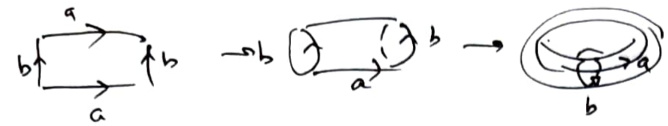
Quotient topology (§22)



Sets

How to form new sets from old?
take a subset $Y \subset X$
intro an equiv. relation $\sim$. form quotient set $X/\sim$
$P(X)$, ...

X - top. space, $\sim$ equiv. relation
let $Y = X/\sim$ Y is a set. Can we turn Y into a top space? Need to define open sets : $U \subset Y$ open ~~iff~~ iff $p^{-1}(U)$ is open in X. $\quad p: X \to Y = X/\sim$

Exercise: this is a topology on Y. Called quotient topology.

Note that $X \xrightarrow{p} X/\sim$ is continuous, surjective | Saturated set.

Suppose $X \xrightarrow{f} Y$ is $\underbrace{\text{continuous, surjective}}$. Is $Y \simeq X/\sim$? In general, no.
$\Rightarrow U \subset Y$ open $\Rightarrow f^{-1}(U)$ open in X.
Because there might be
but there might be
$V \subset X$, $V = f^{-1}(U)$ some $U \subset Y$
V - open in X, U not open in Y.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

$Y = X/\sim$ as sets.

use f to partition X, get $\sim$. form $p: X \to X/\sim$
but open in $X/\sim$

h bijection $X/\sim \to Y$

$f = hp$

Quotient space $X/\sim$ has the finest possible topology among topologies on $X/\sim$ s.t. $X \to X/\sim$ is continuous. Extreme example: Y is indiscrete.

Def $p: X \to Y$ surjective, p is a quotient map if $U \subset Y$ open iff $p^{-1}(U) \subset X$ open

$\Rightarrow$ $A \subset Y$ closed iff $p^{-1}(A)$ closed.

$C \subset X$ is saturated rel. to $p: X \to Y$ if C contains $\forall\, p^{-1}(\{y\})$ it intersects. (has a point in the fiber $\Rightarrow$ contains the entire fiber).

p - quotient map iff p - continuous & maps saturated open (of X) to ~~saturated~~ open of Y.

$f: X \to Y$ open map if $\forall \overset{\text{open}}{U} \subset X$ $f(U)$ is open in Y.
closed map if $\forall$ closed $A \subset X$ $f(A)$ is closed.

__Prop__ ~~of~~ $p: X \to Y$ surjective continuous; either open or closed $\Rightarrow$ p is a quotient map.

How do tell if a map is

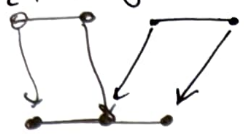
open: ~~$X \to$~~ $X \times Y \to X$ proj. (comp of open is open)

open: inclusion $\underset{\text{open}}{U} \hookrightarrow X$

closed: Inclusion of closed subset ~~X~~ $\hookrightarrow X$

Comp of closed is closed

$\{a, b\} \cup \{c, d\}$



$X \xrightarrow{f} Y$

__Prop__ A continuous map from compact X to Hausdorff Y is closed.

__Prop__ (Corollary) A surjective continuous map from compact X to Hausdorff Y is a quotient map.

__Proof__ $A \subset X$ closed $\Rightarrow$ A - compact (since X compact) $\Rightarrow$ $f(A) \subset Y$ compact.

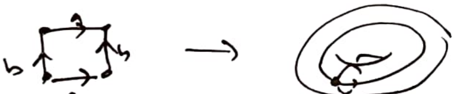
$\Rightarrow f(A) \subset Y$ is closed. $\Rightarrow f$ is closed, f is a quotient map.
(any compact subspace of H. space is closed)

Quotient map.

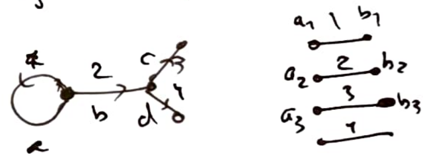
__Examples.__

finite graphs (topologically)

$\coprod_{\alpha \in I} I_\alpha$ + equiv. rel'n on $I \amalg I$.
(glue endpoints)



compact $\qquad$ Hausdorff $\Rightarrow$ closed $\Rightarrow$ quotient



a $a_0 \sim a_1 \sim b_0$,
$b_1 \sim c_0 \sim d_0$

Often, Quotient topology is not Hausdorff.

__Examples:__ G acts on X, ~~equiv~~ by homeomorphisms. a equiv classes

__graphs__

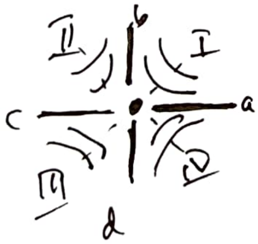
$X = \mathbb{R}$ $\quad G = \mathbb{R}_{>0}$ $\quad x \mapsto \lambda x$
$\lambda \in G$

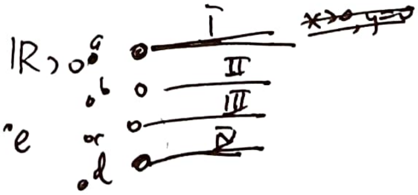
$\mathbb{R}_{<0}$ $\{0\}$ $\mathbb{R}_{>0}$

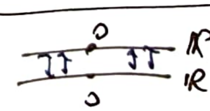
$\mathbb{R}^2$ $\quad (x, y) \sim (\lambda x, \lambda^{-1} y)$
$\lambda \in \mathbb{R}_{>0}$

open $\qquad$ open

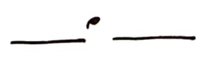
$\{a\}, \{c\}, \{a\} \cup \{c\}$,
$\emptyset, Y$.
not Hausdorff







double point



non-Hausdorff.

line with double point.