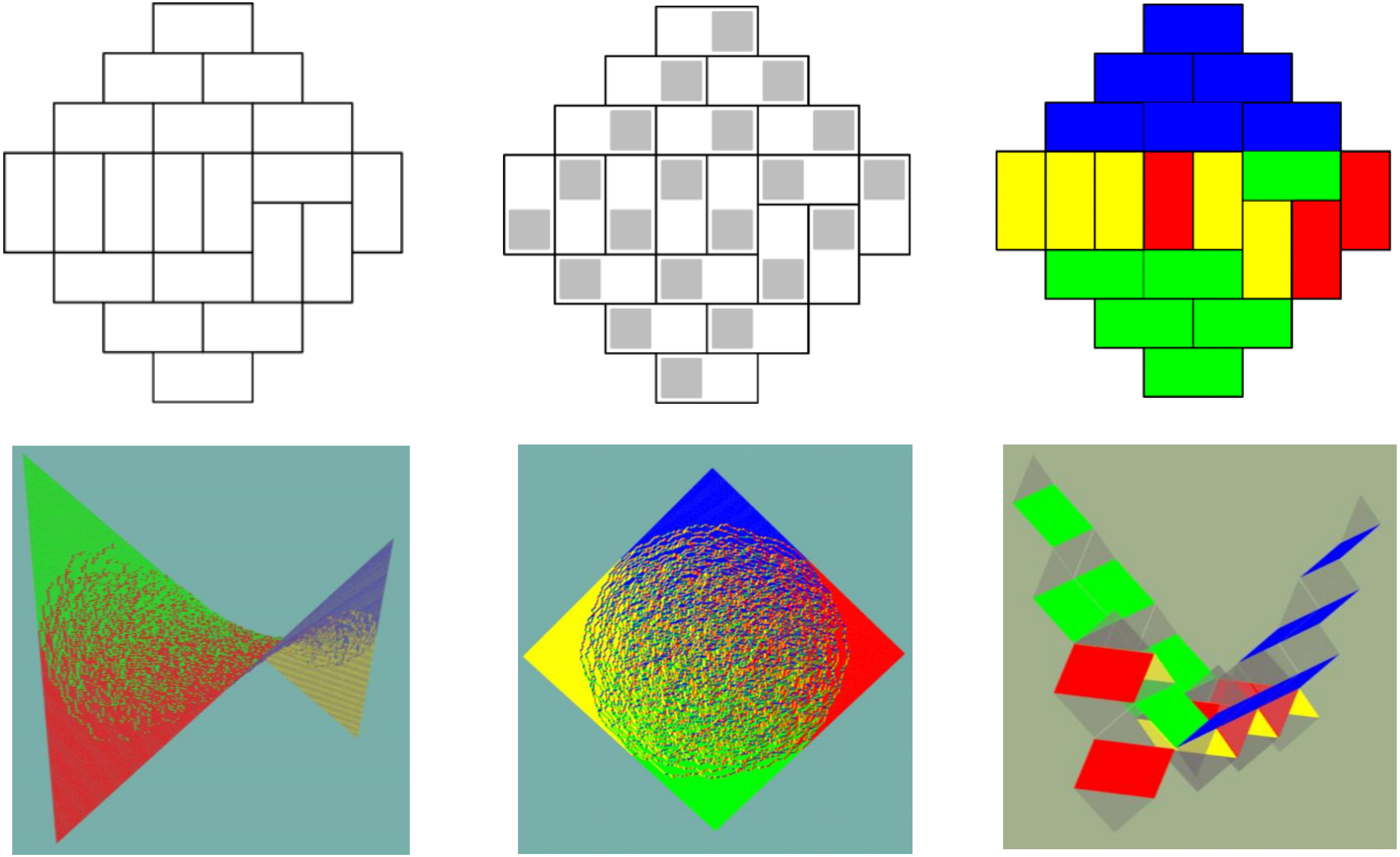
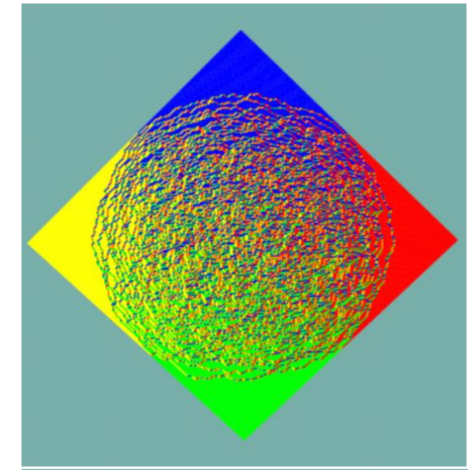
Biased 2×2 periodic Aztec diamond and an elliptic curve

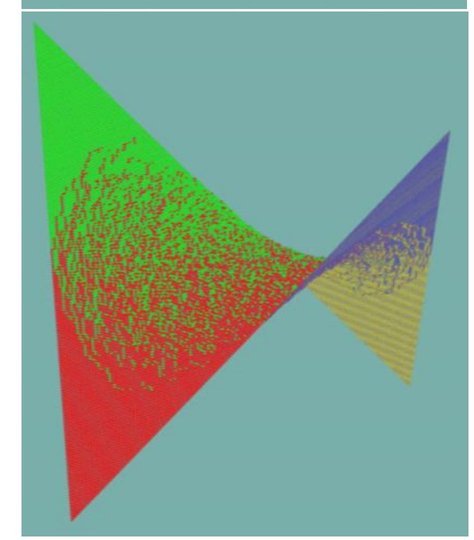
A. Borodin, joint with M. Duits



Random domino tilings of the Aztec diamond is a very well studied model. Here are a few key facts:

