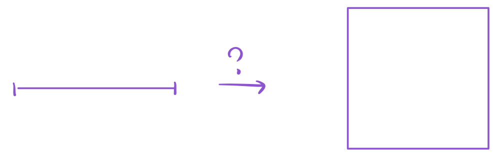
Topology 10/25

Let $I = [0,1]$

Main Question: Does there exist a 

$\underbrace{\text{surjective continuous function } I \longrightarrow I^2}_{\text{"space-filling curve"}}$?

As sets: $\exists$ bijection

E.g. consider $[0,1) \longrightarrow [0,1)^2$ by splitting even/odd digits

$0.31415926 \to 0.3452$

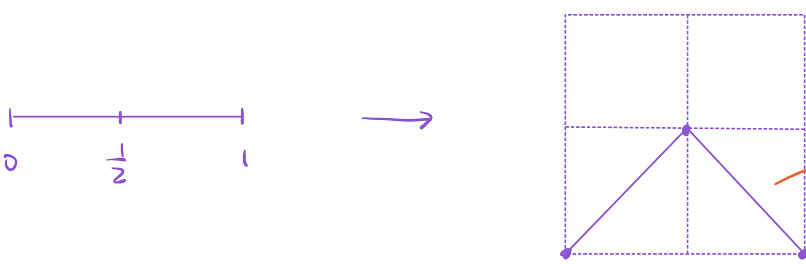
$\to 0.1196$

(not continuous)

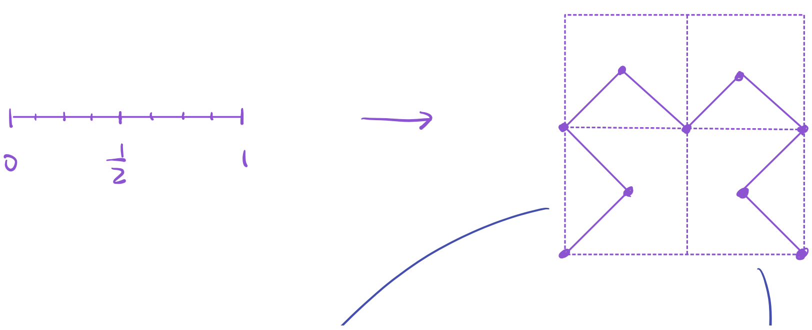
Answer: Yes $\longrightarrow$ Peano curve $f: I \to I^2$

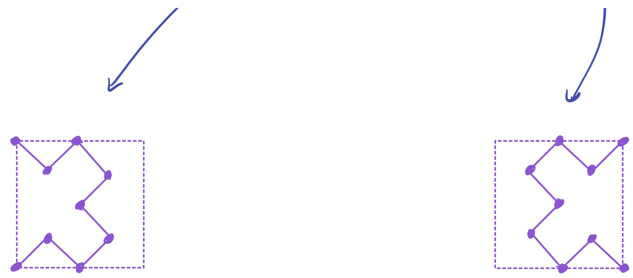
Construction

Plan: Construct f as a limit of $f_0, f_1, f_2, \dots : I \to I^2$

f_0: 

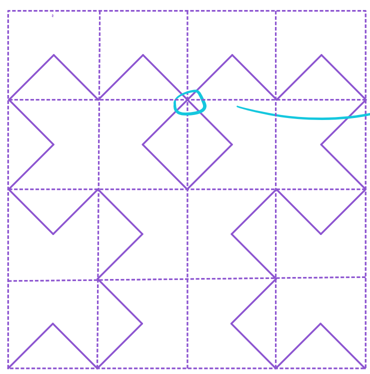
$\Downarrow$ basic operation

f_1: 



apply basic operation to each triangular path

f_2:



not injective

$\longrightarrow 4^2$ triangular paths

$\vdots$

$f_n : I \longrightarrow I^2$ $\longrightarrow 4^n$ triangular paths

End result : $\boxed{f = \lim_{n\to\infty} f_n}$

To show :
- The limit exists.
- f is continuous
- f is surjective

Completeness (X, d) metric space

Def $\{x_n\}$ in X is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N$ s.t. $d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N$

Def X is complete if every Cauchy sequence has a limit point in X

Every X has completion $\hat{X}$, $X \hookrightarrow \hat{X}$ isometric imbedding.

Ex I^2 is complete $\rightarrow$ compact; $\rightarrow$ closed subspace of $\mathbb{R}^2$

Thm X - topological space

(Y, d) - metric space, bounded, complete

Then $\boxed{C(X, Y)} := \{$ continuous functions $X \longrightarrow Y\}$

is a complete metric space under the metric

$$\boxed{\rho(f, g) := \sup_{x \in X} d(f(x), g(x))}$$

↳ sup defined because Y bounded

Proof: Let $\{g_n : X \longrightarrow Y\}$ be a Cauchy sequence in $C(X, Y)$.

