Thm In a linear continuum L, any closed interval is compact in the order topology.

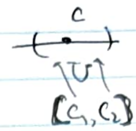
Pf Choose a covering $\mathcal{A}$ of $[a,b]$.

For each x, $a \le x \le b$ consider $[a,x]$. Let $C = \{x \in [a,b] \mid [a,x]$ has a finite subcovering $\mathcal{A}|_{[a,x]}\}$ ($\mathcal{A}$ restricted to $[a,x]$)

$C \subset [a,b]$ Let $c = \sup C = \sup_{x \in C}(x)$

Necessarily $C = [a,c)$ or $[a,c]$ ← U

↖ add open $U \in \mathcal{A}$, $c \in U$ $\Rightarrow$ not possible



~~Get~~ $\Rightarrow$ $c = b$, $C = [a,b]$. □

Thm (27.1) Suppose X - simply ordered, has the least upper bound property. Then any closed interval in X is compact.

A variation on the proof above. See Munkres.

Corollary (27.2) Any closed interval $[a,b] \subset \mathbb{R}$ is compact

$\Rightarrow$ $[a,b]^n \supset I^n$ is compact $I^n \subset \mathbb{R}^n$.

Thm (27.3) $A \subset \mathbb{R}^n$ is compact iff it is closed & is bounded in the Euclidean metric or the square metric ρ.

Pf restrict to ρ $\qquad \rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y)$

Assume A - compact. by (26.3) it is closed (compact subspace of Hausdorff space is closed)

Open sets $\{B_\rho(0,m) \mid m > 0\}$, $m \in \mathbb{Z}_+$. Covers A. Pick a finite subcover $\Rightarrow$ A is bounded. Completes $\Rightarrow$

$\Leftarrow$ $A \subset [-a,a]^n$ for some $a > 0$ (since A bounded)

↑ closed subspace ↑ compact

Thm (26.2) X compact, $A \subset X$ closed $\Rightarrow$ A compact. ($X = [-a,a]^n$)

Examples $C \subset [0,1]$ Cantor set compact, $S^{n-1} = \{v \in \mathbb{R}^n \mid |v| = 1\}$ bounded, closed

Thm (27.4) (Extreme value thm). Let $f: X \to Y$ continuous, Y ordered set in order topology. If X-compact, $\exists c, d \in X$ s.t.

$$f(c) \leq f(x) \leq f(d) \quad \forall x \in X$$

In Calc: special case, $X = [0,1]$, $Y = \mathbb{R}$

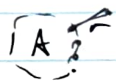
Proof: X compact $\Rightarrow A = f(X)$ compact. Want to show A has the largest element M & the smallest element m. Then $m, M \in A$, $m = f(c)$, $M = f(d)$ some c, d

if A has no largest element $\{(-\infty, a) \mid a \in A\}$ is an open cover of A.
A compact $\Rightarrow$ some finite subcollection $\{(-\infty, a_1), \ldots (-\infty, a_n)\}$ covers A.
Let $a_i = \sup(a_1, \ldots a_n) \Rightarrow a_i \notin (-\infty, a_j)\ \forall j$ contradiction since they cover A
likewise for smallest el-t. D.

Def (X, d) metric space, $A \subset X$ nonempty. $\forall x \in X$ distance from x to A

$$d(x, A) = \inf\{d(x,a) \mid a \in A\}$$

Example $A = \{z \mid |z| < 1, z \in \mathbb{C}\}$



For fixed A, $d(x, A)$ is continuous in x $\quad d(x, A): X \to \mathbb{R}$

$x, y \in X$

$$d(x, A) \leq d(x,a) \leq d(x,y) + d(y,a) \quad \forall a \in A$$

$$d(x, A) - d(x,y) \leq \inf_a d(y,a) = d(y, A)$$

diameter of a bounded $A \subset X$, (X, d) metric

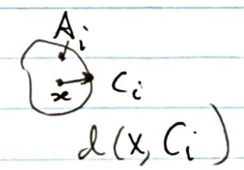
$$\operatorname{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

lemma 27.5 (Lebesgue number lemma) Let $\mathcal{A}$ be open cover of a metric (X, d).
If X compact, $\exists \delta > 0$ s.t. $\forall$ subset of X of diameter $< \delta$ $\exists$ an element of $\mathcal{A}$ containing it

δ is a <u>Lebesgue number</u> for $\mathcal{A}$. (depends on $\mathcal{A}$)