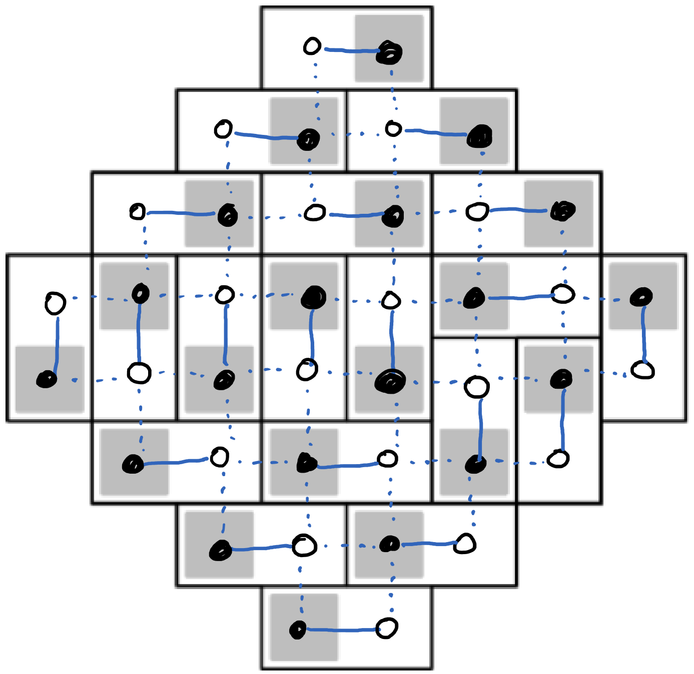
- Total number of tilings is $2^{n(n+1)/2}$.
- There is a sampling algorithm, known as shuffling, that involves only independent Bernoulli $\{0,1\}$ trials.
- The frozen boundary is a circle, called Arctic.
- Local fluctuations are described by an explicit 2-param. family of translation invariant Gibbs measures.
- Frozen edge fluctuations are described by the Airy_2 process.
- Global surface fluctuations are given by the 2d GFF.

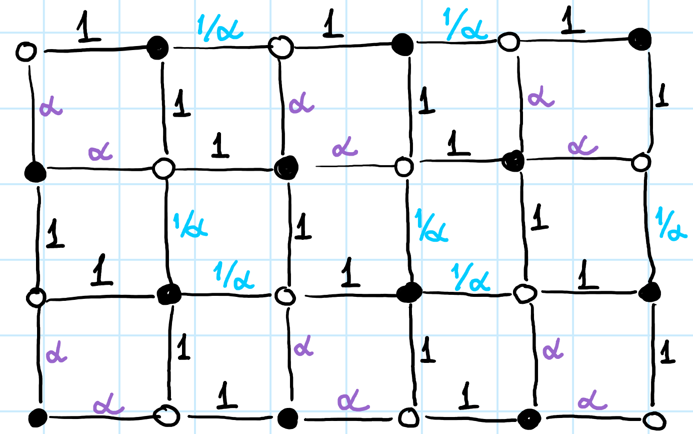


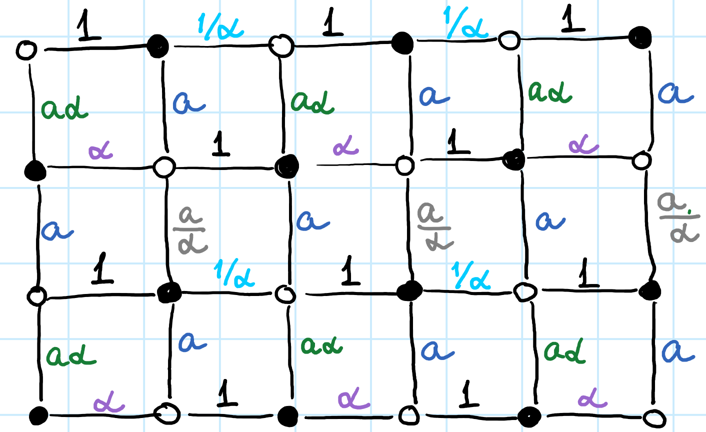


N. Elkies, G. Kuperberg, M. Larsen, J. Propp, 1992.
W. Jockusch, J. Propp, P. Shor, 1995
K. Johansson, 2000.
K. Johansson, 2003
R. Kenyon, A. Okounkov, S. Sheffield, 2003
S. Chhita, K. Johansson, B. Young 2012
A. Bufetov, V. Gorin, 2016
A. Bufetov, A. Knizel, 2016

2×2 – periodic setting

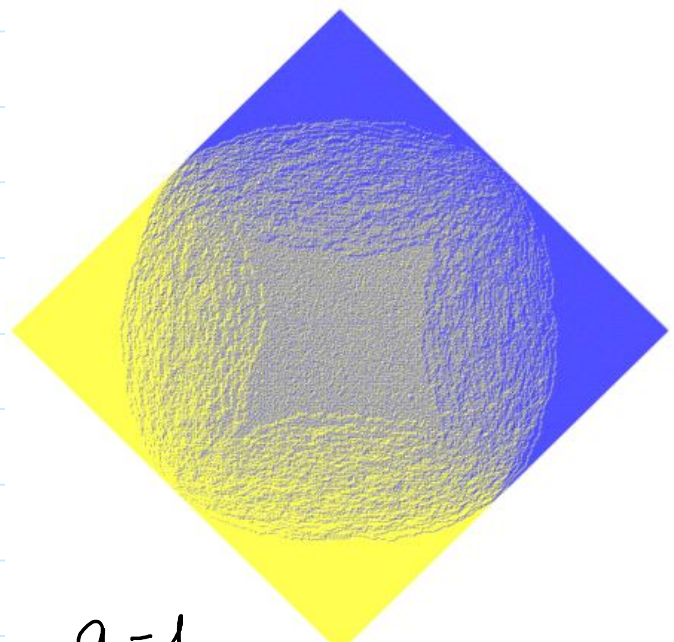




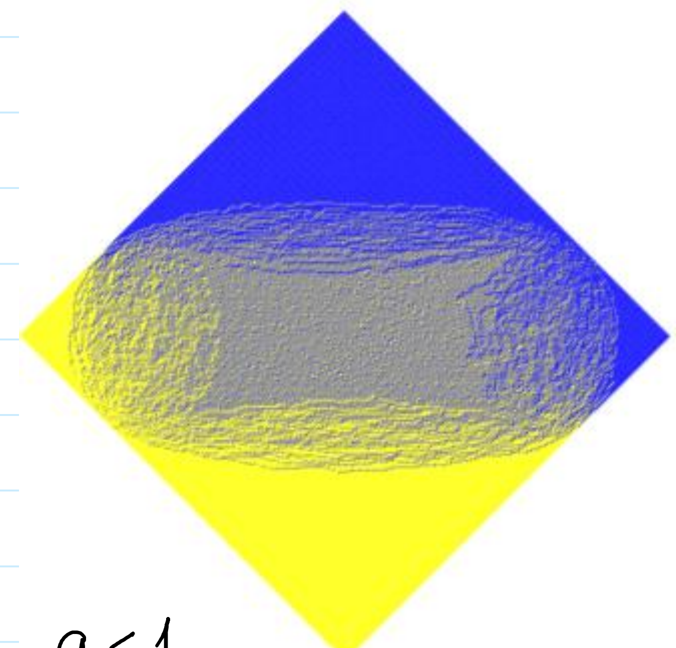


Prob(matching)

