Topology 11/10

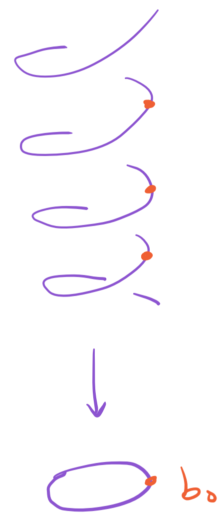
Last time: Liftings

- Given covering $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$, paths and path homotopies can be lifted uniquely
- If E is path-conn., then given $b_0 \in B$, the lifting correspondence

$$\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0) \quad \text{is surjective.}$$

- If E is simply connected, it is bijective.

$$\Rightarrow \quad \pi_1(S^1) = \mathbb{Z}$$



Today: Applications of this result to "classical" topology

Retractions (Recall)

Def X topological space, $A \subseteq X$ subspace

A retraction of X onto A is a continuous map $r : X \longrightarrow A$ s.t. $r|_A = \mathrm{Id}_A$.

If such r exists, we say A is a retract of X.

Ex Fix any $x_0 \in X$, $A = \{x_0\}$

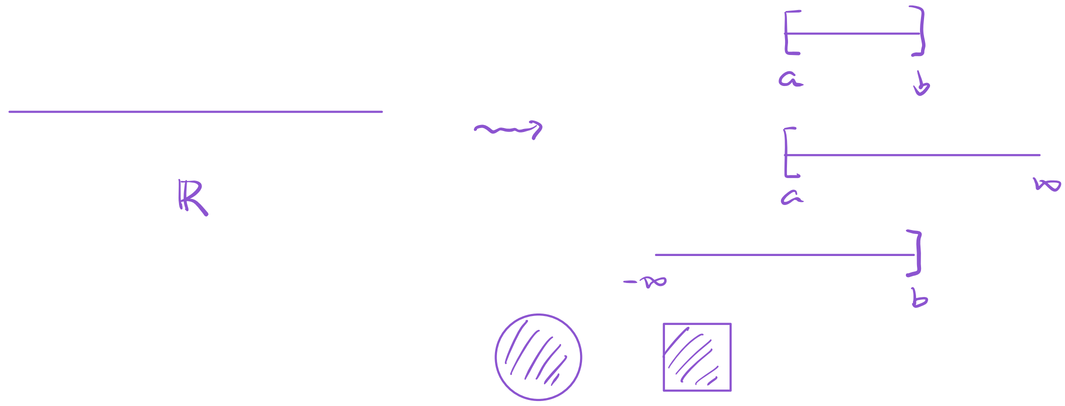
$\Rightarrow$ constant map $r : X \longrightarrow \{x_0\}$ is a retraction.

