Introduction to knot theory, Spring 2012

Homework 7, due Monday, March 12

1. Which of the following are categories? In each example, composition of morphisms is the obvious one.

(a) Objects are finite sets, morphisms are injective maps of sets.
(b) Objects are sets, morphisms are surjective maps of sets.
(c) Objects are abelian groups, morphisms are isomorphisms of groups.
(d) Objects are sets, morphisms are maps of sets which are not surjective.
(e) Objects are topological spaces, morphisms are homeomorphisms.

2. (a) An object I of a category $\mathcal{C}$ is called *initial* if for any object X of $\mathcal{C}$ there exists a unique morphism from I to X. Which of the following categories have initial objects: category of sets, category of groups, category of topological spaces, category of infinite-dimensional vector spaces over a given field (morphisms are linear maps)? Show that any two initial objects in a category are isomorphic.

(b) By analogy, give the definition of a *terminal* object in a category. Which of the categories above admit terminal objects? How do initial and terminal objects compare in a category of sets?

2. Check that complexes of abelian groups and their morphisms constitute a category.

3. Consider a graph with two vertices and three edges connecting these two vertices (each edge connects the same pair of vertices; we say that the graph has multiple edges). Compute homology of this graph.

3. A triangle can be viewed as a triangulated topological space with one 2-simplex, three 1-simplices, and three 0-simplices. Compute homology of this triangulation using the definition of the differential and homology groups, and explain the answer.

4. The two-dimensional torus and the Klein bottle can both be obtained from a square by suitably identifying opposite sides of the square in pairs. Drawing a diagonal in the square will give you a triangulation of each of these two surfaces with two 2-simplices, three

1-simplices and one 0-simplex. For each of the two surfaces, write down the corresponding complex of abelian groups and compute its homology groups directly. Does your answer agree with the results we obtained in class?

5. Determine homology groups of the one-point union of the circle and the projective plane $\mathbb{RP}^2$.

6. Using the theorem that homology groups are invariant under a homotopy equivalence, compute homology of
(a) A plane punctured at n points,
(b) A Möbius band punctured at n interior points,
(c) Open disc in a plane,
(d) Direct product $D^2 \times D^2$ of two closed discs,
(e) Direct product $\mathbb{S}^2 \times [0,1]$ of two-sphere and a closed interval,
(f) Complement of the unknot in $\mathbb{S}^3$,
(g) Complement of the unknot in $\mathbb{R}^3$.

Tentative round table topics and problems.

Category theory. Homology of triangulations.

1. Notion of homotopic morphisms of complexes were defined in class. Check that if f_0, f_1 are homotopic morphisms from X to Y, and g a morphism from Y to Z, then gf_0, gf_1 are homotopic. Same if g is a morphism $W \longrightarrow X$, then $f_0 g, f_1 g$ are homotopic. Conclude that the quotient category of complexes modulo homotopic morphisms can be defined. Show that in the quotient category morphisms between any two objects constitute an abelian group. Do some computations of morhisms in this category. Show that the complex $0 \longrightarrow \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$ is isomorphic to the zero complex.

2. Show that homology group of complexes descend to functors from the homotopy category of complexes to the category of abelian groups.

3. Do computations of homology groups of various triangulations. Tricks to simplify computations (collapsing simplices, etc).