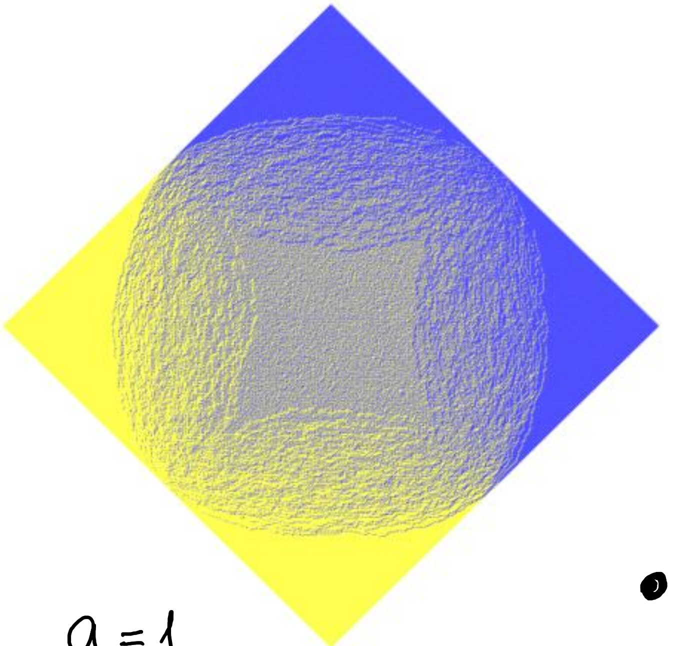
$\sim$ product of edge weights



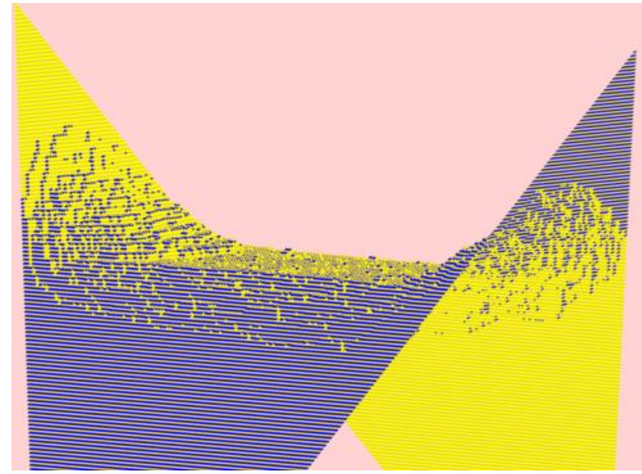
$a = 1$

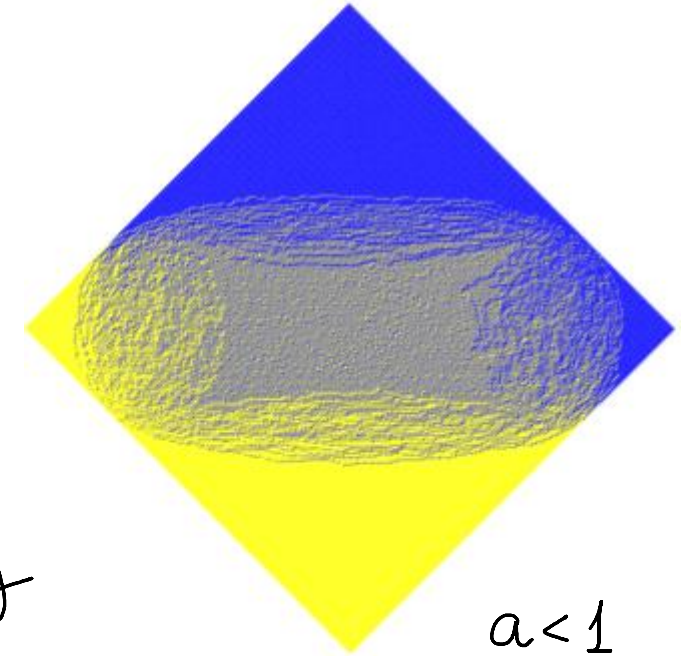


$a < 1$



$a=1$





$a<1$

- Shuffling works for arbitrary edge weights.

  Propp, 2001
  Chhita, the code

- New smooth/gas phase is expected to appear.

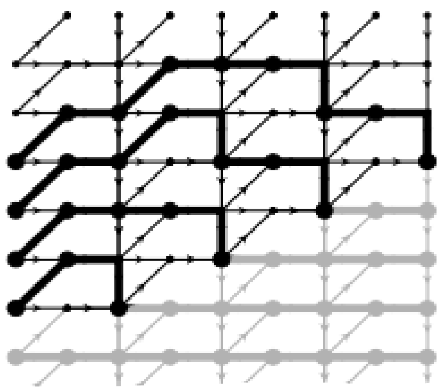
  Kenyon, Okounkov, Sheffield, 2003

- Explicit determinantal correlations
- Various asymptotic questions

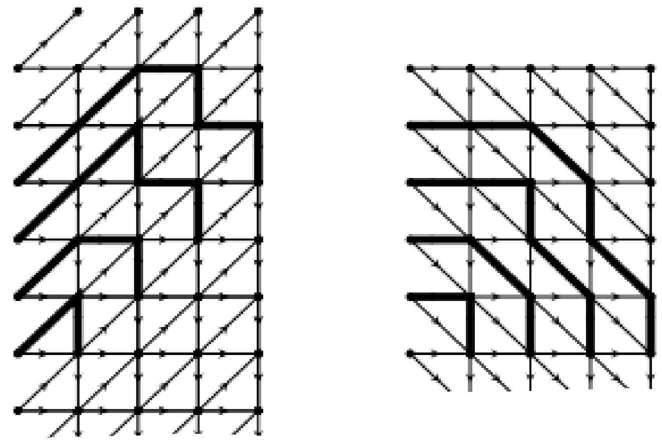
This talk

Chhita-Young 2013, Chhita-Johansson 2014, Beffara-Chhita-Johansson 2016, 2020, Johansson-Mason 2021, Duits-Kuijlaars 2017, Berggren-Duits 2019, Berggren 2019

