THEOREM 2–68. A locally metrizable Hausdorff space is metrizable if and only if it is paracompact.

2–13 Complete metric spaces. The Baire-Moore theorem. We conclude this chapter with several special topics, of which this section is the first. Our considerations here are limited to metric spaces. The results find frequent application in analysis.

Let M be a metric space with metric d. Precisely as is done in the theory of real numbers, a sequence $\{x_n\}$ of points in M is called a *Cauchy sequence* provided that for any positive number ϵ, there is an integer N_ϵ sufficiently large that $d(x_m, x_n) < \epsilon$ whenever m and n exceed N_ϵ. In the real numbers, this *Cauchy condition* is necessary and sufficient for the convergence of the sequence $\{x_n\}$.

A metric space M is *complete* if every Cauchy sequence of points in M has a limit point in M. Thus the real numbers are complete (in the usual metric), but the rational numbers are not. (Indeed, the reals are often defined as a completion of the rationals in the sense of Theorem 2–72 below.) It should be noted immediately that *completeness is not a topological invariant;* it depends upon the chosen metric in the space M. For instance, let $|x - y|$ be the usual metric for the reals E^1, and define the new (but equivalent) metric

$$\rho(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

Each sequence $\{x_n\}$ that satisfied the Cauchy condition in terms of the old metric still does, but the sequence of numbers $\{n\}$ forms a Cauchy sequence in terms of the new metric and, of course, does not converge. A space that is homeomorphic to a complete metric space is called *topologically complete* by some authors.

Our first few theorems relate the property of completeness to matters already familiar.

THEOREM 2–69. Every compact metric space is complete.

Proof: By Theorem 1–23, every infinite subset of a compact space has a limit point. □

THEOREM 2–70. Every closed subspace of a complete metric space is complete.

Proof: Let X be a closed subset of a complete metric space M. Then every Cauchy sequence of points in X has a limit point in M but, since X is closed, the limit point must be in X. □

THEOREM 2–71. If M and N are complete metric spaces, then the product $M \times N$ is complete in the product metric.

Proof: Let d_1 and d_2 be the metrics in M and N, respectively. Then if (x_1, y_1) and (x_2, y_2), x_i in M, y_i in N, are two points in $M \times N$, the *product metric* is given by

$$d[(x_1, y_1), (x_2, y_2)] = [d_1^2(x_1, x_2) + d_2^2(y_1, y_2)]^{1/2}.$$

Now let $\{(x_n, y_n)\}$ be a Cauchy sequence in $M \times N$ (in terms of the product metric). It is easily seen that this implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in M and N respectively and hence converge to points x and y. The point (x, y) in $M \times N$ is then the limit point of the sequence $\{(x_n, y_n)\}$. The details are left as an exercise. $\square$

A metric space M is said to be *isometrically imbedded* in a metric space N if there is a distance-preserving homeomorphism of M into N. In this language, we can state a generalization of the process of completing the rationals by means of Cauchy sequences.

THEOREM 2–72. Any metric space M can be isometrically imbedded in a complete metric space N in such a way that M is dense in N.

Proof: Consider the collection of all Cauchy sequences $\{x_n\}$ in M. Two such sequences $\{x_n\}$ and $\{y_n\}$ will be said to be equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$. (It is easy to see that this is a true equivalence relation.) The equivalence classes of Cauchy sequences in M so obtained form the points of the space N, and we denote such a class by $[\{x_n\}]$. A metric for N may be defined as

$$\rho([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty} d(x_n, y_n),$$

where $\{x_n\}$ and $\{y_n\}$ are any representatives of $[\{x_n\}]$ and $[\{y_n\}]$ respectively. To prove that this definition of ρ is independent of the choice of these representations, let $\{x_n\}$ and $\{x'_n\}$ represent $[\{x_n\}]$, and let $\{y_n\}$ and $\{y'_n\}$ represent $[\{y_n\}]$. Then

$$\lim_{n\to\infty} d(x_n, y_n) \leqq \lim_{n\to\infty} [d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n)] = \lim_{n\to\infty} d(x'_n, y'_n)$$

and

$$\lim_{n\to\infty} d(x'_n, y'_n) \leqq \lim_{n\to\infty} [d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n)] = \lim_{n\to\infty} d(x_n, y_n).$$

A verification that ρ is indeed a metric is left as an exercise.