Pf if $X \in \mathcal{A} \Rightarrow \forall$ pos. # is a Lebesgue # for $\mathcal{A}$. Assume $X \notin \mathcal{A}$.
Choose $\{A_1, \ldots A_n\}$ covers X. Let $C_i = X \setminus A_i$, Let $f: X \to \mathbb{R}$

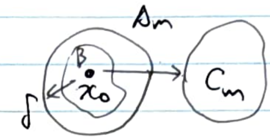
$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$



Claim $f(x)>0\ \forall x$. $x\in X \Rightarrow x\in A_i$ some i. $\Rightarrow \exists\,\varepsilon$ $B_\varepsilon(x)\subset A_i$ $B(x,\varepsilon)$
$\Rightarrow d(x,C_i)\geq\varepsilon,\ f(x)\geq\frac{\varepsilon}{n}. \Rightarrow \boxed{\min_{x\in X} f(x)>0}$ $\delta>0$

f continuous $\Rightarrow$ has ~~max~~ min value δ. Let $B\subset X$, $\operatorname{diam}(B)<\delta$. Let $x_0\in B$.
$\Rightarrow B\subset B_\delta(x_0)$ $B(x_0,\delta)$ $\Rightarrow$ $\delta\leq f(x_0)\leq d(x_0,C_m)$
largest of $d(x_0,C_i)$

want to show $B(x_0,\delta)\subset A_i$ some i



$\Rightarrow B\subset B(x_0,\delta)\subset A_m = X\setminus C_m$. $\square$

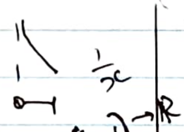
Def $(X,d_X),(Y,d_Y)$ metric spaces. $f:X\to Y$ is called <u>uniformly continuous</u> if $\forall\,\varepsilon>0\ \exists\,\delta>0$ s.t. $\forall\, x_0,x_1\in X$

$$d_X(x_0,x_1)<\delta \Rightarrow d_Y(f(x_0),f(x_1))<\varepsilon$$

Thm 27.6 (uniform continuity thm) Let $f:X\to Y$ be a continuous map from compact metric X to metric Y. Then f is uniformly continuous.

Proof For $\varepsilon>0$ Cover Y by balls $B(y,\varepsilon/2)$. Let A covering of X by inverse images of those balls. Let δ be a Lebesgue # for A.
If $x_1,x_2\in X$, $d_X(x_1,x_2)<\delta \Rightarrow \operatorname{diam}(\{x_1,x_2\})<\delta \Rightarrow \{f(x_1),f(x_2)\}$ is in some ball $B(y,\varepsilon/2)$. $\Rightarrow d_Y(f(x_1),f(x_2))<\varepsilon$.

Counterexample if X not compact $e^x:\mathbb{R}\to\mathbb{R}$

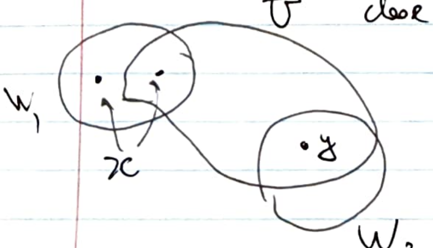


add uniform limit thm here p. 132

Def $x\in X$ is <u>isolated</u> if $\{x\}$ is open in X.

Thm (27.7). If X is a nonempty compact Hausdorff. If X has no isolated points then X is uncountable. $\{0\}\cup\{\frac{1}{n}\}_{n\geq 1}$

Pf: 1) $U\subset X$ open, $U\neq\emptyset$, $\forall x\in X$ $\exists$ nonempty open $V\subset U$, $x\notin\overline{V}$
U choose $y\in U$, $y\neq x$ (x is not isolated)



choose W_1,W_2 disjoint
$V=W_2\cap U$ in U, nonempty (contains y), $x\notin\overline{V}$

2) Assume $f:\mathbb{Z}_+\to X$, show that f is not surjective

$x_n=f(n)$. $U=X \Rightarrow$ pick $V_1\subset X$ nonempty, open, $\overline{V_1}\not\ni x$.
given V_{n-1}, choose V_n - nonempty open $V_n\subset V_{n-1}$,
$\overline{V_n}\not\ni x_n$. $\overline{V_1}\supset\overline{V_2}\supset\ldots$ nonempty closed
$\exists\, x\in\bigcap\overline{V_n}$ $x\neq x_n$ then $\boxed{[0,1]\text{-uncountable}}$.