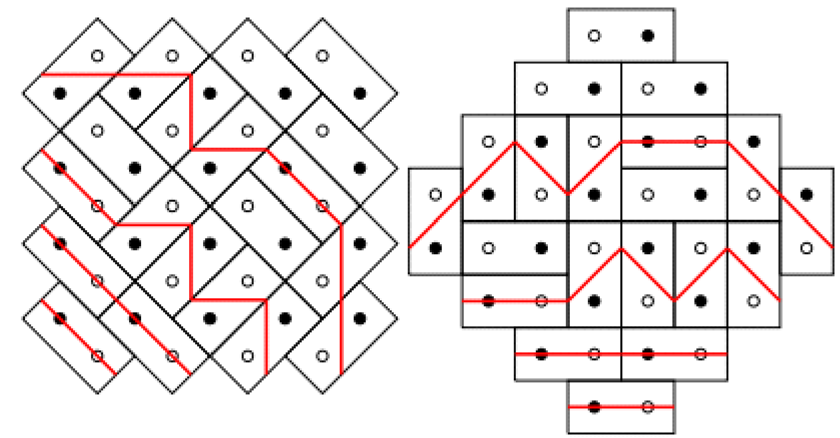
Step 1: Determinantal correlation functions



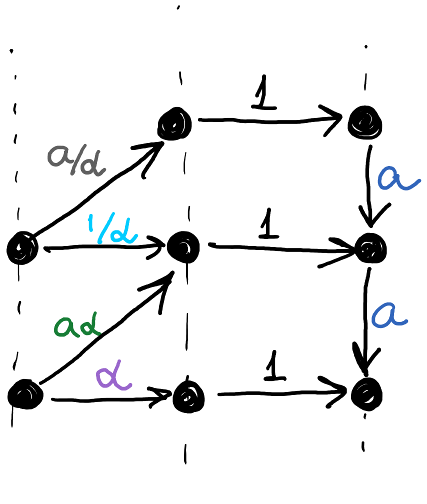
Nonintersecting paths $\longleftrightarrow$ "DR-paths"



Dana Randall

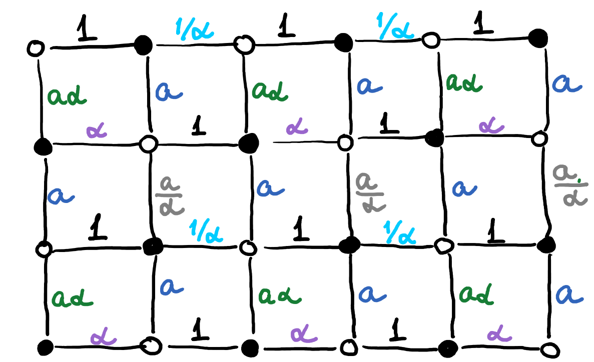


DR-paths $\longleftrightarrow$ Tilings



$$A_{odd} = \begin{bmatrix} \alpha & a\alpha z \\ a/\alpha & 1/\alpha \end{bmatrix} \qquad A_{even} = \frac{1}{1-\frac{a^2}{z}} \begin{bmatrix} 1 & a \\ a/z & 1 \end{bmatrix}$$

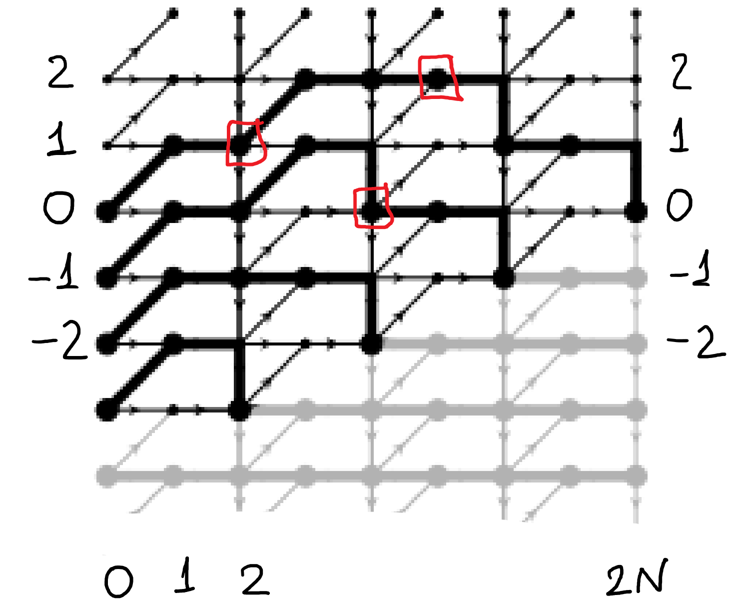
Symbols of column-to-column block-Toeplitz transition matrices



Correlations are encoded by the Wiener-Hopf factorization of $(A_{odd} A_{even})^N$.

Theorem (Berggren-Duits 2019, cf. Duits-Kuijlaars 2017)

$$\mathrm{Prob}\left\{\text{points at } \{(m_k, y_k)\}_{k=1}^{n}\right\} = \det\left[K((m_i,y_i),(m_j,y_j))\right]_{i,j=1}^{n}$$



$$\begin{bmatrix} K((m,2x+1),(m',2x'+1)) & K((m,2x+1),(m',2x')) \\ K((m,2x),(m',2x'+1)) & K((m,2x),(m',2x')) \end{bmatrix} = -\frac{\mathbb{1}_{m'<m}}{2\pi i}\oint_{|z|=1} \frac{A_{(m',m]}(z)\,dz}{z^{x-x'+1}} +$$

$$+\frac{1}{(2\pi i)^2}\oint_{|w|=\rho_1}\oint_{|z|=\rho_2} A_{(m',2N]}(w)\, A_+(w)^{-1} A_-(z)^{-1} A_{(0,m]}(z)\,\frac{w^{x'}}{z^{x}}\,\frac{dz\,dw}{z-w}$$

with $|a|^2<\rho_1<\rho_2<|a|^{-2}$, $A_{(p,q]} = A_{p+1}A_{p+2}\cdots A_q$, $A_{odd} = \begin{bmatrix} \alpha & a\alpha z \\ a/\alpha & 1/\alpha \end{bmatrix}$, $A_{even} = \frac{1}{1-\frac{a^2}{z}}\begin{bmatrix} 1 & a \\ a/z & 1 \end{bmatrix}$,

$$A(z) = A_{(0,2N]}(z) = A_-(z)\,A_+(z)$$

with $A_+^{\pm 1}(z)$ analytic in $|z|\le 1$

$A_-^{\pm}(z)$ analytic in $|z|\ge 1$

$A_-(z) \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as $z\to\infty$.

How to access $A_\pm(z)$?