$\forall \varepsilon > 0$, $\exists N$ s.t. $\rho(g_n, g_m) < \varepsilon \quad \forall n, m \geq N$.

Now $\forall x \in X$, $d(g_n(x), g_m(x)) \leq \rho(g_n, g_m) < \varepsilon$

$\Rightarrow \{g_n(x)\}$ is a Cauchy sequence in Y

Since Y complete, $\exists$ limit, denoted $g(x)$

$\Rightarrow$ function $g : X \longrightarrow Y$

Claim: $\{g_n\}$ converges to g under ρ

Pf: $\forall \varepsilon > 0$,

$\{g_n\}$ Cauchy $\Rightarrow$ $\exists N$ s.t. $\forall n, m \geq N$, $x \in X$

$d(g_n(x), g_m(x)) \leq \rho(g_n, g_m) < \varepsilon/3$

$g_m(x) \longrightarrow g(x)$ $\Rightarrow$ $\exists M$ s.t. $\forall m \geq M$,

$d(g_m(x), g(x)) < \varepsilon/3$

$\Rightarrow d(g_n(x), g(x)) < 2\varepsilon/3 \quad \forall n \geq N, x \in X$ ✓

In fact: Claim $\Leftrightarrow$ $\{g_n\}$ converges uniformly to g

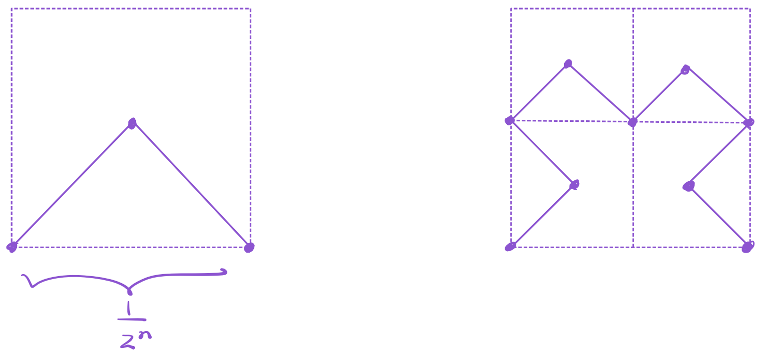
↳ ρ is called uniform metric

By Uniform Limit Theorem, g is continuous, i.e. $g \in C(X, Y)$ $\square$

Back to Peano

Claim 1: $\{f_n\}$ is Cauchy in $C(I, I^2)$ $\xRightarrow{\text{Thm}}$ f exists and continuous

Idea: Consider $f_n \longrightarrow f_{n+1}$



$\forall\, t \in I$, $f_{n+1}(t)$ and $f_n(t)$ lie in the same square of side length $\frac{1}{2^n}$

$\Rightarrow$ under square metric $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$,

$d(f_{n+1}(t), f_n(t)) \leq \frac{1}{2^n}$ $\quad \forall\, t$

$\Rightarrow \rho(f_{n+1}, f_n) \leq \frac{1}{2^n}$

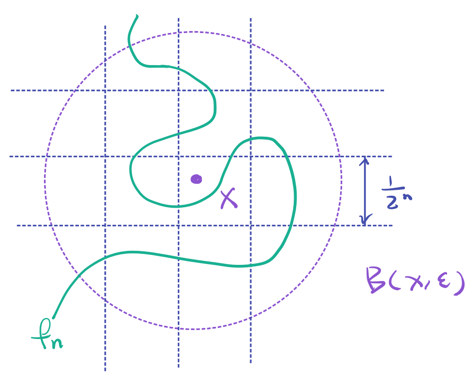
$\forall\, m$, $\rho(f_{n+m}, f_n) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{n+m-1}} \leq \frac{2}{2^n}$ //

Claim 2: f is surjective.

Idea: $f_n(I)$ touches every grid of side length $\frac{1}{2^n}$.

$\forall\, x \in I^2$, $\varepsilon > 0$

$B(x, \varepsilon)$ intersects $f_n(I)$ for n sufficiently large



$\Rightarrow x \in \overline{f(I)} = f(I)$

$\uparrow$

I compact

f continuous

//

Theorem 44.1. *Let $I = [0, 1]$. There exists a continuous map $f : I \to I^2$ whose image fills up the entire square I^2.*

The existence of this path violates one's naive geometric intuition in much the same way as does the existence of the continuous nowhere-differentiable function (which we shall come to later).

Proof. *Step 1.* We shall construct the map f as the limit of a sequence of continuous functions f_n. First we describe a particular operation on paths, which will be used to generate the sequence f_n.

