lecture 10 Topology, Fall 2022

Compactness (§26-28 in [M])

Def A collection $\mathcal{A}$ of subsets of X covers X / (a covering of X) if union of its elements is X

Open covering if each subset is open

Def X is compact if $\forall$ open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Examples 1) $\mathbb{R}$, $(0,1)$ are not compact

2) $X = \{0\} \cup \{\frac{1}{n}\}_{n \geq 1}$ compact

3) X, finite complement topology - compact.

Remark: When testing for compactness, can restrict to open coverings of X with all subsets from a fixed basis $\mathcal{B}$.

Prop Let $Y \subset X$. Y is compact iff every covering of Y by sets open in X contains a finite subcollection that covers Y.

Thm (26.2) X - compact, $Y \subset X$ closed $\Rightarrow$ Y compact

Covering $\{U_\alpha\}$ of $Y \Rightarrow \exists$ open sets $V_\alpha \subset X$, $U_\alpha = V_\alpha \cap Y$

$\{V_\alpha\}_\alpha \cup \{X \setminus Y\}$ covers X. Choose a finite subcollection that covers.

Thm (26.3) $\forall$ compact subspace of a Hausdorff space is closed.

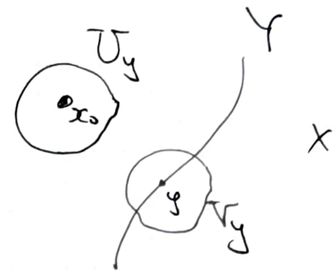
(for a non-Hausdorff counterexample use indiscrete or the finite complement topology)

Proof: see next page.

<u>Proof</u>: $Y \subset X$ Hausdorff
compact

need to show $X \setminus Y$ open. Let $x_0 \in X \setminus Y$.

$\forall y \in Y$ choose disjoint neighbourhoods
U_y and V_y of x_0, y



$\{V_y\}_y$ covers Y. Choose a finite subcovering $\{V_{y_1}, \ldots, V_{y_n}\}$.

$U = U_{y_1} \cap \ldots \cap U_{y_n}$ is a neighbourhood of x_0 disjoint from Y $\Rightarrow$ Y is closed in X. $\square$

Lemma (26.4) Y compact $\subset X$ Hausdorff, $x_0 \notin Y \Rightarrow$
$\exists$ disjoint open $U \ni x_0$, $V \supset Y$ subsets of X. $\square$

<u>Application of thm 26.2</u>: Once we prove that $[a,b]$ is compact $\Rightarrow$
$\forall$ closed subset of $[a,b]$ is compact

<u>Thm</u> ~~The~~ image of a compact ~~set~~ space under a continuous map is compact.
<u>Pf</u>: $f: X \to Y$, $f(X) \subset Y$. Take a covering $\mathcal{A}$ of $f(X)$
Look at covering $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ of X. Choose a finite subcovering by $f^{-1}(A_1), \ldots, f^{-1}(A_n)$. Then $f(X)$ is covered by $A_1, \ldots, A_n$. $\square$

Thm (26.6) If $f: X \to Y$ is bijective & continuous, X - compact, Y - Hausdorff $\Rightarrow$ f is a homeomorphism.

Pf To show f^{-1} continuous, enough to show $A \subset X$ closed $\Rightarrow$ $f(A) \subset Y$ closed.

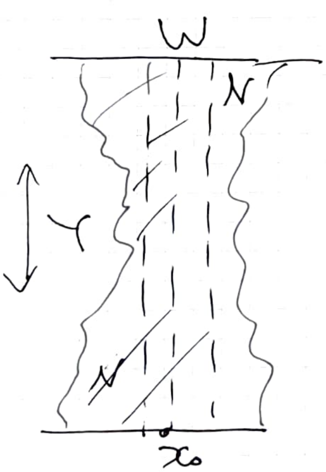
$A \subset X$ closed $\Rightarrow$ A - compact $\Rightarrow$
$f(A)$ compact, $f(A) \subset Y$ - Hausdorff
$\Rightarrow f(A)$ closed. $\square$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f^{-1}"] \\ A \arrow[u, phantom, "\cup"] \arrow[r, bend right] & f(A) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

