Lie Groups: Fall, 2024
Primer on ODE's and Integral Curves of Vector Fields

September 9, 2024

Definition 0.1. Let χ be a smooth vector field on a smooth manifold M. An *intgral curve* for χ is a smooth curve defined on an open interval $I \subset \mathbb{R}$ is a smooth $\gamma\colon I \to M$ with the property that for all $t \in I$, $\gamma'(t) = \xi(\gamma(t))$.

Theorem 0.2. *Let M be a smooth manifold and χ a smooth vector field. For any $p \in M$ and $t_0 \in \mathbb{R}$ there is an $\epsilon > 0$ and an integral curve $\gamma\colon (t_0 - \epsilon, t_0 + \epsilon) \to M$ for χ with $\gamma(t_0) = p$. This curve is unique and the curve varies smoothly with the vector field and initial condition.*

Explanation of Uniqueness and Smooth Variation Suppose that I, J are open intervals in $\mathbb{R}$ and $t_0 \in I \cap J$. Suppose that we have an integral curve γ_I, resp. γ_J, for χ, defined on I, resp. J, with $\gamma_I(t_0) = \gamma_J(t_0)$. Then γ_I and γ_J agree on $I \cap J$. In particular, there is a maximal interval of definition $(-a, b)$ for any integral curve of of χ with given initial condition at some $\{t_0\}$. (Here, a, b are positive real numbers or ∞.)

Suppose we have a smooth map Ψ from a finite dimensional manifold N to the space of vector fields on M. This means that restricting to any local coordinate system $(y^1, \ldots, y^k)$ for N and any local coordinate system $(x^1, \ldots, x^n)$ for M the map of the map Ψ is given by

$$\Psi(y^1, \ldots, y^k) = \sum_{i=1}^{n} f_i(x^1, \ldots, x^n, y^1, \ldots, y^k) \frac{\partial}{\partial x^i}$$

with the f_i being smooth functions of $n + k$ variables. Suppose also that we have a smopth map $I\colon N \to M$ Then there is an open neighborhood ν of $N \times \{0\}$ in $N \times I$ and a smooth map $\widetilde{\gamma}\colon \nu \to M$ such that the restriction, γ_n, of $\widetilde{\gamma}$ to $(\{n\} \times R) \cap \nu$ is an integral curve for $\Psi(n)$ with $\gamma_n(0) = I(n)$.