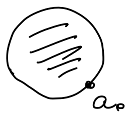
Lecture, Nov 15

$A \overset{i}{\subset} X \qquad X \xrightarrow{p} A \ni a_0 \qquad \pi_1(X, a_0) \to \pi_1(A, a_0)$

$p i = id_A$ surjection



disk D^2

S^1 boundary circle

$\nexists$ retraction $D^2 \to S^1$

$\pi_1(D^2, a_0) \to \pi_1(S^1, a_0)$

$\cong \qquad \cong$

$1 \qquad \mathbb{Z}$

$\{0,1\}$

I connected,

no retraction $I \to \{0,1\}$

not connected

no retraction onto

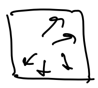
$D^n \to S^{n-1}$

$D^n \subset \mathbb{R}^n$

$= \{ v \mid |v| \leq 1 \}$

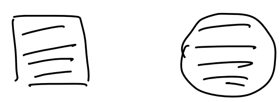
$\partial I^2 = A$

I^2

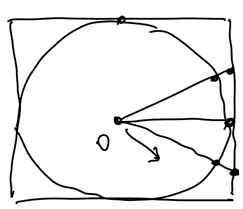


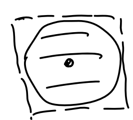
A

$I^2 \xrightarrow{?} A$



D^2 is homeomorphic to I^2

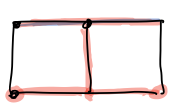




Ex Complete this argument, get a homeomorphisms

$$\begin{array}{ccc} D^2 & \to & I^2 \\ \cup & & \cup \\ S^1 & \to & P \end{array}$$ perimeter

topologically, pair (D^2, S^1) is "the same" as (I^2, P)



$A \subset X$

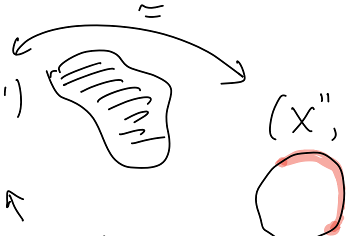
(X, A)

(X', A')

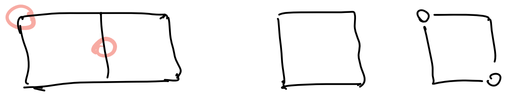
$(X, A) \simeq (X', A')$

topologically, $\exists$ no difference

$\simeq$

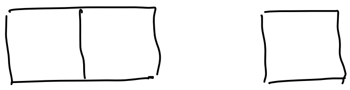


(X'', A'')



Conn

2 conn comp



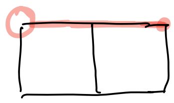
$\pi_1(X)$ - free group on 2 generators

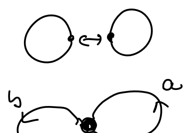
$\pi_1(S^1) \cong \mathbb{Z}$

$\mathbb{R}^2$

$A \subset X$

<u>Exercise</u>: find all retracts of X.