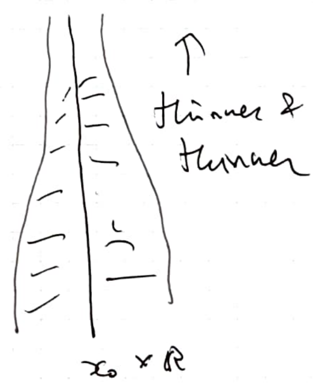
Remark: Under these assumptions, if we try to give X finer topology than what it inherits from Y, the topology would have too many open sets & X won't be compact.

Thm Finite products of compact spaces are compact.
Enough to do the case of 2 spaces.

Tube lemma If X, Y given, Y - compact, $x_0 \in X$ & $N \subset X \times Y$ open, $N \supset x_0 \times Y$. $\Rightarrow \exists$ neighbourhood W of x_0 in X s.t. $N \supset W \times Y$



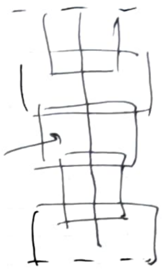
fails in Y not compact, $\mathbb{R} \times \mathbb{R}$



<u>Proof of lemma</u> Cover $x_0 \times Y$ by basis elements $U \times V$. $x_0 \times Y \simeq Y$ is compact. $\exists$ subcover $x_0 \times Y$ by finitely many $U_1 \times V_1, \ldots, U_n \times V_n$ s.t. $x_0 \in U_i$. Let $W = U_1 \cap \ldots \cap U_n \Rightarrow$ W open, contains $x_0 \Rightarrow U_1 \times V_1, \ldots, U_n \times V_n$ covers $W \times Y$ □.

<u>Proof of theorem</u>. X, Y compact, $\mathcal{A}$-open covering of $X \times Y$. For $x_0 \in X$, $x_0 \times Y$ compact $\Rightarrow$ fin. many elements $A_1 \ldots A_m$ of $\mathcal{A}$ that cover it $N = A_1 \cup \ldots \cup A_m$ open set containing $x_0 \times Y$ $\Rightarrow N \supset W \times Y$ a tube

$U_i \times V_i$



x_0

$\forall x \in X$ choose neighborhood W_x of x s.t. tube $W_x \times Y$ is covered by finitely many elements of $\mathcal{A}$

$\{W_x\}_{x \in X}$ is an open covering of $X \Rightarrow$ choose finite subcovering $\Rightarrow$ get a finite covering of $X \times Y$. □

<u>Remark</u> Product of ∞ many compact spaces is compact. (example: Cantor set)

<u>Def</u> A collection $\mathcal{C}$ of subsets of X has a finite intersection property if $\forall$ finite subcollection $\{C_1 \ldots C_n\}$ the intersection $C_1 \cap \ldots \cap C_n \neq \emptyset$. $\Big\} \Leftrightarrow$ complements don't cover X

<u>Thm</u> X compact iff $\forall \mathcal{C}$ with finite intersection property $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

<u>Pf</u>: Given coll $\mathcal{A} \Rightarrow$ coll $\mathcal{C} = \{X \setminus A \mid A \in \mathcal{A}\}$ coll. of complements

(1) $\mathcal{A}$ coll. of open $\Leftrightarrow$ $\mathcal{C}$ coll of closed

(2) $\mathcal{A}$ covers $X \Leftrightarrow \bigcap_{C \in \mathcal{C}} C = \emptyset$.

X compact. Contrapositive "Given any collection of open sets that covers X, then some finite subcollection covers X.

Special case: nested sequence $C_1 \supset C_2 \ldots \supset C_n \supset C_{n+1} \supset \ldots$ of closed sets

if $C_n \neq \emptyset \Rightarrow \mathcal{C} = \{C_n\}_{n\geq 1}$ has finite int. property $\Rightarrow$

$\bigcap_{n\geq 1} C_n \neq \emptyset$. X compact