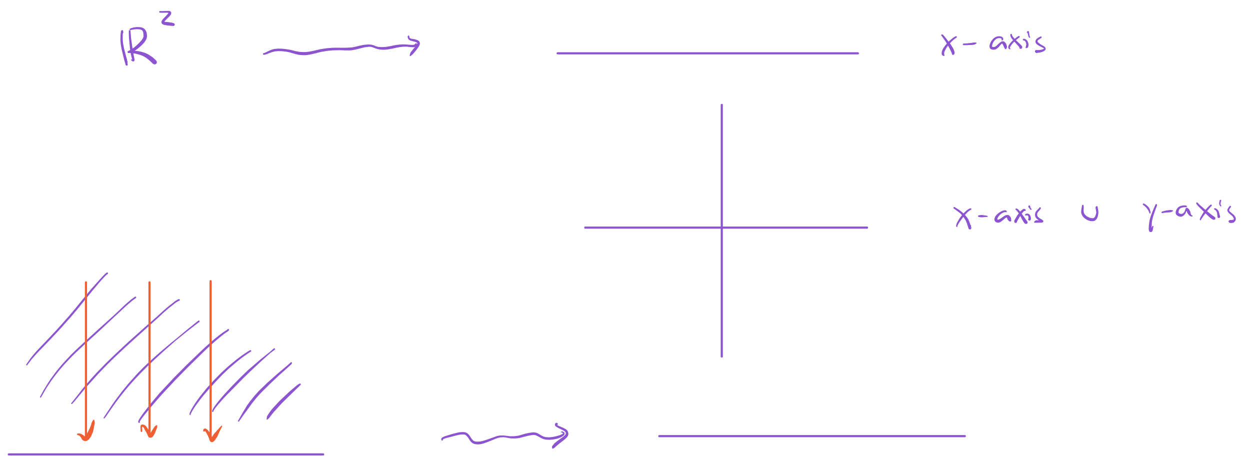
$$x \longmapsto x_0$$

Ex $\quad r: X \times Y \longrightarrow \{x_0\} \times Y$ is a retraction

$\qquad (x, y) \longmapsto (x_0, y)$

Ex





Recall Let $A \subseteq X$ be a retract.

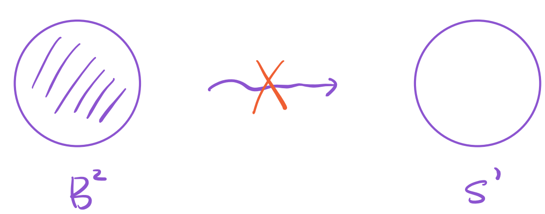
① If X is connected / path-connected / compact, then same is true for A.

② If X is Hausdorff, then A is closed in X.

Focus today:

Thm (No-retraction)

S^1 is not a retract of B^2.



(Harder to prove: need to show no retraction exists

Intuition: can't "punch a hole" in the middle of B^2)

Strategy: look at induced maps on fundamental groups

Lemma If $A \subset X$ is a retract, and let $j: A \to X$ denote the inclusion, then $j_*: \pi_1(A) \to \pi_1(X)$ is injective.

Pf Let $r: X \to A$ be a retraction. Then

$r \circ j : A \to X \to A$ is the identity on A.

$\Rightarrow r_* \circ j_* : \pi_1(A) \to \pi_1(X) \to \pi_1(A)$ is the identity on $\pi_1(A)$

$\downarrow$ j_* injective $\searrow$ r_* surjective $\square$

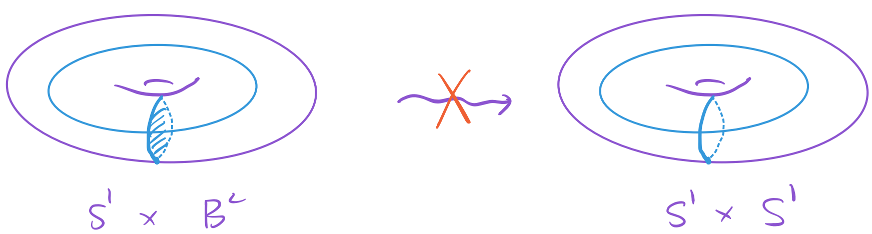
Proof of No-retraction Thm If $S^1 \subset B^2$ is a retract,

by Lemma: $j_*: \pi_1(S^1) \to \pi_1(B^2)$ is injective.

