Representations of finite groups, spring 2016

Homework 4, due Wednesday February 17 before class.

1. Which of the following rings are semisimple (one of the many equivalent conditions that a ring be semisimple is that every module is completely reducible)? Give brief justifications.

$$\mathbb{H}, \quad \mathbb{C}[x], \quad \mathbb{R}[x]/(x^3), \quad \mathbb{H}\times\mathbb{R}, \quad \mathbb{C}[x]/(x^2)\times\mathbb{R}, \quad \mathbb{Z}/4, \quad \mathbb{Z}/15.$$

2. In this exercise you prove that complete reducibility property is inherited by submodules. Suppose module V is completely reducible and W is a submodule. Assume K is a submodule of W and let K' be a complement of K in V. Show that $K'\cap W$ is a complement of K in W.

3. (a) Determine all isomorphism classes of semisimple $\mathbb{C}$-algebras of dimension 9 over $\mathbb{C}$.
(b) Using classification of division algebras over $\mathbb{R}$ stated in class (there are only three: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$), list all possible semisimple $\mathbb{R}$-algebras of dimension 4 (over $\mathbb{R}$).

4. (20 points) (a) Show that any idempotent e in the direct product ring $R_1\times R_2$ has the form (e_1, e_2), where e_i is an idempotent in R_i. Generalize to product of finitely many rings. Use this to classify idempotents in a direct product of fields $F_1\times\cdots\times F_n$.

(b) The idempotents we found in $\mathbb{C}[C_n]$ are

$$e_k = \frac{1}{n}\sum_{m=0}^{n-1}(\zeta g)^m, \quad \zeta = e^{\frac{2\pi i k}{m}},$$

where g is a generator of the cyclic group C_n. Express ge_k as a multiple of e_k. Then show that these idempotents are mutually orthogonal:

$$e_k e_\ell \;=\; e_k \;\text{ if }\; k=\ell, \text{ otherwise } 0,$$

and that they add up to one, $\sum_{k=0}^{n-1} e_k \;=\; 1$.

(c) We proved in class that the group algebra $\mathbb{C}[C_n]$ is a product of $\mathbb{C}$'s. Determine how many idempotents does $\mathbb{C}[C_n]$ have and explain how to obtain them from $e_0, \ldots, e_{n-1}$.