**Representations of finite groups, spring 2016**
**Homework 6, due Wednesday March 2 before class.**

Reading: Chapter 4 of Steinberg. Optional reading: Section 18.3 of Dummit and Foote.

1. (20 points, this is exercise 4.12 in Steinberg). Find all irreducible representation of the quaternionic group $Q_8$ and write down the character table of $Q_8$.

(a) Start by recalling generators $i, j, k$ of $Q_8$ and defining relations. Check that the map $\rho$ in (Steinberg, exercise 4.12) gives a representation of $Q_8$ and compute its character $\chi_\rho$. Check irreducibility of this representation by verifying $(\chi_\rho, \chi_\rho) = 1$.

(b) Find the commutator subgroup $[Q_8, Q_8]$ and the quotient $Q_8/[Q_8, Q_8]$ by the commutator. Classify one-dimensional representations of $Q_8$.

(c) Determine conjugacy classes in $Q_8$.

(d) Write down the character table of $Q_8$ and the unitary matrix associated to the character table. Check orthogonality relations of the 2nd kind for several pairs of columns (this is a quick way to scan for possible errors).

(e) Compare this character table with the character table for the dihedral group $D_8$ obtained in class.

(*Optional:*) Find a natural construction of the homomorphism $\rho$ from the realization of $Q_8$ as a subgroup of $\mathbb{H}^*$, where $\mathbb{H}$ is the ring of quaternions.

2. (10 points) (a) Representation $W$ of the symmetric group $S_3$ has character

$$\chi_W(1) = 7, \qquad \chi_W((12)) = -1, \qquad \chi_W((123)) = 4.$$

Find multiplicities of irreducible representations of $S_3$ in $W$ and its dimension.

(b) Does there exist a representation of $S_3$ with the character

$$\chi_V(1) = 2, \qquad \chi_V((12)) = 2, \qquad \chi_V((123)) = -1?$$

3.(10 points) (a) Character $\chi_V$ of a complex representation of a finite group $G$ has inner product $(\chi_V, \chi_V) = 2$. What can you say about decomposition of $V$ into irreducibles? Same question if $(\chi_V, \chi_V) = 3$.

(b) Suppose that (complex) representation $V$ of a finite group $G$ has $(\chi_V, \chi_W) = 0$ for any irreducible representation $W$ of $G$. What can you say about $V$?