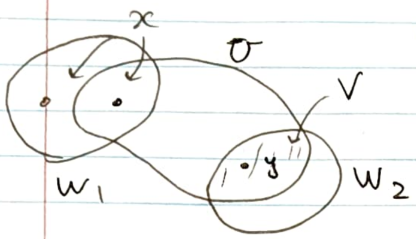
(end of §27). **Def** x is isolated if $\{x\}$ is open in X.

Thm (27.7) Assume X is nonempty compact Hausdorff. If X has no isolated points then X is uncountable.

Pf: 1) Let $U \subset X$ open, $U \neq \emptyset$. $\forall x \in X$ $\exists$ nonempty open $V \subset U$, $x \notin \overline{V}$



choose $y \in U$, $y \neq x$ (possible since x is not isolated)
choose W_1, W_2 disjoint
$V = W_2 \cap U$ in U, nonempty (contains y), $x \notin \overline{V}$

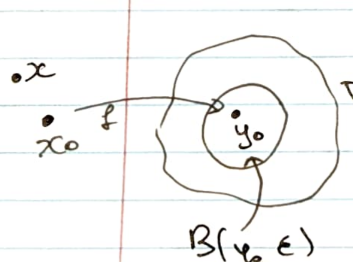
2) Assume $f: \mathbb{Z}_+ \to X$. Show that f is not surjective.
Let $x_n = f(n)$. Start with $U = X \Rightarrow$ pick $V_1 \subset X$ nonempty, $x_1 \notin \overline{V_1}$.
given V_{n-1}, choose V_n - nonempty open $V_n \subset V_{n-1}$, $x_n \notin \overline{V_n}$
$\overline{V_1} \supset \overline{V_2} \supset \dots$ nonempty closed, X compact $\Rightarrow \exists\, x \in \bigcap_n \overline{V_n}$, $x \neq x_n\ \forall n$.

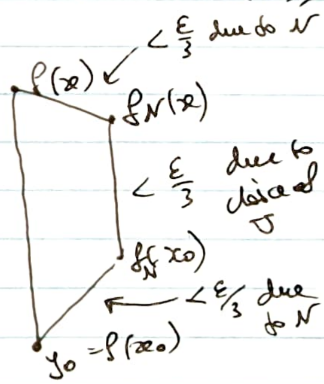
Examples a) $[0,1]$, Cantor set satisfy conditions of Thm 27.7
b) $\{0\} \cup \{\frac{1}{n}\}_{n \geq 1}$ compact, countable, has isolated points.

(end of §21) **Def** Let $f_n : X \to Y$ be a sequence of functions from X to a metric space (Y, d). f_n converges uniformly to $f: X \to Y$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t.
$$d(f_n(x), f(x)) < \varepsilon \quad \forall n > N,\ \forall x \in X.$$

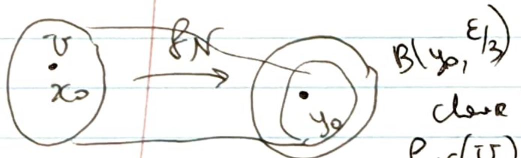
Thm (21.6, Uniform limit theorem) Let $f_n : X \to Y$ be a sequence of continuous functions from top. space X to metric (Y, d). If $\{f_n\}$ converges uniformly to f then f is continuous.

Pf $X \xrightarrow{f} Y$. pick open $V \subset Y$. To check that $f^{-1}(V)$ is open, pick $x_0 \in X$ s.t. $y_0 = f(x_0) \in V$. $f(x_0) \in V$. Pick $\varepsilon > 0$ st ball $B(y_0, \varepsilon) \subset V$. We need to find an open neighbourhood U of x_0 s.t. $f(U) \subset B(y_0, \varepsilon)$.
pick N s.t. $d(f(x), f_n(x)) < \frac{\varepsilon}{3}$ $\forall n \geq N$, $\forall x \in X$





$\Rightarrow d(f(x), f(y_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$
$\forall x \in U$



choose U
$f_N(U) \subset B(y_0, \frac{\varepsilon}{3})$

§28 Variations on Compactness

Remember: limit point x of A: $\forall$ open $U \ni x$, $(U \setminus \{x\}) \cap A \neq \emptyset$

limit point of a sequence $\{x_n\}_{n \geq 1}$, $x = \lim_{n\to\infty} x_n$ if $\forall$ open neighbourhood U of x, $x_n \in U$ for $n \gg 0$.

In a bad space $\{x_n\}$ may have more than one limit point

X Hausdorff $\Rightarrow$ $\{x_n\}$ has at most one limit point

Def X is limit point compact if any infinite subset $A \subset X$ has a limit point in X (enough to restrict to infinite countable A).

Thm X - compact $\Rightarrow$ X is limit point compact

Pf: If $A \subset X$ has no limit point, for each $x \in X$ pick open $U_x \ni x$ s.t. $(U_x \setminus \{x\}) \cap A = \emptyset$. $\{U_x\}_x$ is a covering of X. Each U_x contains at most 1 point of A

Take finite subcovering $\{U_{x_i}\}_{i=1}^n$ $\Rightarrow$ $|A| \leq n$. $\Box$

X is sequentially compact if every sequence of points has a convergent subsequence

Thm Let X be metrizable. TFAE

(1) X is compact

(2) X is limit point compact

(3) X is sequentially compact

(1) $\Rightarrow$ (2) see above $A = \{x_n\}$ set, $A \subset X$

(2) $\Rightarrow$ (3) pick a sequence $\{x_n\}$, $n \in \mathbb{Z}_+$. If $|A| < \infty$ $\to$ have a limit point

if $|A| > \infty$ $\Rightarrow$ A has a limit point. Use this point to construct a convergent subsequence.

(3) $\Rightarrow$ (1) longer proof, see Munkres §28 p. 180-181.

Complete metric spaces (§43 in Munkres, for this topic I also follow Hocking-Young, Topology, 2-13 p81-85.)

(X,d) metric $\{x_n\}$ is called a Cauchy sequence if $\forall \varepsilon>0$ $\exists N$ s.t. $d(x_n, x_m)<\varepsilon$ $\forall n,m \geq N$.

X is complete if every Cauchy sequence of points of X has a limit point in X. This limit point is necessarily unique.

Completeness is defined for a given metric. $\mathbb{Q}$ with the usual metric is not complete, with discrete metric $d(a,b)=1$ $\forall a,b\in\mathbb{Q}$, $a\neq b$ is complete

Thm Every compact metric space is complete
follows from an earlier theorem

Thm Every closed subspace of a complete metric space is complete.

Pf Use cauchy sequences.

Thm if X, Y - complete metric spaces $\Rightarrow$ $X\times Y$ is complete in each of the standard metrics.

Thm For any metric (X,d) $\exists$ a completion. $\hat{X}$ - a complete metric space that contains X as a dense ~~set~~ subspace

$X \subset \hat{X}$ isometric imbedding