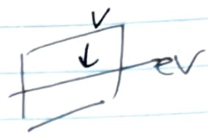
let X be a set with some structure $e: X \to X$ s.t. $e^2 = e$ and e preserves the structure is called an <u>idempotent</u> map or a <u>projection</u>.

V - vect space, $e: V \to V$ $e^2 = e$. $eV \subset V$ a subspace
$V = eV \oplus (1-e)V$
Im e here



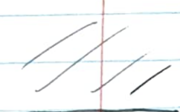
Comm ring $R \ni e$ $\Rightarrow R = Re \times R(1-e)$

a weaker decomposition holds for general rings.

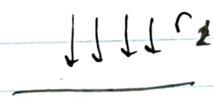
1) X - top. space, a continuous map $r: X \to X$ is called a <u>retraction</u> if $r^2 = r$. $A = r(X)$

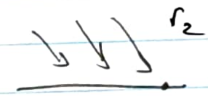
2) $A \subset X$ is called <u>a retract</u> of X iff $\exists$ a map $r: X \to A$, $r|_A = id_A$

Each retraction r gives rise to a retract $A = r(X)$. Different retractions can have the same A.



$\mathbb{R}^2_+ = \{(x,y) \mid y \geq 0\}$
$\cup$
$\mathbb{R} = \{(x,0)\}$
a retract





<u>Special case</u> : projection onto a direct summand
$X \times Y$
$\cup$
$X \times Y \to X \times \{y_0\}$
$(x,y) \longmapsto (x, y_0)$

1) $\forall$ point $x \in X$ gives a retraction $r_x: X \to \{x\}$

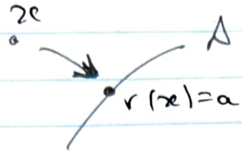
2) $x, y \in X$ $x \neq y$ $A = \{x,y\}$. When is there a retraction $X \xrightarrow{r} A \subset X$?
$r^{-1}(x), r^{-1}(y)$ gives a separation. Iff $\exists$ a separation of X s.t. x, y are in different subspaces.
X connected $\Rightarrow \{x,y\}$ is not a retract.

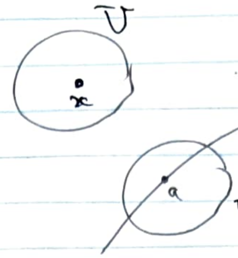
<u>Prop</u> 1) X - connected, $A \subset X$ retract $\Rightarrow A$ is connected
2) replace <u>connected</u> with <u>path-connected</u> above
3) replace <u>connected</u> with <u>compact</u> above

A better version of (3)

Prop X - Hausdorff, $A \subset X$ retract $\Rightarrow A$ is closed in X.

Proof





U, V - disjoint, open

A $\Rightarrow V \cap A$ open in $A \Rightarrow$

$r^{-1}(V \cap A)$ open in X
$\Downarrow$
x

Let $U' = U \cap r^{-1}(V \cap A)$
$\uparrow$
open neighborhood of x

if $y \in U' \cap A \Rightarrow y = r(y) \in V \cap A$
contradiction since U, V disjoint

$\Rightarrow U'$ disjoint from $A \Rightarrow A$ is closed in X. $\square$

Classification of retracts of $\mathbb{R}$: must be connected, path-connected, closed

$A \subset \mathbb{R}$ retract $\Rightarrow A = \mathbb{R}$ or $(-\infty, a]$ or $[a, +\infty)$ or $[a, b]$ (closed intervals only)

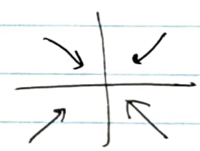
(a,b), $(a,b]$ do not work

Exercise: find retracts of some finite topological spaces.

$\{a, c\}$ not a retract

Are $\{a, b\}$, $\{b, c\}$ retracts?

Examples of retracts of $\mathbb{R}^2$: a point, $\mathbb{R}$, unit disc, ▤



$\{(x,0)\}_x \cup \{(0,y)\}_y$

union of coordinate axes

gluing of continuous maps $X = A \cup B$ (A, B closed in X)

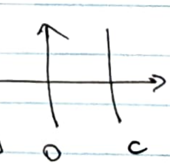
$f: X \to Y$
$f|_A$ continuous
$f|_B$ cont.
$\Rightarrow f$ continuous

Linear maps are continuous

why is r a continuous map?

○ $S^1 \subset \mathbb{R}^2$ not a retract, but harder to prove.

x-axis $\cup$ y-axis $\cup$ $\{x = c\}$ $c \neq 0$

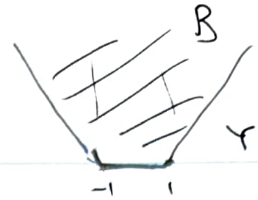


$X = \{(x,y) \mid 0 \le x \le 1, y \ge 0\}$
$A = \sqcup$ $A \subset X$
$x = 0$ or 1 or $y = 0$.

