Poincaré Duality and the Lefschetz Fixed Point Theorem

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1 Poincaré Dulality

Let M be a closed, oriented manifold of dimension n. For any point $p \in M$ the relative homology $H_n(M, M \setminus \{p\})$ is identified with $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. From the long exact homology sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\})$ and the fact that $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to S^{n-1} we see that $H_n(M, M \setminus \{p\}) \cong \mathbb{Z}$. An *local orientation* of the manifold at p determines a generator for this homology group, i.e., picks out an isomorphism from $\mathbb{Z} \to H_n(M, M \setminus \{p\})$. Geometrically, we can view this as follows. For any relative n-cycle whose simplices are transverse to p we have the degree of the cycle at p; i.e, the points of the cycle that map to p counted with sign, determined by the relative orientation of the n-simplex containing point compared with the orientation of M at p.

Theorem 1.1. *If M is a closed oriented connected n-manifold, then $H_n(M) \cong \mathbb{Z}$ genrated by a fundamental cycle $[M]$ and for each point $p \in M$, the image of $[M]$ in $H_n(M.M \setminus \{p\})$ gives the isomorphism to $\mathbb{Z}$ determined by the orientation of M at p*

When M is smooth, the fundamental cycle can be obtained by taking a smooth triangulation of M. The result holds however for all topological manifolds.

Recall that there is a cap product $H^i(X) \otimes H_n(X) \to H_{n-i}(X)$ induced by the pairing

$$\langle \varphi, \sigma \rangle = \varphi(fr_i(\sigma)) bk_{n-i}(\sigma).$$

Here $\sigma \colon \Delta^n \to X$ is a singular n-simplex; that is to say, a generator of $Sing_n(X)$ and $\varphi \in Sing^i(M) = \mathrm{Hom}(Sing_i(M), \mathbb{Z})$. The notations $fr_i(\sigma)$ and $bk_{n-i}(\sigma)$ refer respectively to the front i face and the back $(n-i)$ face of σ. Direct computation shows that this pairing passes to a pairing of cohomology and homology to give values in homology.

Theorem 1.2. *(Poincaré Duality) Let M be a closed, connected n-manifold with fundamental class $[M]$. Then for each $0 \le i \le n$ the map $H^i(M) \to H_{n-i}(M)$ that sends $[\varphi]$ to the class of $\varphi \cap [M]$ is an isomorphism.*

In particular we have isomorphism

$$\cap[M]\colon H^i(M)/\text{Torsion} \to H_{n-i}(M)/\text{Torsion}.$$

By the universal coefficient theorem there is a natural isomorphism

$$H^i(M)/\text{Torsion} = \text{Hom}(H_i(M)/\text{Torsion}, \mathbb{Z})$$

so that we can view Poincaré duality modulo torsion as a collection of pairings

$$(H_i(M)/\text{Torsion}) \otimes (H_{n-i}(M)/\text{Torsion}) \to \mathbb{Z}$$

for $0 \le i \le n$. These pairings are *non-degenerate* in the sense that their adjoints make each lattice the dual of the other. They are called the *homological intersection pairings*.

There is a geometric description of this pairing: Given two homology classes $a \in H_i(M)$ and $b \in H_{n-i}(M)$ there are cycle representative α and β so that the only points of intersection between α and β occur in the interior of i simplices of α and $n-i$ simplices of β. Furthermore, near each point x of intersection α and β are both smooth maps and are transverse at x in the sense that $T_xM = T_x\alpha \oplus T_x\beta$. We assign a sign to each point x of intersection by comparing direct sum of the the orientation of the i-simplex of α at x followed by the orientation of the $n-i$-simplex of β at x with the ambient orientation of M are x. The *geometric intersection*, or just the *intersection* for short, of the two cycles is the sum over the points of intersection of the sign of the local intersection. The geometric intersection pairing is a homological invariant and agrees with the homological intersection pairing.

