**Introduction to knot theory, Spring 2012**

**Homework 2, due Monday, February 6**

1. (20 points) The mirror image $K^!$ of a knot $K$ is given by reflecting $K$ about a plane in $\mathbb{R}^3$. Let $D$ be a diagram of $K$, $D_1$ be the diagram obtained from $D$ by inverting all crossings, and $D_2$ - diagram given by reflecting $D$ about a line in the plane. Show that both $D_1$ and $D_2$ are diagrams of $K^!$. Use this to prove that coloring groups $C(K)$ and $C(K^!)$ are naturally isomorphic. Conclude that $\tau_n(K) = \tau_n(K^!)$ for any $n$.

2. (10 points) Prove that the knot $K\#K^!$ is amphichiral (equal to its mirror) for any knot $K$. First, show that $(K\#L)^! = K^!\#L^!$.

3. (30 points) Compute the coloring groups $C(K)$ for the trefoil $3_1$, figure-eight knot $4_1$, and the five-crossing knot $5_2$ (get diagrams for these knots from the Rolfsen knot table). Using the coloring groups, determine $\tau_n(3_1)$ and $\tau_n(4_1)$ for all $n$ (hint: the answer will depend on $n$, for instance, in the trefoil case, you might want to distinguish two cases: 3 divides $n$ or its does not). Give a example of a knot $K$ and numbers $n, m$ such that $\tau_{nm}(K) \neq \tau_n(K)\tau_m(K)$ (recall that this is true if $n$ and $m$ are relatively prime).

4. (10 points) Suppose that a knot $K$ is the closure of braid $\sigma$. Explain how to construct a braid whose closure is $K^!$.

5. (20 points) (a) Draw the closure of the 3-stranded braid $\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$. Check that the closure is a 2-component link and compute the linking number of the two components.
(b) Given a 3-stranded braid $\sigma$, how can you quickly tell if its closure is a link or a knot? Try your method on the following braids:

$$\sigma_1^{41}\sigma_2^{-73}, \qquad (\sigma_1^2\sigma_2)^{1000}, \qquad (\sigma_1\sigma_2^{-1}\sigma_1)^{51}, \qquad (\sigma_2\sigma_1)^{211}.$$

In each case, determine whether the closure is a knot, a 2-component link, or a 3-component link.

5. (10 points) Give an example of a braid whose closure is the figure-eight knot.

**Extra credit:** Give an example of 3-stranded braids $\sigma$ and $\tau$ such that each of the closures $\widehat{\sigma}$, $\widehat{\tau}$ and $\widehat{\sigma\tau}$ is the unknot. Next, determine whether the closure of the braid $\sigma^{-1}\tau$ is the unknot.