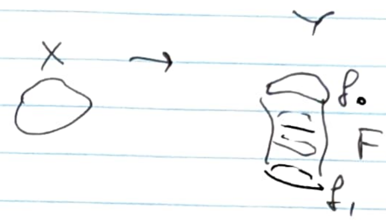
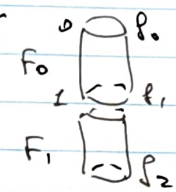
$X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$ $f_0 \simeq f_1$ homotopic if $\exists$ a continuous map $F: X \times I \to Y$, s.t. $f_0 = F|_{\{0\}}$, $f_1 = F|_{\{1\}}$

$f_0(x) = F(x,0)$, $f_1(x) = F(x,1)$



Prop This is an equivalence relation on Maps(X,Y) $=$ $C(X,Y)$

Proof: $f_0 \simeq f_1$, $f_1 \simeq f_2$ via F_0, F_1. $F_0, F_1 : X \times I \to Y$



$$F(x,t) = \begin{cases} F_0(x, 2t), & 0 \le t \le \frac{1}{2} \\ F_1(x, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

(use pasting lemma for continuity).

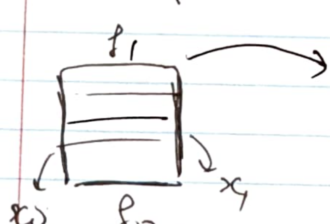
f is null-homotopic if it's homotopic to the constant map.

Example $X = \{p\}$ point. Equivalence classes (maps up to homotopy) $\{p\} \to Y$ are ~~parametrized~~ enumerated by path-conn. components of Y.

| Maps into $\mathbb{R}^n$, into convex subspaces of $\mathbb{R}^n$

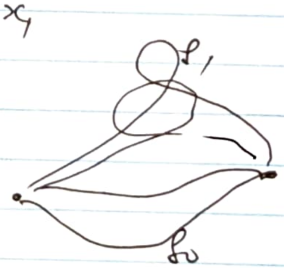
A path from x_0 to x_1, $x_i \in X$ is a continuous map $f: [0,1] \to X$, $f(i) = x_i$, $i = 0,1$.
x_0 - ~~...~~ initial.

Maps f_0, f_1 are path-homotopic if they can be connected by a continuous family of paths, endpoints fixed.



$F: I \times I \to X$

$F(x,0) = f_0(x)$ $\quad F(0,y) = x_0$
$F(x,1) = f_1(x)$ $\quad F(1,y) = x_1$

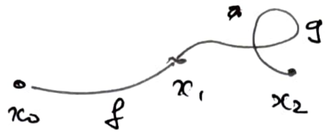


F path homotopy between f_0, f_1

$f_0 \simeq_p f_1$

Prop Path homotopy is an equivalence relation

Composition

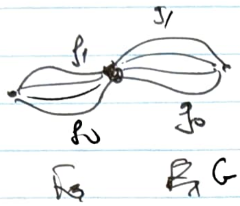


$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

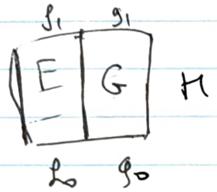
$f \to [f]$
path-homotopy class

Prop Composition descends to a well-defined map on path-homotopy classes

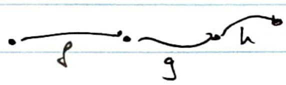
$$f_0 \cdot g_0 \simeq_p f_1 \cdot g_1$$



$\cancel{F(s,t)}$ $H(s,t) = \begin{cases} F(2s,t), & 0 \le s \le \frac{1}{2} \\ G(2s-1,t), & \frac{1}{2} \le s \le 1 \end{cases}$



$[f] \cdot [g]$ defined if $f(1) = g(0)$



Thm 1) Associativity $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$

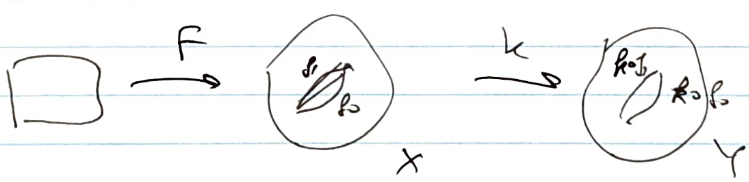
2) e_x constant path $e_x : I \to X$

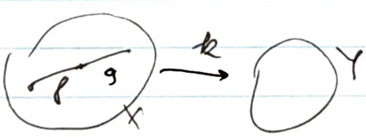
$$[f] \cdot [e_{x_1}] = [f] \qquad [e_{x_0}] \cdot [f] = [f]$$

3) Inverse $f : x_0 \to x_1$, $\bar{f}$ $\bar{f}(s) = f(1-s)$ reverse of f

$$[f] \cdot [\bar{f}] = [e_{x_0}] \qquad [\bar{f}] \cdot [f] = [e_{x_1}]$$

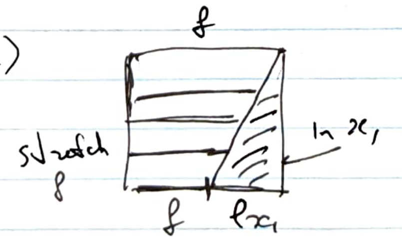
Pf



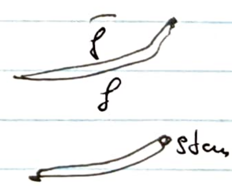


$k \circ (f \cdot g) = (k \circ f) \cdot (k \circ g)$

2)



3)



b) Associativity

$\to I \to X$

$[X, Y]$ set of homotopy classes of maps $X \to Y$

I - contractible: $id: I \to I$ is null-homotopic.

X - contractible if $id_X: X \to X$ is null-homotopic

I, $\mathbb{R}$ - contractible, $\mathbb{R}^n$ - contr. ~~conv~~ convex region of $\mathbb{R}^n$ is contractible

Y - contractible $\Rightarrow$ $[X, Y]$ consists of 1 element

X - contractible $\Rightarrow$ $[X, Y]$ elements are in a bijection with p.c.c. of Y

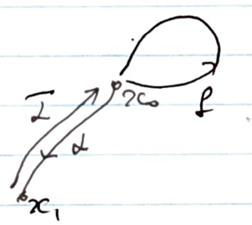
§52 Fund group

loop based at x_0 : $f\,[0,1] \to X$ $\quad f(0) = f(1) = x_0$.

$f \circ g$ composition $\quad [f]$  deformations $\quad [e_{x_0}]$ identity

$[\bar{f}]$ $\qquad [\bar{f}] \circ [f] = [e_{x_0}]$

$\pi_1(X, x_0)$ fundamental group $\qquad$ · X-contractible $\Rightarrow$ $\pi_1(X, x_0) = 1$



$$\hat{\alpha}$$
$$\pi_1(X, x_0) \to \pi_1(X, x_1)$$
$$f \longmapsto \bar{\alpha} \circ f \circ \alpha$$
$$[f] \longmapsto [\bar{\alpha}] \circ [f] \circ [\alpha] = [\bar{\alpha} f \alpha]$$
$$[g] \longmapsto$$
$$[f \circ g] \longmapsto$$

<u>Prop</u> $\hat{\alpha}$ is a group isomorphism
inverse isom given by $\hat{\bar{\alpha}}$

<u>Corollary</u>: if x_0, x_1 are connected by a path, $\pi_1(X, x_0)$, $\pi_1(X, x_1)$ are isomorphic.