'(G)$ is the volume of G with respect ω', resp. ω, the measure and the notion of integration are determined up to a multiplicative constant

Since left and right multiplication commute, right multiplication by an element $g \in G$ produces a new left-invariant measure $R(g)^*\omega$, which as we have seen is a positive multiple of ω. But since right multiplication by g is an orientation-preserving diffeomorphism of G to itself, $\int_G R(g)^*\omega = \int_G \omega$. This implies that $R(g)^*\omega = \omega$. Thus, the left-invariant measure and integration are also right-invariant. That is to say we have constructed a bi-invariant measure and a bi-invariant notion of integration.

Left- and right-invariant measures exist more generally on topological groups and are called *Haar measures*. By the same argument, for a compact topological group, a left invariant Haar measure is automatically right-invariant, so one has bi-invariant Haar measures.

2 Invariant metrics for G-spaces

Fix a bi-invariant measure on G as in the previous section, leading to a bi-invariant integration over G. Suppose that $G \times V \to V$ is a real linear representation of G on a finite dimensional vector space. Then there is a positive definite, symmetric inner product on V that is invariant under G in the sense that for all $g \in G$ and all $v, w \in V$ we have

$$\langle gv, gw \rangle = \langle v, w \rangle.$$

Begin with an arbitrary positive definite symmetric inner product $B(\cdot,\cdot)$ on V and define

$$\langle v, w \rangle = \int_G B(gv, gw) d\mathrm{vol}_G.$$

Clearly, $\langle v, w \rangle$ is bilinear, symmetric and positive definite.

Claim 2.1. *$\langle \cdot,\cdot \rangle$ is G-invariant.*

Proof. By the right-invariance of integration

$$\langle hv, hw \rangle = \int_G B(ghv, ghw) d\mathrm{vol}_G = \int_G B(gv, gw) d\mathrm{vol}_G.$$

□

Corollary 2.2. *There is a symmetric, positive definite inner product on $\mathfrak{g}$ that is invariant under the adjoint representation.*

3 Complete Reducibility

Definition 3.1. An $\mathbb{R}$-linear G-module is *simple* if it has no $\mathbb{R}$-linear G-submodules except $\{0\}$ and the entire $\mathbb{R}$-module. A finite dimensional $\mathbb{R}$-linear representation of G is *completely reducible* if it can be written as a direct sum of simple $\mathbb{R}$-linear G-modules. A Lie group G is *reductive* if every finite dimensional $\mathbb{R}$-linear G-module is completely reducible.

Corollary 3.2. *Any compact Lie group is reductive. That is to say, let $G \times V \to V$ be a finite dimensional real linear representation of G. Then there is a direct sum decomposition $v = \oplus_{i \in I|} V_i$ where the V_i are invariant under the G-action and each V_i is a simple G-module.*

Proof. Suppose that V is an $\mathbb{R}$-linear representation of G and $W \subset V$ is a proper G-submodule. Take a positive definite, symmetric G-invariant inner product on V and consider $W^\perp \subset V$. Since the inner product is positive definite, $V = W \oplus W^\perp$. Since the inner product is G-invariant, $W^\perp$ is a G-submodule. Using this result and induction on dimension we see that V is completely reducible.

This proves that G is a reductive group. $\square$

Replacing the symmetric positive definite inner product by a positive definite hermitian inner product, one can prove in the same manner that any finite dimensional $\mathbb{C}$-linear representation of G is completely reducible.