Representations of finite groups, spring 2016

Homework 3, due Wednesday February 10 before class.

Everywhere below, F denotes a field.

1. (a) Which of the following modules over $\mathbb{Z}$ are (I) cyclic, (II) simple(irreducible)? Explain.

$$\mathbb{Z}/7, \quad \mathbb{Z}, \quad \mathbb{Z}/3 \oplus \mathbb{Z}/5, \quad \mathbb{Z} \oplus \mathbb{Z}/3, \quad \mathbb{Z}/3 \oplus \mathbb{Z}/9.$$

(b) Give an example of a module over the ring $F[x]/(x^2)$ which is cyclic but not irreducible. Explain why the module is not irreducible.

2. Which of the following rings are division rings? When a ring is not a division ring, give an example of a nonzero element without a two-sided inverse.

$$F, \; F\times F, \; F[x], \; F[x]/(x^2), \; F[x]/(x-2), \; F[x]/(x^2-x), \; \mathrm{Mat}_2(F).$$

3. (a) Prove that the field of real numbers $\mathbb{R}$ is the center of the ring of quaternions $\mathbb{H}$.

(b) Check that quaternionic conjugation (denoted by bar)

$$\overline{a + bi + cj + dk} = a - bi - cj - dk$$

is an antiautomorphism of the ring $\mathbb{H}$ of quaternions, in particular, $\overline{(q_1 q_2)} = \overline{q_2} \cdot \overline{q_1}$. Conclude that $\mathbb{H}^{op} \cong \mathbb{H}$. Conjugating twice gives identity, so we can also call it antiinvolution.

4. Prove that a module over a division ring D is cyclic if and only if it is irreducible. Show that any irreducible module over a division ring is isomorphic to the regular representation of D (which we also write as ${}_D D$, a copy of D on which ring D acts by left multiplication).

5. (a) Use Zorn's lemma to prove the existence of a complementary subspace: Given a vector space V over F and a subspace $W \subset V$, there exists a subspace $W' \subset V$ such that $V = W + W'$ and $W \cap W' = 0$.

(b) Explain where in the proof of Maschke's theorem, in the case the representation V is infinite-dimensional, we need to use Zorn's lemma.