Next, define the mapping h that carries a point x in M onto the equivalence class of all Cauchy sequences in M that converge to x. This class is not empty, for if we set $x_n = x$ for all n, then $\{x\}$ is such a sequence. It is easily seen that h is an isometry of M into N, as required. That $h(M)$ is dense in N will follow from the arguments below.

We show that N is complete. To do so, let $\{[\{x_{m,n}\}]_m\}$ be a Cauchy sequence in N, and choose a representative $\{x_{m,n}\}$ for each "point" of the sequence. We obtain an array of sequences

$$\begin{array}{l} x_{1,1}, x_{1,2}, x_{1,3}, \ldots \\ x_{2,1}, x_{2,2}, x_{2,3}, \ldots \\ x_{3,1}, x_{3,2}, x_{3,3}, \ldots \\ \vdots \end{array}$$

For the kth sequence there is, by definition, an integer n_k such that $d(x_{k,n_k}, x_{k,i}) < 1/k$ whenever $i > n_k$. We can then define a Cauchy sequence $\{y_n\}_k$ in M where each $y_n = x_{k,n_k}$. From the definition of ρ we see that

$$\rho([\{x_{k,n}\}], [\{y_n\}_k]) < \frac{1}{k}.$$

Therefore

$$\lim_{k\to\infty} \rho([\{x_{k,n}\}], [\{y_n\}_k]) = 0,$$

and the two sequences are equivalent in N.

But since $\{[\{y_n\}_k]\}$ is a Cauchy sequence in N, it follows that, given ϵ, there is an integer K such that $\rho([\{y_n\}_k], [\{y_n\}_l]) < \epsilon$ whenever k and l exceed K. But this implies that $d(x_{k,n_k}, x_{l,n_l}) < \epsilon$ whenever $k, l > K$. Thus the sequence $\{x_{k,n_r}\}$ is a Cauchy sequence in M. That the sequences $\{x_{k,n_k}\}_k$ of constants converge to this diagonal sequence is immediate. Therefore the sequence $[\{x_{k,n_k}\}]_k$ in N has a limit point in N, and so does the equivalent sequence $\{[\{x_{m,n}\}]_m\}$. This proves that N is complete and moreover that every point of N is the limit of a Cauchy sequence (in N) of constant sequences (in M). It follows that $h(M)$ is dense in N. $\square$

Most of the results of this section find their primary use in analysis. However, the next result, together with Theorem 2–79, provides the basis for an important imbedding property in topology (see Theorem 3–62).

THEOREM 2–73. *If M and N are metric spaces, and if N is bounded and complete, then the function space N^M of all continuous mappings of M into N is complete in the metric $\rho(f, g) = \sup_x d(f(x), g(x))$, where d is the metric in N.*

Proof: Let $\{f_n\}$ be a sequence of continuous mappings of M into N that have the property that, given $\epsilon > 0$, there is an integer K such that $\rho(f_n, f_m) < \epsilon$ whenever $m, n > K$. For a fixed point x in M, the sequence of points $\{f_n(x)\}$ then forms a Cauchy sequence in N since $d(f_n(x), f_m(x)) \leq \rho(f_n, f_m)$. Because N is assumed to be complete, there is a point $f(x)$ in N such that $\lim_{n\to\infty} f_n(x) = f(x)$. Therefore we have a function f of M into

N defined by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all x in M.

To complete the proof, we must show that f is continuous. To do so, we use the customary $(\epsilon - K)$-argument. That is, given $\epsilon > 0$, there is an integer K such that $\rho(f_n, f) < \epsilon/3$ whenever $n > K$. For such a value of n, there is a positive number such that $d(f_n(x), f_n(y)) < \epsilon/3$ whenever $d_1(x, y) < \delta$ (d_1 is the metric in M). Hence we have

$$\begin{aligned} d(f(x), f(y)) &\leqq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ &\leqq \rho(f, f_n) + d(f_n(x), f_n(y)) + \rho(f_n, f) < \epsilon \end{aligned}$$

whenever $d_1(x, y) < \delta$. This proves that f is continuous. $\square$

The requirement that N be bounded in Theorem 2–73 is needed only to show that $\rho(f, g)$ exists. Our argument above actually proves the following.