2 Lefschetz Fixed Point Theorem

2.1 The homology class of the diagonal in $M \times M$.

Let M be a closed oriented n-manifold. The first thing to understand is the homology class in $H_n(M \times M)$ represented by the diagonal $\Delta \subset M \times M$. Of course by the Künnith theorem

$$H_*(M \times M; \mathbb{Z})/\text{Torsion} \cong (H_*(M)/\text{Torsion}) \otimes (H_*(M)/\text{Torsion}).$$

Proposition 2.1. *For each i fix a basis $\{x_{i,1} \dots, x_{i,k_i}\}$ for $H_i(M)/\mathrm{Torsion}$. Notice that k_i is the i^{th} Betti number of M. Modulo torsion, the homology class of the diagonal is given by*

$$[\Delta] = \sum_{i=0}^{n} \sum_{j=1}^{k_i} x_{i,j} \otimes x^*_{i,j}.$$

Proof. We begin with a claim.

Claim 2.2. *If $a \in H_i(M)$ and $b \in H_{n-i}(M)$ then $(-1)^{(n-i)} a \cdot b = (a \times b) \cdot \Delta$.*

Proof. We can assume that cycle representatives α and β for a and b meet generically in a finite set of points in the interior of the top dimensional simplices and at the intersections the cycles α and β are smooth with transverse intersections. The local sign of a point x of intersection is computed by comparing the orientation of $T_x\alpha$ followed by the orientation of $T_c\beta$ with the ambient orientation of M.

The points of intersection of $(\alpha \times \beta) \cdot \Delta$ are exactly the points of intersection of α and β. At a point x of intersection the tangent space $T_x\alpha$ lies in the first factor and the tangent space to $T_x\beta$ lies in the second factor. The orientation of $M \times M$ along the diagonal is given by $\big(T(M \times M)/T\Delta\big) \oplus T_{(x,x)}\Delta$ where the second factor is given the orientation of M coming from projection of Δ to the second factor and the first factor is given the orientation coming from the isomorphism to TM obtained by projection to the first factor. The latter projection is given by $(\tau_1, \tau_2) \mapsto \tau_1 - \tau_2$. Thus, under this projection $(T_x\alpha \otimes \{0\}) \oplus (\{0\} \otimes T_x\beta)$ maps to $T_x\alpha \oplus T_x\beta = T_xM$ by a map that is multiplication by -1 on $T_x\beta$ and the identity on $T_x\alpha$. Thus,

$$(\alpha \times \beta) \cdot \Delta = (-1)^{(n-i)} \alpha \cdot \beta.$$

This completes the proof of the claim. □

Given a basis $x_{i,1}, \dots, x_{i,k_i}$ for $H_i(M)/\mathrm{Torsion}$ we set $x^*_{i,1} \dots, x^*_{i,k_i}$ equal to the dual basis of $H_{n-i}(M)/\mathrm{Torsion}$ under the intersection pairing. Set

$$A = \sum_{i=0}^{n} \sum_{j=1}^{k_i} x_{i,j} \otimes x^*_{i,j}.$$

Since the intersection pairing is non-degenerate modulo torsion we prove the proposition by showing that A has the same intersection with any n-dimension class of the form $\alpha \otimes \beta$ as $[\Delta]$ does. We take a class $\alpha \otimes \beta$ with α of degree i and β of degree $n - i$.

Then $\beta = \sum_r \beta_r x_{n-i,r}$ for appropriate constants β_j. Then

$$
\begin{aligned}
&(\alpha \otimes \beta) \cdot \big(\sum_{i=0}^{n} (\sum_{j=1}^{k_i} x_{i,j} \otimes x^*_{i,j}\big) \\
&= \big(\alpha \otimes \sum_r \beta_r x_{n-i,r}\big) \cdot \big(\sum_{j=1}^{k_{n-i}} x_{n-i,j} \otimes x^*_{n-i,j}\big) \\
&= (-1)^{(n-i)} \sum_{j,r} (\alpha \cdot x_{n-i,j}) \otimes (\beta_r x_{n-i,r} \cdot x^*_{n-i,j}) \\
&= (-1)^{(n-i)} \sum_{j=1}^{k_{n-i}} \alpha \cdot x_{n-i,j} \otimes \beta_j \\
&= (-1)^{(n-i)} \alpha \cdot \sum_j \beta_j x_{(n-i,j)} = (-1)^{(n-i)} \alpha \cdot \beta
\end{aligned}
$$