Step 2: Matrix re-factorization

Elementary step: $\begin{bmatrix} \alpha & \gamma z \\ \beta & \delta \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b/z & d \end{bmatrix} = \begin{bmatrix} d & xc \\ b/zx & a \end{bmatrix} \begin{bmatrix} \delta x & \gamma z \\ \beta & \alpha/x \end{bmatrix}$, $x = \dfrac{\alpha a + \gamma b}{\delta d + \beta c}$.

same eigenvalue

same eigenvalue

$$\begin{bmatrix} \alpha & a\alpha z \\ a/\alpha & 1/\alpha \end{bmatrix} \begin{bmatrix} 1 & a \\ a/z & 1 \end{bmatrix} = P_{0,-}(z)\, P_{0,+}(z), \qquad P_{0,-}P_{0,+} \xrightarrow{\text{swap}} P_{0,+}P_{0,-} \overset{\text{re-factorize}}{=\!=\!=} P_{1,-}P_{1,+}$$

$$(P_{0,-}P_{0,+})^N = P_{0,-}P_{0,+}P_{0,-}P_{0,+}\cdots P_{0,-}P_{0,+} = P_{0,-}(P_{0,+}P_{0,-})^{N-1}P_{0,+} = P_{0,-}(P_{1,-}P_{1,+})^{N-1}P_{0,+} =$$

$$P_{1,-}P_{1,+} \xrightarrow{\text{swap}} P_{1,+}P_{1,-} \overset{\text{re-factorize}}{=\!=\!=} P_{2,-}P_{2,+}$$

$$= P_{0,-}P_{1,-}(P_{2,-}P_{2,+})^{N-2}P_{1,+}P_{0,+} = \dots = \underbrace{(P_{0,-}P_{1,-}\cdots P_{N-1,-})(P_{N-1,+}\cdots P_{1,+}P_{0,+})}_{\text{Wiener-Hopf factorization}}$$

Berggren-Duits, 2019

Step 3: Linearization

$$P_-(z)\,P_+(z) = P(z) = \begin{bmatrix} a_{11} & a_{12}+b_{12}z \\ a_{21}+b_{21}z^{-1} & a_{22} \end{bmatrix} \longmapsto \widetilde{P}(z) = P_+(z)P_-(z) = P_+(z)\,P(z)\,P_+^{-1}(z).$$

Central idea of Integrable Systems: Represent a nonlinear flow as a compatibility condition of linear problems (Lax pair).

$$\begin{cases} \big(P(z)-w\big)\,\Psi(z,w) = 0 \\ \widetilde{\Psi}(z,w) = R(z)\,\Psi(z,w) \end{cases}$$

Solution of $\det R(z)=0$ is the $|z|<1$ part of $\det P(z)=0$

If $\widetilde{\Psi}$ satisfies $\big(\widehat{P}(z)-w\big)\widetilde{\Psi}=0$ with similar $\widehat{P}(z)$, then $R(z)=P_+(z)$ and $\hat{P}=\tilde{P}$ (up to conjugations by scalar diagonal matrices).

Compare KdV equation $\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$ is a compatibility condition for its Lax pair $\left(\frac{\partial^2}{\partial x^2}+u\right)\varphi = \lambda\varphi$, $\frac{\partial \varphi}{\partial t} = \left(\frac{\partial^3}{\partial x^3} + \frac{3}{2}u + \frac{3}{4}u_x\right)\varphi$.

Bird's eye view on linearization

$$P(z) = \begin{bmatrix} a_{11} & a_{12}+b_{12}z \\ a_{21}+b_{21}z^{-1} & a_{22} \end{bmatrix}$$

The space of our flow

$$\begin{cases} (P(z)-w)\,\Psi(z,w)=0 \\ \tilde{\Psi}(z,w)=P_+(z)\,\Psi(z,w) \end{cases}$$

The Lax pair

$$P(z) \longmapsto P_+(z)\,P(z)\,P_+^{-1}(z).$$

The flow

Note that $\det(P(z)-w)$ does not change under the evolution (isospectrality).

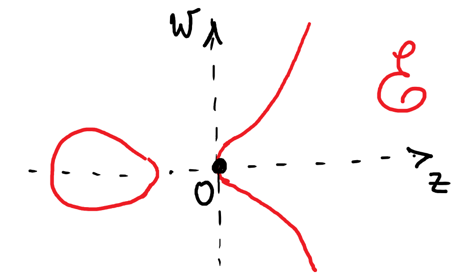
Hence, $\{(z,w)\in\mathbb{C}^2 : \det(P(z)-w)=0\}$ is an invariant. Its natural compactification $\mathcal{E}$ is the spectral curve; it has genus 1 (elliptic curve).

- Normalize $\Psi(z,w)=\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}$ by $\Psi_1(z,w)+\Psi_2(z,w)\equiv 1$.
- One shows that $\Psi_1(z,w), \Psi_2(z,w)$ span the space of meromorphic functions on $\mathcal{E}$ with two fixed simple poles.
- One zero of Ψ_1 is at 0, one zero of Ψ_2 is at ∞.
- The second zero of $\Psi_{1/2}$ evolves by linear shifts on $\mathcal{E}$.
- The linearity follows from singularity structure of P_+ and Abel's theorem.

Moser-Veselov, 1991
finite-gap method, 1976+
Dubrovin, Its, Krichever

Getting hands dirty

$$P(z) = \begin{bmatrix} a_{11} & a_{12} + b_{12} z \\ a_{21} + b_{21} z^{-1} & a_{22} \end{bmatrix}, \quad \underbrace{\operatorname{Tr}(P(z)) = 2c_1, \ \det(P(z)) = -c_2 \frac{(z-z_1)(z-z_2)}{z}}_{\text{fixed}}.$$



$$\det\left(P(z) - c_1\left(1 + \tfrac{w}{z}\right)\right) = 0 \iff c_1(w^2 - z^2) = c_2\, z(z-z_1)(z-z_2).$$

For each $P(z)$ there exists a unique $(x,y) \in \mathcal{E}$ such that

$$\left[P(x) - c_1\left(1 + \tfrac{y}{x}\right)\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives a parametrization of $P(z)$ by points of $\mathcal{E}$ (up to diagonal conjugation):

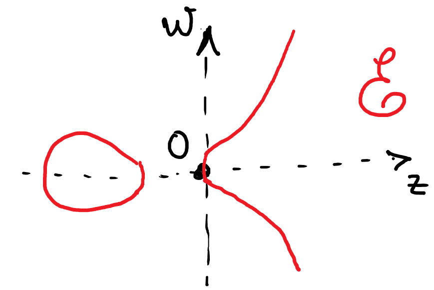
$$P(z) = \begin{bmatrix} c_1(1 - y/x) & u(z-x) \\ \frac{c_2}{u}\left(1 - \frac{z_1 z_2}{zx}\right) & c_1(1 + y/x) \end{bmatrix}, \quad (x,y) \in \mathcal{E}, \quad u \in \mathbb{C}^*.$$

Hence, $(x,y) \in \mathcal{E}$ is a zero of $\Psi_1(z,w)$: $\left(P(z) - c_1\left(1 + \frac{w}{z}\right)\right)\Psi(z,w) = 0$.
We want to understand its evolution.