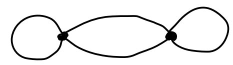


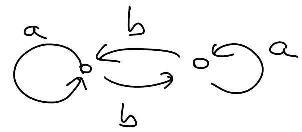
$S^1 \sqcup S^1$

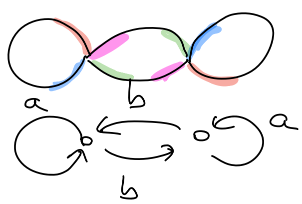
$\downarrow q$

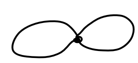
figure 8

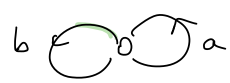
X

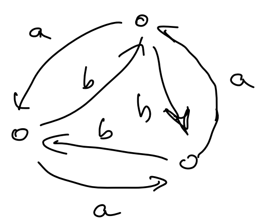


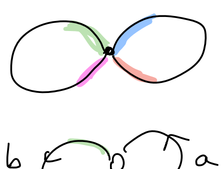










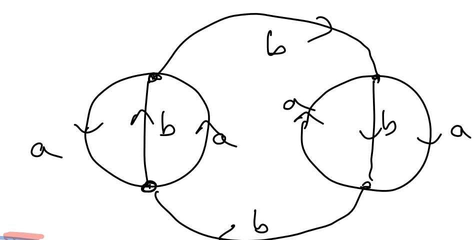


X

$\cancel{b^{-1}b}$, $\cancel{a^{-1}a}$

$\cancel{bb^{-1}}$ $\cancel{aa^{-1}}$

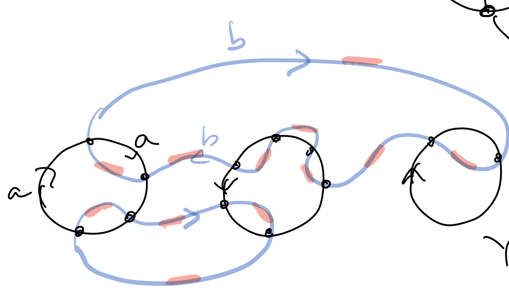
$aba^2b^{-3}a^{-2}b\ldots$



a, b free,

$\mathbb{Z} * \mathbb{Z}$ not abelian

free group grows fast

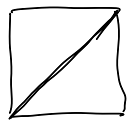


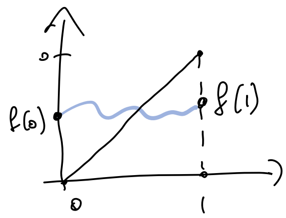
Y

$\pi_1(Y) \longrightarrow \pi_1(X)$ as an index 12 subgroup

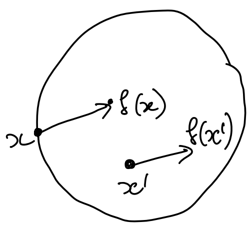
Brauwer fixed point theorem $\forall$ $f: B^2 \to B^2$ has a fixed point
continuous D^2

$f: I \xrightarrow{f} I$





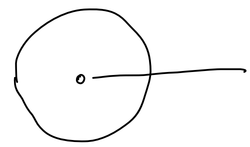
$I \to I^2 \setminus$ diagonal
miss diagonal



$f(x) \neq x \quad \forall x \in B^2$

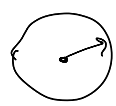
$v(x) = f(x) - x$ is not zero

$v(x') = f(x') - x'$



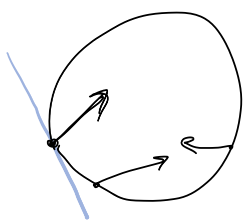
$-g(x') = \dfrac{-(f(x') - x')}{|f(\cancel{x}') - x'|}$ vector is a map $B^2 \to \mathbb{R}^2 \setminus \{0\} \simeq S^1 \times \mathbb{R}_{>0}$

rescale $g(x) = \dfrac{v(x)}{|v(x)|} \in S^1$

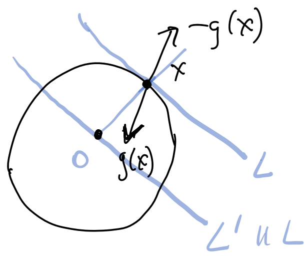


$\dfrac{v(x)}{|v(x)|}$

$\overset{\shortparallel}{-\dfrac{v(x)}{|v(x)|}}$ $\quad B^2 \xrightarrow{g} S^1 \qquad g$ on S^1



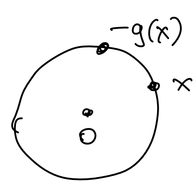
$g|_{S^1}$ map from S^1 to S^1



vectors $-g(x)$ and x point in the same direction relative to L'

we can create a homotopy from $-g$ map on S^1 and identity.