Begin with an arbitrary closed interval $[a, b]$ in the real line and an arbitrary square in the plane with sides parallel to the coordinate axes, and consider the triangular path g pictured in Figure 44.1. It is a continuous map of $[a, b]$ into the square. The operation we wish to describe replaces the path g by the path g' pictured in Figure 44.2. It is made up of four triangular paths, each half the size of g. Note that g and g' have the same initial point and the same final point. You can write the equations for g and g' if you like.

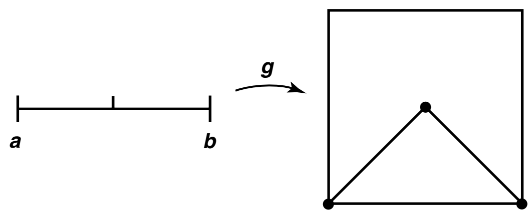


Figure 44.1

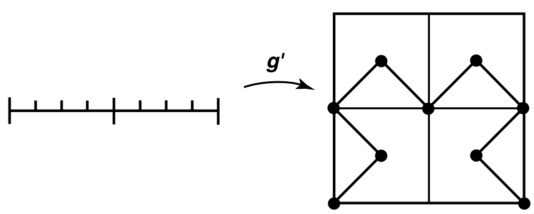


Figure 44.2

This same operation can also be applied to any triangular path connecting two adjacent corners of the square. For instance, when applied to the path h pictured in Figure 44.3, it gives the path h'.

Step 2. Now we define a sequence of functions $f_n : I \to I^2$. The first function, which we label f_0 for convenience, is the triangular path pictured in Figure 44.1, letting $a = 0$ and $b = 1$. The next function f_1 is the function obtained by applying the operation described in Step 1 to the function f_0; it is pictured in Figure 44.2. The next function f_2 is the function obtained by applying this same operation to each of the four

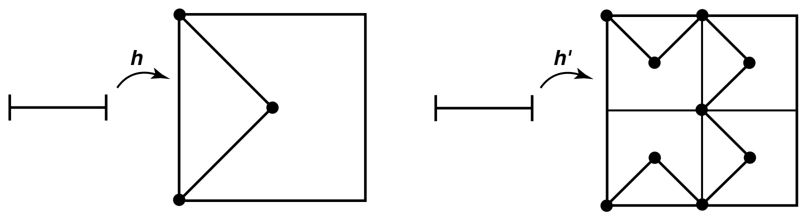


Figure 44.3

triangular paths that make up f_1. It is pictured in Figure 44.4. The next function f_3 is obtained by applying the operation to each of the 16 triangular paths that make up f_2; it is pictured in Figure 44.5. And so on. At the general step, f_n is a path made up of 4^n triangular paths of the type considered in Step 1, each lying in a square of edge length $1/2^n$. The function f_{n+1} is obtained by applying the operation of Step 1 to these triangular paths, replacing each one by four smaller triangular paths.

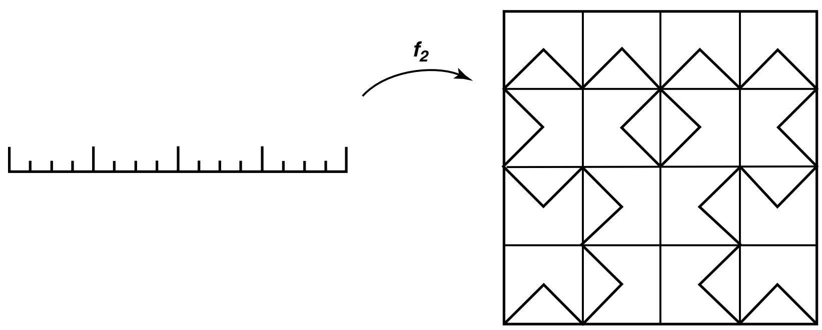


Figure 44.4

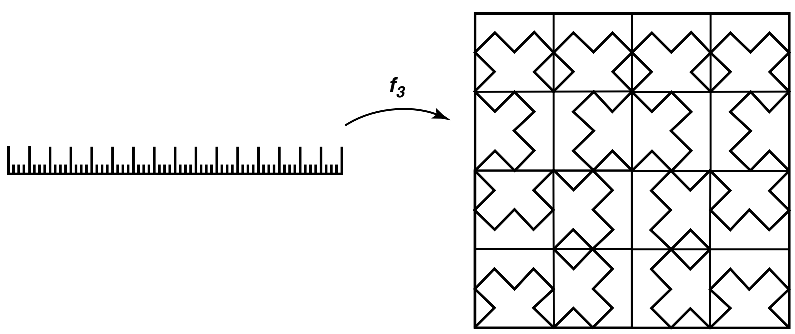


Figure 44.5