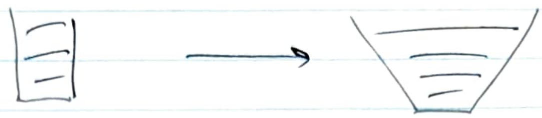
~~Contine~~ Is A a retract of X? what happens at ∞?

Difference between metric & top. space

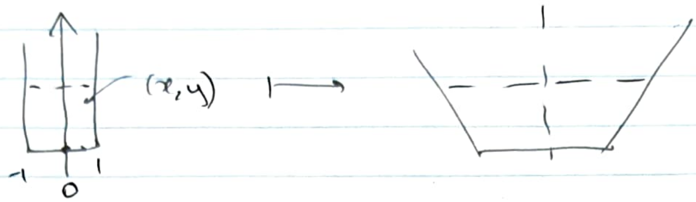
Claim (X, A) is homeomorphic to (Y, B)



$B \subset \mathbb{R}^2$ $\quad B = \{(x,y) \mid -1 \le x \le 1, y \ge 0$ or $x \ge 1, y \ge x-1$ or $x \le -1, y \ge -1-x \}$



stretch



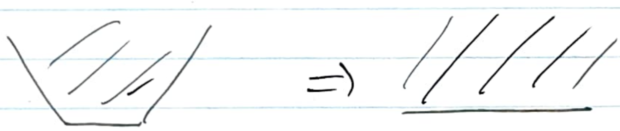
$(x, y) \longmapsto (x(1+y), y)$

continuous bijections

$\left(\frac{a}{1+b}, b\right) \longleftarrow (a,b)$ $\quad$ b does not change, shrink a

□

Claim (Y, B) homeomorphic to $(\mathbb{R}^2_+, \text{x-axis})$



linear map

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

continuous bijections. □ + gluing

$\Rightarrow$  retracts onto . what about  ? 

homeomorphic to 

Unit disk w/o a boundary point is homeomorphic to $\mathbb{R}^2_+$

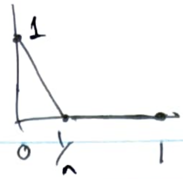
 

$D^2 \setminus \{(0,1)\} \cong \mathbb{R} \times [0,2) \cong \mathbb{R}^2_+$

$[0,2) \cong \mathbb{R}_+$

Do missing part of lect 11 (after Lebesgue # lemma)

Motivation: $f_n : [0,1] \to \mathbb{R}$



$$f_n(x) = \begin{cases} 1 - nx, & 0 \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n} \end{cases}$$

$\lim_{n\to\infty} f_n = f$

$$\lim f_n(x) = f(x) = \begin{cases} 1, & x = 0 \\ 0 & 0 < x \le 1. \end{cases}$$

f is <u>not</u> continuous at 0.

For f to be continuous need a condition on $\{f_n\}$

Def: $f_n : X \to Y$ be a sequence of f's from X to a metric space (Y, d)

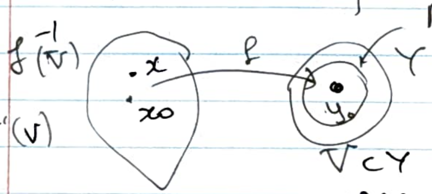
f_n <u>converges uniformly</u> to $f : X \to Y$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t.

$$d(f_n(x), f(x)) < \varepsilon \quad \forall n > N, \ \forall x \in X.$$

Thm (Uniform limit theorem)

Let $f_n : X \to Y$ be a sequence of <u>continuous</u> functions from top. space X to (Y, d) metric. If $\{f_n\}_n$ converges <u>uniformly</u> to f then f is <u>continuous</u>

want $U \ni x_0$, $f(U) \subset V$



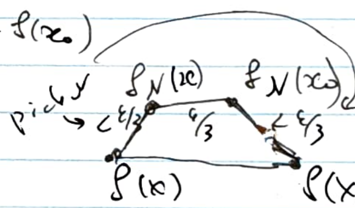
$B(y_0, \varepsilon) \subset V$

$y_0 = f(x_0)$

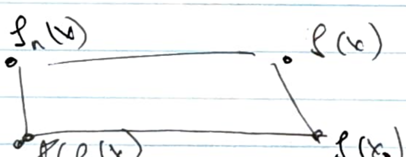
1) Pick N

$d(f(x), f_n(x)) \le \frac{\varepsilon}{3}$

$\forall n \ge N \ \forall x$



1) pick V
2) pick x_0
3) want U
$f(U) \subset V$



$d(f(x), f(x_0)) < \varepsilon$ if each of the three intervals $< \frac{\varepsilon}{3}$

2) pick U: $f_N(U) \subset B(y_0, \frac{\varepsilon}{3}) \Rightarrow \forall x \in U \ d(f(x), f(x_0)) < \varepsilon$

want to find an open neigh U of x_0 s.t. $f(U) \subset B(y_0, \varepsilon)$