Topology, fall 2022

Homework 6, due Thursday October 27.

Read Munkres section §28. We stated but did not prove Theorem 28.2 there that for metrizable spaces all three versions of compactness are equivalent.

I-II. Work through the proof of Theorem 28.2 in Munkres.
I. (10 points) Read the proof of the implication $(2) \Longrightarrow (3)$ and explain the implication in your own words, adding a picture or pictures to visualize what happens in the proof.
II. (10 points) Read the proof of the implication $(3) \Longrightarrow (1)$ and draw pictures to help you understand parts of the proof. Provide explanatory notes to your pictures.

III. (10 points) Exercise 2 on page 181.

IV. A continuous map is called *closed* if it takes closed sets to closed sets. Exercise 6 on page 171 gives a sufficient criterion for a map to be closed, see a proof *here*, for instance.
(10 points) Explain why the following maps are not closed. All topologies are the usual ones.

(a) Projection $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \mapsto x$.

(b) Inclusion $[0, 1) \subset \mathbb{R}$.

V. (10 points) Which of the following subspaces A of $\mathbb{R}^2$ with the usual topology are compact? Briefly explain.

(a) $A = \{(x, y) | x + y \in \mathbb{Z}\}$.

(b) $A = \{(x, y) | x^2 + y^2 = 3 \text{ and } y \geq 0\}$

(c) $A = \{(x, y) | x \in \mathbb{Q}, 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

We discussed complete metric spaces following Hocking and Young "Topology", see *this link* or find it on our webpage (Section 43 in Munkres covers similar material). You can also read about complete metric spaces *here* and *here*.
We proved that a compact metric space is complete, consequently the closed interval $[a, b]$ with the usual metric is complete.

VI. (10 points) Use this to prove that $\mathbb{R}$ is complete.

VII. (10 points) Review the results on complete metric spaces and determine which of the following subspaces of $\mathbb{R}^2$ are complete in the metric induced by the standard Euclidean metric of $\mathbb{R}^2$:

(a) The unit circle $\mathbb{S}^1 = \{(x, y)|x^2 + y^2 = 1\}$.

(b) The open half-plane $\mathbb{R}^2_{>0} := \{(x, y)|y > 0\}$

(b) The set of points on the unit circle with the rational coordinate $\{e^{i\pi\phi}|\phi \in \mathbb{Q}\}$.

In each case describe the completion of the corresponding space.