Corollary 2–74. If N in Theorem 2–73 is complete (but not necessarily bounded), then the space of bounded continuous mappings of M into N is complete.

We remark that *every metric space with metric d has a metric d' that is bounded and that does not alter Cauchy sequences.* One such metric may be obtained by replacing the original values $d(x, y)$ by values $d'(x, y)$, defined by $d'(x, y) = d(x, y)$ if $d(x, y) < 1$ and by $d'(x, y) = 1$ if $d(x, y) \geqq 1$. We leave it to the reader to verify that d' is a metric.

A metric space M with metric d is said to be *totally bounded* if, given any positive number r, M is the union of finitely many sets of d-diameter less than r.

Theorem 2–75. A metric space is compact if and only if it is complete and totally bounded.

Proof: From Theorem 2–69 we know that a compact metric space is complete. And such a space must also be totally bounded or else the covering by open spherical neighborhoods of some radius r would not have a finite subcovering. Hence the condition is necessary.

To prove sufficiency, we take advantage of Exercise 2–21 and prove that a complete and totally bounded metric space M is countably compact. To do so, let $\{x_n\}$ be any sequence of points of M. Now M is a union of a finite number of sets $X_{1,1}, \ldots, X_{1,n_1}$ of diameter < 1. At least one of these sets, say $X_{1,1}$, contains an infinite number of points x_n. Let x_{k_1} be the first point of $\{x_n\}$ in $X_{1,1}$. Again, M is a union of a finite number of sets $X_{2,1}, \ldots, X_{2,n_2}$ of diameter $< \frac{1}{2}$, and one of these, say $X_{2,1}$, has the property that $X_{1,1} \cap X_{2,1}$ contains infinitely many points of the sequence $\{x_n\}$. Choose x_{k_2} as the first point of $\{x_n\}$, with $k_2 > k$, and lying in $X_{1,1} \cap X_{2,1}$. In general, we consider M as a finite union of sets of diameter $< 1/i$ and choose a new point x_{k_i}, $k_i > k_{i-1} > \cdots > k_2 > k_1$, of the sequence $\{x_n\}$ lying in $\bigcap_{j=1}^{i} X_{i,1}$. Since for any $k_j > k_i$, the points x_{k_i} and x_{k_j} lie together in a set of diameter $< 1/i$, the subsequence $\{x_{k_i}\}$ which we have extracted is a Cauchy sequence. Since M is assumed to be complete, this subsequence converges to a point of M and hence the sequence $\{x_n\}$ has a limit point. $\square$

Some new (to us) terminology is often seen in analysis. A subset of a space S is called a *G_δ-set* if it is the countable intersection of open sets, and is called an *F_σ-set* if it is the countable union of closed sets. It is obvious that a subset is a G_δ-set if and only if its complement is an F_σ-set. As a point of interest, the genesis of these terms is as follows. The G in G_δ stands for the German word *Gebiet* (open set), and the δ means *Durchschnitt* (intersection). The F in F_σ comes from the French word *fermé* (closed), and the σ stands for *sum*, which many authors use in place of *union*.

THEOREM 2–76 (Alexandroff). Every G_δ-set in a complete metric space is homeomorphic to a complete space (or is topologically complete).

Proof: Let Q be a G_δ-set in the complete metric space M. We show that a new (but equivalent) metric can be placed upon Q so that Q is complete in terms of the new metric. By definition, $Q = \bigcap_{i=1}^{\infty} U_i$, where each U_i is open in M. As in Section 2–3, we consider the distance $d(x, M - U_i)$ for each point x in U_i and define a function $f_i\colon U_i \to E'$ by

$$f_i(x) = \frac{1}{d(x, M - U_i)}.$$

Now let $\varphi_i(x, y)$ be the real function defined on $U_i \times U_i$ by

$$\varphi_i(x, y) = \frac{|f_i(x) - f_i(y)|}{1 + |f_i(x) - f_i(y)|}.$$

The function φ_i will in general not be a metric for U_i, because it is possible to have $\varphi_i(x, y) = 0$ without having $x = y$. However, we do have

$$\varphi_i(x, y) + \varphi_i(y, z) \geqq \varphi_i(x, z),$$

for all x, y, z in U_i. Since

$$|f_i(x) - f_i(y)| + |f_i(y) - f_i(z)| \geqq |f_i(x) - f_i(z)|,$$