□

2.2 The Graph of f and its Intersection with the Diagonal

Now let $f : M \to M$ be a smooth map. We compute the homological intersection of the graph of f with Δ. The graph of f is the image of Δ under the diffeomorphism $\mathrm{Id}_M \times f : M \times M \to M \times M$. It follows that the homology class of the graph of f is given by

$$[\Gamma(f)] = \sum_{i=0}^{n} \sum_{j=1}^{k_i} x_{i,j} \otimes f_*(x^*_{i,j}).$$

We compute $[\Gamma(f)] \cdot [\Delta]$. According to the Claim 2.2 it is given by

$$\sum_{i=0}^{n} (-1)^{(n-i)} \sum_{j=1}^{k_i} x_{i,j} \cdot f_*(x^*_{i,j}).$$

Let $(f^{(n-i)}_{j,r})$ be the matrix for f_* in the basis $\{x^*_{i,j}\}$ for $H_{n-i}(M)/\mathrm{Torsion}$.

That is to say $f_*(x^*_{i,j}) = \sum_r f^{(n-i)}_{r,j} x^*_{i,r}$. Then

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{(n-i)}\sum_{j=1}^{k_i} x_{i,j}\cdot f_*(x^*_{i,j}) \\
&= \sum_{i=0}^{n}(-1)^{(n-i)}\sum_{j} f^{(n-i)}_{j,j} = \sum_{i=0}^{n}(-1)^{n-i}\mathrm{Tr}(f^{(n-i)}) = \sum_{i=0}^{n}(-1)^{i}\mathrm{Tr}(f^{i}) \\
&= \sum_{i=0}^{n}(-1)^{i}\mathrm{Tr}(f_*\colon H_i(M)/\mathrm{Torsion} \to H_i(M)/\mathrm{Torsion})
\end{aligned}$$

Now let us compute the geometric signed intersection $\Gamma(f)\cdot\Delta$ assuming that these two sum manifolds meet transversely. These submanifolds meet transverely if and only if $T_{(x,x)}\Gamma(f)$ and $T_{(x,x)}\Delta$ intersect only in zero. That happens if and only if there is no non-zero tangent vector $\tau \in T_xM$ with $D_xf(\tau) = \tau$. This is equivalent to the statement that 1 is not an eigenvalue of D_xf or that $(\mathrm{Id} - D_xf)\colon T_xM \to T_xM$ is an isomorphism.

For a transversal fixed point $f(x) = x$, the local intersection of $\Gamma(f)\cdot\Delta$ at (x,x) is ±1 and is computed by taking the degree of the map $T_{(x,x)}\Gamma(f) \to T_xM$ given by the projection $(\tau_1,\tau_2) \in T_xM \oplus T_xM$ is mapped to $\tau_1 - \tau_2 \in T_x(M)$. Since the $T_{(x,x)}\Gamma(f)$ is given by the equation $\tau_2 = D_xf(\tau_1)$, the local intersection number at (x,x) is the degree of the isomorphism $(\mathrm{Id}-D_xf)\colon T_xM \to T_xM$. This degree is given by the sign of $\det(\mathrm{Id}-D_xf)$. The intersection $\Gamma(f)\cdot\Delta$ is the sum over the points of intersection of these local intersection signs

Since for transverse submanifolds, the homological intersection is equal to the sum of the local intersection numbers over the points of intersection, we have proved:

Theorem 2.3. *The homological intersection of the graph of f with the diagonal, $[\Gamma(f)]\cdot[\Delta]$, is equal to*

$$L(f) = \sum_{i=0}^{n}(-1)^{i}\mathrm{Tr}(f_*\colon H_i(M)/\mathrm{Torsion} \to H_i(M)/\mathrm{Torsion}).$$

Suppose that f is a smooth map with isolated fixed points $\{x_i\}$ and with $\Gamma(f)$ meeting Δ transversely at each (x_i,x_i). Then the sign of the intersection of $\Gamma(f)\cdot\Delta$ at (x_i,x_i) is ±1 and is given by

$$\mathrm{sign}(\det((\mathrm{Id} - dD_{x_i}f)\colon T_{x_i}M \to T_{x_i}M)),$$

so that

$$L(f) = \Gamma(f) \cdot \Delta = \sum_{x_i} \operatorname{sign}(\det((\mathrm{Id} - dD_{x_i} f) \colon T_{x_i} M \to T_{x_i} M)),$$

where the sum ranges over the fixed points x_i *of* f*. In particular, if* f *has no fixed points, then* $L(f) = 0$*.*