But this is impossible: $\pi_1(S^1) = \mathbb{Z}$

$\pi_1(B^2)$ is trivial. $\square$

Exercise $S^1 \times S^1$ is not a retract of $S^1 \times B^2$



Null homotopies

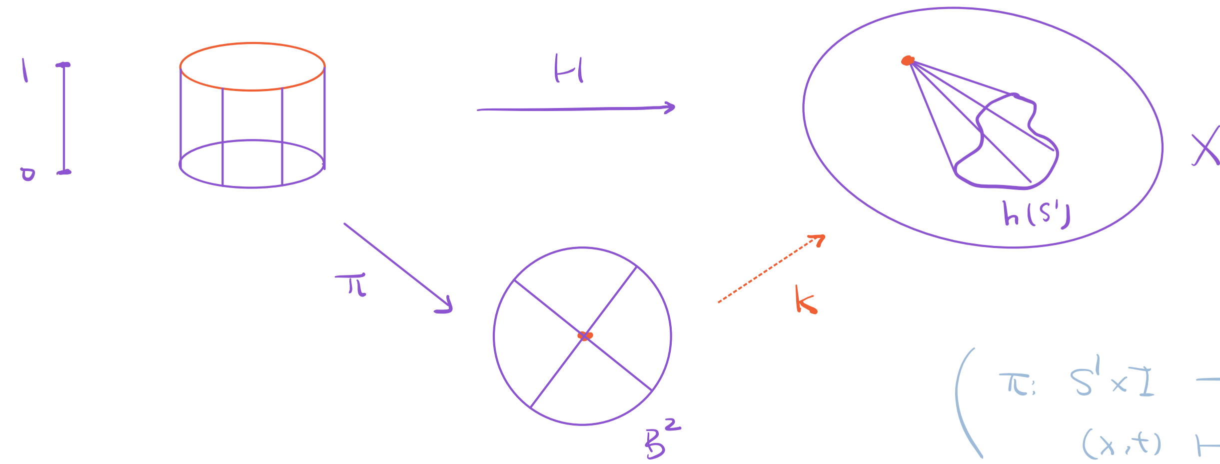
Lemma Let $h: S^1 \to X$ be continuous. Then the following are equivalent:

① h is nullhomotopic.

② h extends to a map $k: B^2 \to X$.

③ $h_*: \pi_1(S^1) \to \pi_1(X)$ is trivial.

Proof (① ⇒ ②) Let $H: S^1 \times I \longrightarrow X$ be a nullhomotopy.



$$\left(\begin{aligned} \pi: S^1 \times I &\longrightarrow B^2 \\ (x, t) &\longmapsto (1-t)x \end{aligned} \right)$$

Now, $B^2 = S^1 \times I \,/\, S^1 \times \{1\}$,

and $H\big|_{S^1 \times \{1\}}$ is constant

$\Rightarrow$ H factors through the quotient, inducing $k: B^2 \longrightarrow X$ //

(② ⇒ ③) Consider $h = k \circ j$:

$$\begin{tikzcd} S^1 \arrow[rr, "h"] \arrow[dr, "j"'] & & X \\ & B^2 \arrow[ur, "k"'] & \end{tikzcd}$$

inclusion

$\Rightarrow h_* = k_* \circ j_*$

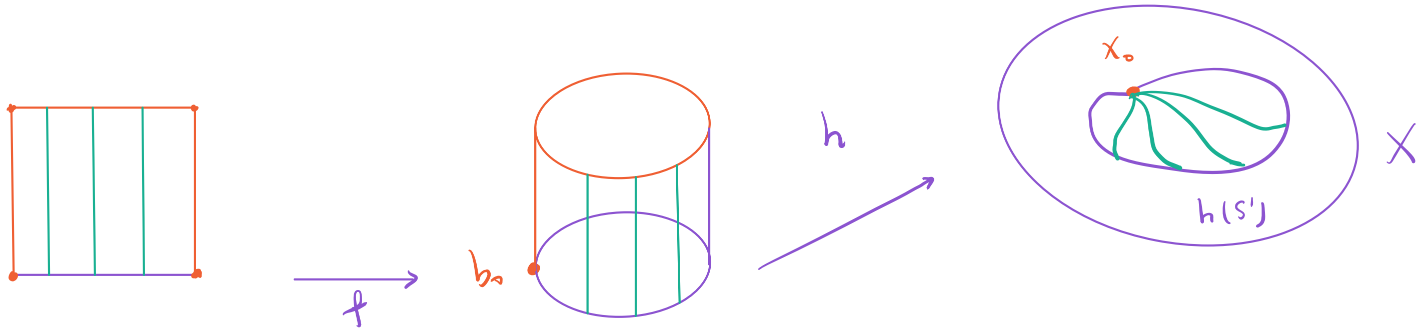
$$\begin{tikzcd} \pi_1(S^1) = \mathbb{Z} \arrow[rr, "h_*"] \arrow[dr, "j_*"'] & & \pi_1(X) \\ & \pi_1(B^2) = 1 \arrow[ur, "k_*"'] & \end{tikzcd}$$

Since $\pi_1(B^2)$ is trivial, j_* is trivial, then so is h_* //

(③ ⇒ ①) Fix base points $b_0 \in S^1$, $x_0 = h(b_0) \in X$.