$S^1 \underset{id}{\overset{-g}{\rightrightarrows}} S^1$



move $-g(x)$ to x linearly in t $\quad t \in [0,1]$

this gives a homotopy $F: S^1 \times I \to S^1$

$F(x, 0) = -g(x), \quad F(x,1) = x$

$-g(x)$ is homotopic to id. $\qquad \pi_1(S^1) \xrightarrow{id} \pi_1(S^1)$

$-g(x)$ induces an isom. of fund group $\qquad \mathbb{Z} \xrightarrow{id} \mathbb{Z}$

get an extension of $-g(x)$ to a continuous map

$$\begin{tikzcd} B^2 \arrow[r, "-g"] & S^1 \\ S^1 \arrow[u, hook] \arrow[ur, "-g(x)"'] & \\ x \arrow[u, phantom, "\in"] \arrow[uur, mapsto] & \end{tikzcd}$$

B^2 contractible

B^2

$$\begin{tikzcd} \pi_1(B^2, x_0) \arrow[r] & \pi_1(S^1, -g(x_0)) \\ \pi_1(S^1, x_0) \arrow[u] \arrow[ur, "\simeq"] & \mathbb{Z} \end{tikzcd}$$

$\pi_1(S^1, x_0) \cong \mathbb{Z}$

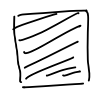
$0+0=0$

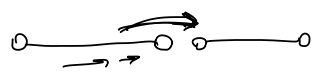
$$\begin{tikzcd} B^2 & \pi_1(B^2) = 0 \arrow[r] & \pi_1(S^1) \cong \mathbb{Z} \\ S^1 \arrow[u, hook] & \pi_1(S^1) \cong \mathbb{Z} \arrow[u] \arrow[ur, "id"'] & \end{tikzcd}$$

Contradiction

$S^1 \hookrightarrow D^2$

$$\begin{tikzcd} \pi_1(S^1) \arrow[r] & \pi_1(D^2) \arrow[r, "\exists ?"] & \pi_1(S^1) \\ \mathbb{Z} \arrow[r] \arrow[rr, bend right, "\text{id not possible}"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$





$(0,1)$

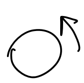
$X \xrightarrow{f} X$

$\exists$ a fixed point

usually take X compact

$(0,1) \simeq \mathbb{R}$ $\qquad \mathbb{R} \to \mathbb{R}$

$x \mapsto x+a$

S^1 

open $B^2 \simeq \mathbb{R}^2$

not homeomorphic

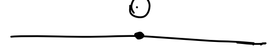
$\mathbb{R}^2 \neq \mathbb{R}^3$

$\mathbb{R}^2 \setminus \{0\} \qquad \mathbb{R}^3 \setminus \{0\}$

π_1 nontrivial $\qquad \pi_1$ trivial

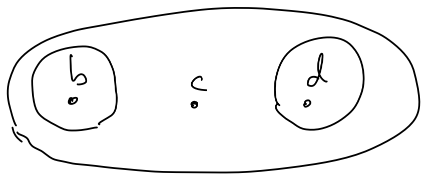
<u>Quotient topology</u>

$b \qquad c = \{0\} \qquad d \qquad \mathbb{R}$

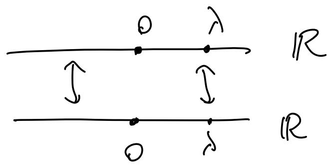


$b = \{a \mid a < 0\} \qquad \{a \mid a > 0\} = d$

3 equivalence classes $\qquad$ Quotient $\qquad$ q. topology.



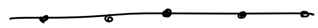
topology of orbits



$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}$

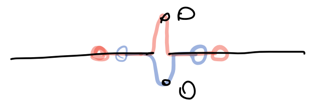
$\mathbb{R}$ action of group $G = \mathbb{R}_{>0}$

λ



$t \to \lambda t$ $a \to \lambda a$

orbits of group action



not hausdorff