Getting hands dirty

$$P(z)=\begin{bmatrix} a_{11} & a_{12}+b_{12}z \\ a_{21}+b_{21}z^{-1} & a_{22}\end{bmatrix}=\begin{bmatrix} c_1\left(1-y/x\right) & u(z-x) \\ \frac{c_2}{u}\left(1-\frac{z_1z_2}{zx}\right) & c_1\left(1+y/x\right)\end{bmatrix},\qquad \det(P(z))=-c_2\frac{(z-z_1)(z-z_2)}{z}$$

$$c_1(w^2-z^2)=c_2\,z(z-z_1)(z-z_2).$$



Hence, $(x,y)\in\mathcal{E}$ is a zero of $\Psi_1(z,w)$: $\left(P(z)-c_1\left(1+\frac{w}{z}\right)\right)\Psi(z,w)=0$.
We want to understand its evolution.

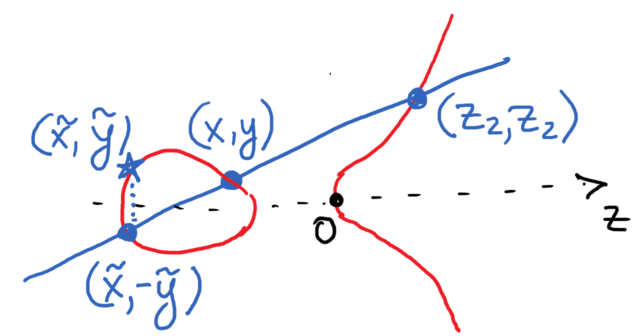
$$P(z)=P_-(z)P_+(z),\quad \det P_-(z_1)=0,\quad \det P_+(z_2)=0;$$

$$P_+(z)=\begin{bmatrix}\alpha & \beta z\\ \gamma & \delta\end{bmatrix}\Rightarrow \tilde{\Psi}_1(z,w)=P_+(z)\Psi(z,w)=\underbrace{\alpha(a_{12}+b_{12}z)+\beta(-a_{11}z+c_1(w-z))}_{\text{linear in }(z,w)}$$

A linear function has three zeros on $\mathcal{E}$. Two are apparent: $(z,w)=(x,-y)$ or $(z_2,-z_2)$.
Both of them are zeros of both $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ $\Rightarrow$ $\tilde{\Psi}\propto\begin{bmatrix}0\\1\end{bmatrix}$.
It is the remaining zero $(\tilde{x},\tilde{y})$ that we are after.

Colinearity: $(\tilde{x},\tilde{y})\oplus(x,-y)\oplus(z_2,-z_2)=0$

or $\quad(\tilde{x},\tilde{y})=(x,y)\oplus(z_2,z_2)$.



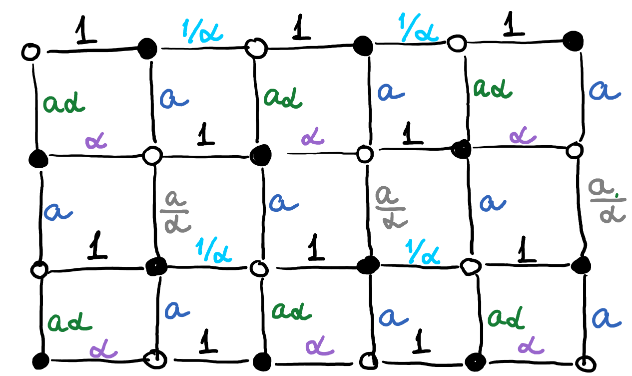
Simplification for a periodic flow

$$\begin{bmatrix} \alpha & a\alpha z \\ a/\alpha & 1/\alpha \end{bmatrix}\begin{bmatrix} 1 & a \\ a/z & 1 \end{bmatrix} = P_{0,-}(z)\,P_{0,+}(z), \qquad P_{0,-}\,P_{0,+} \xrightarrow{\text{swap}} P_{0,+}\,P_{0,-} \underset{}{\overset{\text{re-factorize}}{=\!=}} P_{1,-}\,P_{1,+}$$

$$P^N = \left(P_{0,-}\,P_{0,+}\right)^N = \left(P_{0,-}\,P_{1,-}\cdots P_{N-1,-}\right)\left(P_{N-1,+}\cdots P_{1,+}\,P_{0,+}\right)$$

Wiener-Hopf factorization

The elliptic curve is $w^2 = z^2 + \dfrac{4z(z-a^2)(z-a^{-2})}{(a+a^{-1})^2(\alpha+\alpha^{-1})^2}$.



The dynamics is the shift by $(z,w) = (a^{-2}, a^{-2})$.

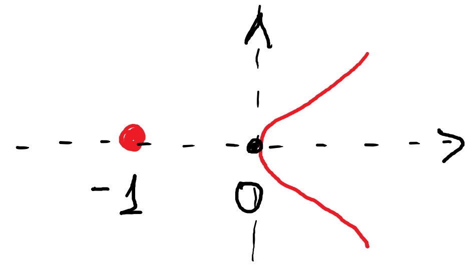
If this shift has a finite order d then

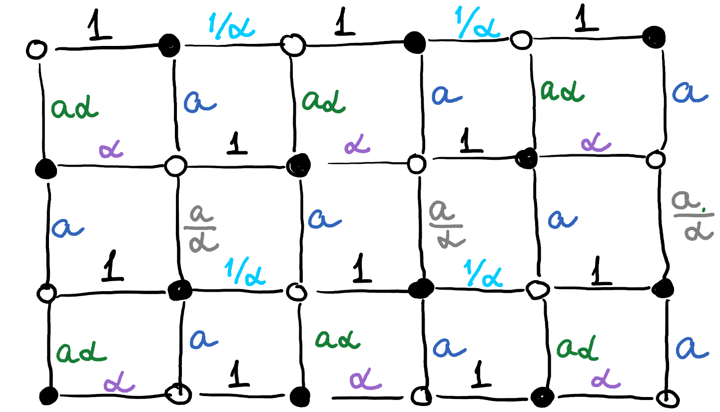
$$P(z),\quad P_-^{(d)}(z) := P_{0,-}(z)\cdots P_{d-1,-}(z),\quad P_+^{(d)}(z) := P_{d-1,+}(z)\cdots P_{0,+}(z)$$

are all diagonalizable by the same eigenbasis consisting of meromorphic functions on the spectral curve with known singularities. The integrand for the correlation kernel becomes scalar, with various powers of the eigenvalues.

