lecture 9, Topology, fall 2022

Last time: 1) separation of a topological space
2) Connected spaces (no separation)

Thm L is a linear continuum (in the order topology) $\Rightarrow$ L is connected, intervals & rays of L are connected $\square$

Corollary: $\mathbb{R}$, $[a,b]$, $[a,b)$ connected.

Thm (24.3) (Intermediate value theorem) Let $f: X \to Y$ continuous, X connected, Y ordered set in the order topology. For $a,b \in X$ & $r \in Y$, $f(a) < r < f(b)$ $\exists c \in X$, $f(c) = r$

In calculus: special case $X = [a,b]$, $Y = \mathbb{R}$

Proof $\overset{a}{\overset{\pi}{A}} = f(X) \cap (-\infty, r)$, $\overset{b}{\overset{\pi}{B}} = f(X) \cap (r, +\infty)$

disjoint, nonempty, open in $f(X)$.

If $\nexists c$, $f(c) = r \Rightarrow f(X) = A \cup B$ separation. $\square$

Prop (Example 1 in [M]) Ordered square is connected.

I_o^2 is a linear continuum: least upper bound property, $A \subset I^2$

$\pi_1 : I \times I \to I$ projection. Let $b = \sup(\pi_1(A))$.

Case 1: $b \in \pi_1(A) \Rightarrow A \cap b \times I$ - nonempty, has least u.b $\times c$ $= \sup(A)$

Case 2 $b \notin \pi_1(A). \Rightarrow b \times 0$ is the least u.b. for A ..

[C]onnectedness in $\mathbb{R}$ is more special than in I_o^2.

Def $x, y \in X$. A path from x to y in X is a continuous map $f: [a,b] \to X$ s.t. $f(a) = x$, $f(b) = y$. X is path-connected if every pair of points of X can be joined by a path in X.

Prop X - path-connected $\Rightarrow$ X is connected.
Otherwise, $\exists$ a separation $X = A \cup B$. take $x \in A, y \in B$, path $f: [a,b] \to X$.
$f([a,b])$ is connected. Contradiction. $\square$.

Prop 1) Convex subspace of $\mathbb{R}^n$ is path-connected.
$A \subset \mathbb{R}^n$
$f(t) = tw + (1-t)v$

2) Star-shaped subspace of $\mathbb{R}^n$ is path-connected

$\exists v \in A \ \forall w \in A \quad [v, w] \subset A$

$\mathbb{R}^n \setminus \{0\}$ path-conn

$\mathbb{R}^n \setminus \{p_1 \dots p_k\}$ path-conn.

$S^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$ path-conn, $n > 1$

Prop I_o^2 is connected but not path-conn.

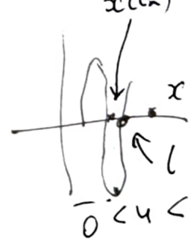
1×1

0×0

Otherwise $\exists$ a path $f [a,b] \to I_o^2$, $f(a) = 0 \times 0$, $f(b) = 1 \times 1$.

$\forall x \in I \quad U_x = f^{-1}(x \times (0,1))$, U_x open in $[a,b]$, $U_x \cap U_y = \emptyset$ $x \neq y$.
($x \times (0,1)$ open in I_o^2)

each U_x contains a rational # in $[a,b]$ $\Rightarrow$ Contradiction. $\square$.



$x(t_n)$

$x(\frac{1}{n})$

$0 < u < x(\frac{1}{n})$

Example topologist's sine curve (example 7 p.156)

$S = \{ x \times \sin(\frac{1}{x}) \mid 0 < x \leq 1 \}$ $\quad S \sqcup 0 \times [-1,1]$.
$\qquad \overset{\parallel}{\bar{S}}$ closure - connected

$\uparrow f$
$[a, c]$

$x(t_n) = u$
$\sin(\frac{1}{x}) = (-1)^n$
$f(a) = (0,1)$.

$\{ t \mid f(t) \in 0 \times [-1,1] \}$ closed in $[a,c]$ $\Rightarrow$ has largest el't b.

$f(t) = (x(t), y(t))$ $\quad x(0) = 0$, $x(t) > 0$ $t > 0$

$f: [b,c] \to \bar{S}$ $\quad b \to I$, $(b,c] \to S$. $\quad [b,c] \sim [0,1]$.

$y(t_n)$ does not converge (± 1)

Connected components

<u>Def</u> fix X. $x \sim y$ if $\exists$ connected $A \subset X$, $x, y \in A$.

<u>Prop</u> this is an equivalence relation

refl, symmetric $\quad x \sim y, y \sim z \quad x, y \in A, \; y, z \in B \Rightarrow A \cup B$

(A connected, B connected; $A \cup B$ connected)

Equivalence classes under $\sim$ are called <u>connected components</u> of X.

If $X = C \cup D$ is a separation, some c. components belong to C, some to D.

if $X = C \cup D$ is a separation, $X \underset{\text{homeomorphic}}{\simeq} C \sqcup D$.

Given 2 separations, can form a common refinement

$X = C_1 \cup D_1$

$X = C_2 \cup D_2$

$X = (C_1 \cap D_1) \cup (C_1 \cap D_2) \cup (C_2 \cap D_1) \cup (C_2 \cap D_2)$

Iterate $\quad X = X_1 \cup \dots \cup X_n$. If cannot refine $\Rightarrow$ X has fin. many conn. components

both open/closed

then $X \overset{h}{\simeq} X_1 \sqcup \dots \sqcup X_n$

Otherwise X has ∞ many conn. components.

If $\{X_\alpha\}_{\alpha \in J}$ connected components of X, have a bijection

$\coprod_{\alpha \in J} X_\alpha \to X$, but not a homeomorphism, in general.

Example: $\mathbb{Q}$, C cantor set - <u>totally disconnected</u>, each point is its own connected component, also $\{\frac{1}{n}\}_{n \geq 1} \cup \{0\} \subset \mathbb{R}$.

$\mathbb{R} \setminus \mathbb{N}$ - conn. components?

Likewise, can decompose X into path-connected components

<u>Prop</u> Each conn. component is a disjoint union of path-conn. components.

what are PCC's of I_o^2? of top. sine curve?

<u>Thm</u> (25.1) Components of X are connected disjoint subspaces of X whose union is X, s.t. each nonempty connected subspace of X intersects only one of them.