Let $f: I \longrightarrow S^1$ represent $1 \in \pi_1(S^1, b_0)$

Since h_* is trivial, $h_*[f]$ is the identity of $\pi_1(X, x_0)$



$\Rightarrow$ $\exists$ null homotopy $F : I \times I \longrightarrow X$ between $h \circ f$ and the constant map to x_0

$F(0,t) = F(1,t) = F(s,1) = x_0 \quad \forall\ s,t$

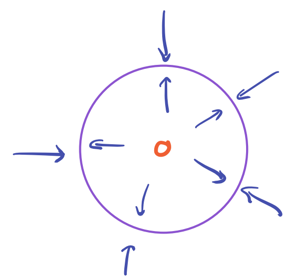
$\Rightarrow$ Under the quotient $S^1 \times I = I \times I / (0,t) \sim (1,t)$,

F induces a null homotopy $S^1 \times I \longrightarrow X$ between h and the constant map. $\square$

Corollary ① The identity map $S^1 \longrightarrow S^1$ is not null homotopic.

② The inclusion $S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ is not null homotopic.

Proof of ②: S^1 is a retract of $\mathbb{R}^2 \setminus \{0\}$:



$\Rightarrow$ $\pi_1(S^1) \longrightarrow \pi_1(\mathbb{R}^2 \setminus \{0\})$ is injective, and thus nontrivial. $\square$

Fixed points

A point $x \in X$ is a fixed point of $f : X \longrightarrow X$ if $f(x) = x$

Warm-up Every continuous $f : [0,1] \longrightarrow [0,1]$ has a fixed point.

(Consider $f(x) - x$ and apply IVT)

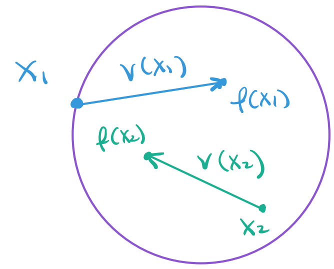
Brouwer Fixed-point Thm

Every continuous $f: B^2 \longrightarrow B^2$ has a fixed point.

Proof Suppose otherwise that $f(x) \neq x \quad \forall\, x \in B^2$.

Define $v: B^2 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ $\longrightarrow$ nowhere-vanishing vector field.

$$v(x) = f(x) - x$$



Observation: $\forall\, x \in S^1$, $v(x)$ can't point <u>directly outward</u>

i.e. $v(x) = f(x) - x = a\,x$ for some $a \in \mathbb{R}_{>0}$

(otherwise $f(x) = (1+a)\,x \notin B^2$)

Denote $w = v|_{S^1} : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$

Since v is an extension of w to B^2, by earlier Lemma, w is null homotopic.

Moreover, $-w$ is also homotopic to the inclusion $j: S^1 \to \mathbb{R}^2 \setminus \{0\}$,

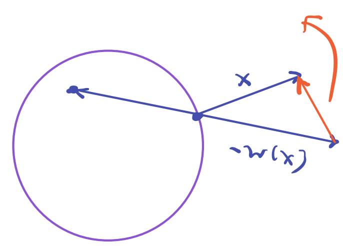
via $F: S^1 \times I \longrightarrow \mathbb{R}^2 \setminus \{0\}$

$$F(x,t) = t\,x - (1-t)\,w(x)$$

homotopy avoids 0

<u>Check</u>: $\forall\, x \in S^1$, $t \in (0,1)$, $0 \neq F(x,t)$

Otherwise, $w(x) = \underbrace{\frac{t}{1-t}}_{>0}\, x$.



$\Longrightarrow$ j is null homotopic, a contradiction. $\square$

didn't get to cover in class