Simplification for a periodic flow

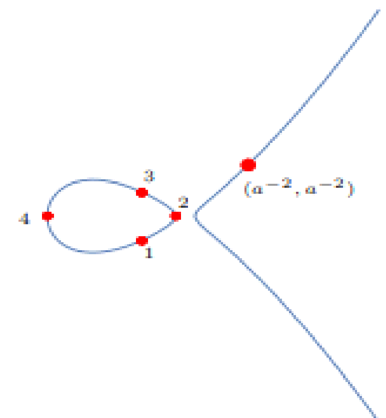
- The simplest case is $\alpha=1$;
  2×2 periodicity disappears;
  compact oval collapses;
  the flow is constant.





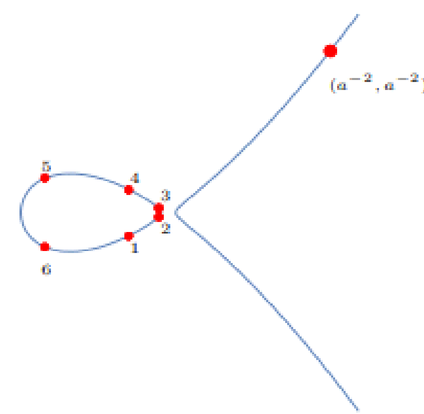
- The next case is $a=1$.
  This is period 4 situation:

$$\left(-1,-\frac{1-\alpha^2}{1+\alpha^2}\right)\mapsto(-\alpha^2,0)\longmapsto\left(-1,\frac{1-\alpha^2}{1+\alpha^2}\right)\mapsto(-\alpha^{-2},0),$$

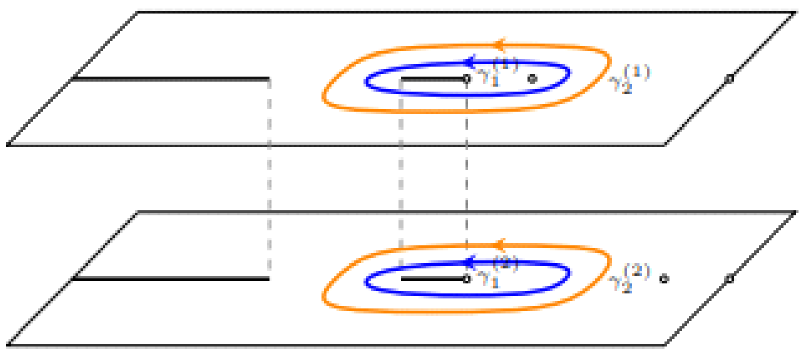


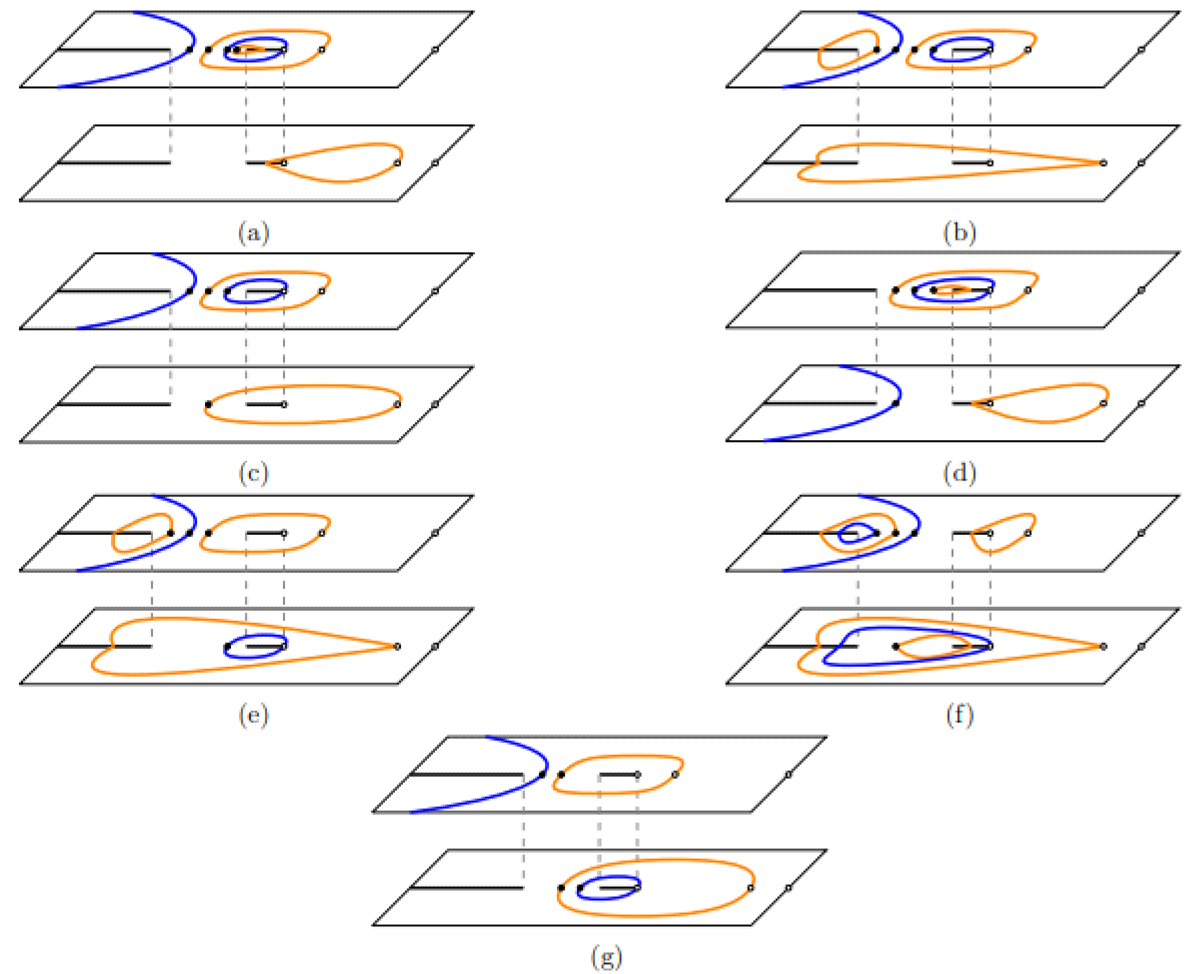
Chhita-Young 2013, Chhita-Johansson 2014,
Beffara-Chhita-Johansson 2016, 2020, Johansson-Mason 2021,
Duits-Kuijlaars 2017, Berggren-Duits 2019, Berggren 2019

- Even periods appear to be simpler.
  For $d=6$, $a^2=\alpha/(1+\alpha+\alpha^2)$.



