Representations of finite groups, spring 2016

Homework 2, due Wednesday February 3 before class.
Everywhere below, F denotes a field.

1. Prove:
(a) The composition $\beta\alpha$ of R-module homomorphisms $\alpha : M \longrightarrow N$ and $\beta : N \longrightarrow K$ is an R-module homomorphism.
(b) If M is a submodule of N and N a submodule of K, then M is a submodule of K.

2. (a) Find all idempotents in the following rings:
F, $\mathbb{Z}/15, \mathbb{Z}/4 \times \mathbb{Z}/4$.
(b) Prove that 0 and 1 are the only idempotents in the ring $F[x]/(x^n)$.
(c) Prove that 0 and 1 are the only idempotents in the group algebra $F[C_\infty]$ of the infinite cyclic group (recall that this group algebra is isomorphic to the algebra of Laurent polynomials in one variable $F[x, x^{-1}]$).

3. Given an R-module M, explain how to realize any submodule $N \subset M$ as the kernel of some homomorphism from M into an R-module.

4. Suppose there is a chain of R-modules $K \subset N \subset M$. Explain how to define a surjective module homomorphism $M/K \longrightarrow M/N$. Can you describe the kernel of this homomorphism?

5. The center of a ring R consists of elements that commute with every element of R:

$$Z(R) := \{a \in R | ab = ba \quad \forall b \in R\}.$$

We proved in class that $Z(R)$ is a commutative unital ring.
a) Show that the center of the ring $\mathrm{Mat}_n(F)$ of $n \times n$ matrices with coefficients in a field F consists of multiples of the identity matrix. (Hint: write down the condition that a matrix commutes with all elementary matrices e_{ij}.)
b) Show that $Z(R_1 \times R_2) = Z(R_1) \times Z(R_2)$ for any rings R_1, R_2. Here $\times$ denotes the direct product of rings.

6. Give an example of a $\mathbb{Z}$-module which has exactly three proper submodules.

7. (a) Simplify the following element of the group algebra $\mathbb{C}[\mathbb{Z}/3]$ of the cyclic group $C_3 \cong \{1, g, g^2\}$:

$$(2 - 3g + g^2)(1 + 2g + g^3 + g^4).$$

(b) Simplify the following element of the group ring $\mathbb{Z}[S_3]$ of the symmetric group S_3:

$$(1 + 2(13) - (132))((12) + (123) + 3(23)).$$

Optional problems: Do not write solutions to these problems, these are for you to think through as additional exercises.

1. Let I, J be left ideals in a ring R. Which of the following sets are left ideals in R?

$$I \cap J, \quad I + J, \quad IJ, \quad R \setminus I, \quad RI, \quad IR, \quad I \setminus J.$$

IJ denotes the set of finite sums of products ij, over $i \in I, j \in J$.

2. Find all ideals in the ring $F[x]/(x^n)$, where F is a field.

3. Give an example of a left ideal in the matrix ring $R = \mathrm{Mat}(n, F)$ that is not a right ideal. Can you show that the only two-sided ideal of R is the zero ideal?