We were able to use steepest descent arguments for any finite period d.
Here is what possible contour deformations look like for the smooth/gas region.

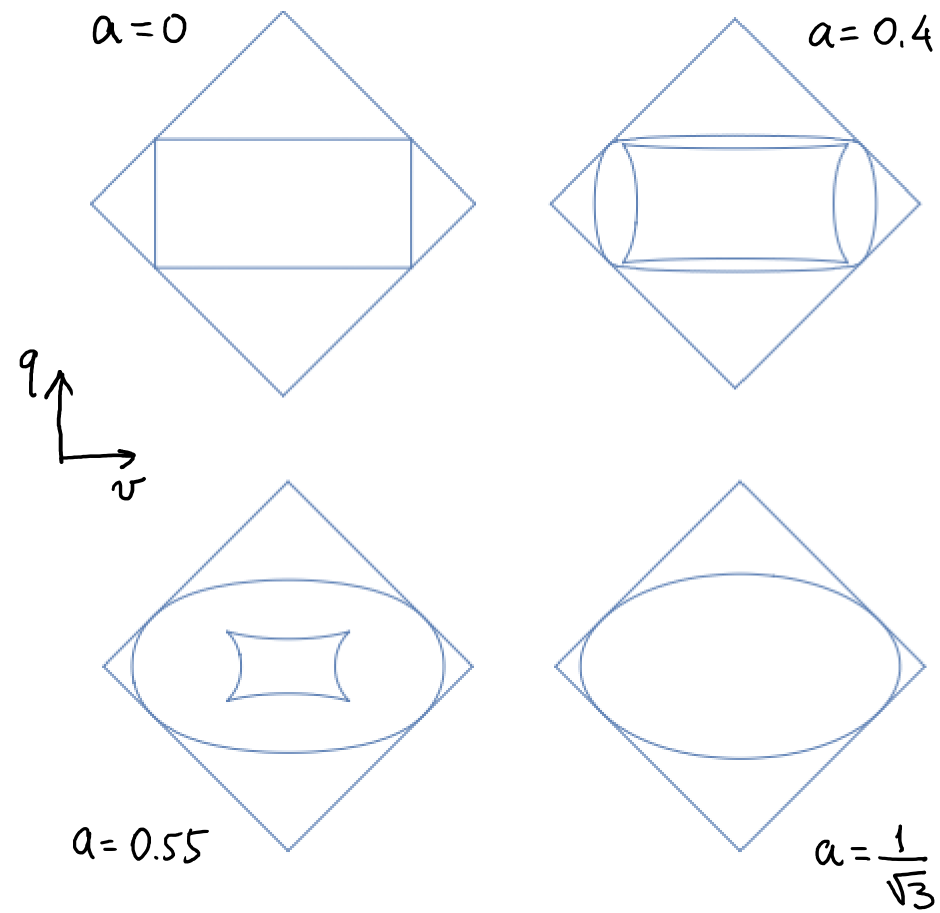




cf. Duits–Kuijlaars 2017
Berggren 2019

And here is the phase-separating curve for the period 6 case; $a^2 = \alpha/(1+\alpha+\alpha^2)$:

$$
\begin{aligned}
0 = {} & 16-336a^4+1440a^8+7776a^{12}-34992a^{16}-104976a^{20}-288q^2+6336a^4q^2-45504a^8q^2+124416a^{12}q^2- \\
& 209952a^{16}q^2+419904a^{20}q^2+1296q^4-32400a^4q^4+242352a^8q^4-587088a^{12}q^4+839808a^{16}q^4- \\
& 629856a^{20}q^4+23328a^4q^6-303264a^8q^6+769824a^{12}q^6-909792a^{16}q^6+419904a^{20}q^6+104976a^8q^8- \\
& 314928a^{12}q^8+314928a^{16}q^8-104976a^{20}q^8-72v^2+1152a^4v^2-1224a^8v^2-43200a^{12}v^2+75816a^{16}v^2+ \\
& 419904a^{20}v^2-157464a^{24}v^2+1296q^2v^2-20088a^4q^2v^2+119880a^8q^2v^2-527472a^{12}q^2v^2+ \\
& 1283040a^{16}q^2v^2-997272a^{20}q^2v^2+472392a^{24}q^2v^2-5832q^4v^2+81648a^4q^4v^2-367416a^8q^4v^2+ \\
& 863136a^{12}q^4v^2-833976a^{16}q^4v^2+734832a^{20}q^4v^2-472392a^{24}q^4v^2+52488a^4q^6v^2-472392a^8q^6v^2+ \\
& 944784a^{12}q^6v^2-524880a^{16}q^6v^2-157464a^{20}q^6v^2+157464a^{24}q^6v^2+81v^4-1215a^4v^4-3483a^8v^4+ \\
& 79461a^{12}v^4-2349a^{16}v^4-750141a^{20}v^4+570807a^{24}v^4-59049a^{28}v^4-1458q^2v^4+ \\
& 21870a^4q^2v^4-158922a^8q^2v^4+867510a^{12}q^2v^4-1963926a^{16}q^2v^4+1418634a^{20}q^2v^4-301806a^{24}q^2v^4 \\
& +118098a^{28}q^2v^4+6561q^4v^4-98415a^4q^4v^4+452709a^8q^4v^4-387099a^{12}q^4v^4-610173a^{16}q^4v^4 \\
& +964467a^{20}q^4v^4-269001a^{24}q^4v^4-59049a^{28}q^4v^4+5832a^8v^6-52488a^{12}v^6-128304a^{16}v^6 \\
& +734832a^{20}v^6-717336a^{24}v^6+157464a^{28}v^6+52488a^8q^2v^6-472392a^{12}q^2v^6+944784a^{16}q^2v^6 \\
& -524880a^{20}q^2v^6-157464a^{24}q^2v^6+157464a^{28}q^2v^6+104976a^{16}v^8-314928a^{20}v^8+314928a^{24}v^8-104976a^{28}v^8.
\end{aligned}
$$



This is a degree 8 curve in (q,v) with parameter a in the coefficients.

Some further work:

- Identification of the re-factorization flow with that arising from domino shuffling.

  Chhita-Duits 2022

  ...

- Extending analysis to arbitrary shifts and handling more general periodicity, with more smooth/gas regions.

  Berggren-B. 2022+