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Integrable Partial Differential Equations :
Geometry, Asymptotics, and Numerics

Integrability of an integro-differential Painlevé equation

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Isomonodromic Deformations, Painlevé equations & Integrable Systems
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Plan of the seminar

- A generalization of the Tracy–Widom distribution appearing in KPZ.
- Three different Riemann–Hilbert approaches.
- A "classical" Riemann–Hilbert problem.

  (plus a discrete version : work in progress)
- An operator–valued Riemann–Hilbert problem.

"A Riemann–Hilbert Approach to the Lower Tail of the Kardar–Parisi–Zhang Equation"
with T. Claeys, Comm. in Pure and Applied Math. 2021.

"Airy kernel determinant solutions to the KdV equation and integro-differential Painlevé equations"
with T. Claeys and G. Ruzza, Comm. in Math. Phys. 2021.

"Momenta spacing distributions in anharmonic oscillators and the higher order finite temperature Airy kernel"
with T. Bothner and S. Tarricone, Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques. 2021.

A deformation of the Tracy–Widom (GUE) distribution

$$K_T^{\mathrm{Ai}}(\lambda,\mu) := \int_{-\infty}^{\infty} \sigma(T^{1/3}r)\mathrm{Ai}(\lambda+r)\mathrm{Ai}(\mu+r)\mathrm{d}r; \quad \sigma(r) := \frac{1}{1+\exp(-r)};$$

$$\begin{aligned} Q(s,T) &:= \det\left(\mathrm{Id} - \chi_{[-s,\infty)} K_T^{\mathrm{Ai}} \chi_{[-s,\infty)}\right) \\ &= 1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[-s,\infty)^k} \det\left(K_T^{\mathrm{Ai}}(\lambda_i,\lambda_j)\right)_{i,j=1}^{k} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_k. \end{aligned}$$

Applications :

Johansson : MNS matrix model, deformed GUE (2007)

Amir-Corwin-Quastel : Narrow wedge solution of the KPZ equation (2010)

Dean-Le Doussal-Majumdar-Schehr : Fermionic systems at finite temperature (2015)

Betea-Bouttier : Periodic Schur process (2019)

From Gumbel to Tracy–Widom

Let $X_1, \ldots, X_N$ i.i.d. random variables, $X_i \sim \mathcal{N}(0, 1/2)$.

$$\lim_{N\to\infty} \left[\frac{\max(X_1, \ldots, X_N) - a_N}{b_N} \leq s \right] = F_G(s) = \exp\left(-\mathrm{e}^{-s}\right)$$

$$a_N = \sqrt{\log N} - \frac{\log(4\pi \log N)}{4\sqrt{\log N}}, \quad b_N = \frac{1}{2\sqrt{\log N}}$$

Theorem (Johansson, 2005) :

$Q(s, T)$ interpolates between Tracy–Widom and Gumbel :

$$\lim_{T\to\infty} Q(-s, T) = F_{\mathrm{TW}}(s), \qquad \lim_{T\to 0+} Q\left(-s_T, T\right) = F_G(s),$$

where $s_T = \dfrac{s}{T^{1/3}} - \dfrac{1}{2T^{1/3}} \log(4\pi T)$. □

The integro-differential Painlevé II equation

Proposition (Amir-Corwin-Quastel, 2010) :

$$Q(-s,T) = \exp\left[-\int_s^\infty (r-s)\left(\int_{\mathbb{R}} u^2(x,r)\sigma'(T^{1/3}r)\mathrm{d}r\right)\mathrm{d}x\right]$$

with $u(s,\lambda)$ satisfying the equation

$$\frac{\partial^2}{\partial s^2}u(s,\lambda) = \left[s+\lambda+2\int_{\mathbb{R}} u^2(s,r)\sigma'(T^{1/3}r)\mathrm{d}r\right]u(s,\lambda), \qquad \leftarrow \text{Integro-differential Painlevé II}$$

and boundary condition

$$u(s,\lambda) \overset{s\to\infty}{\sim} \mathrm{Ai}(s+\lambda).$$

Integrable kernels (Its-Izergin-Korepin-Slavnov)

Given a contour $\Sigma \subseteq \mathbb{C}$ and an operator $K : L^2(\Sigma) \longrightarrow L^2(\Sigma)$ with kernel

$$K(x,y) = \frac{\mathbf{f}^\top(x)\mathbf{h}(y)}{x-y},$$

the resolvent $L := (\mathrm{Id} - K)^{-1}K$ can be computed solving the Riemann-Hilbert problem (on Σ) with jump matrix

$$J(z) := \mathbf{1} - 2\pi\mathbf{i}\mathbf{f}(z)\mathbf{h}^\top(z).$$

$Q(s,T)$ as the determinant of an integrable kernel

$$K_T^{\mathrm{Ai}}(u,v) := \int_{-\infty}^{\infty} \sigma(T^{1/3}r)\mathrm{Ai}(u+r)\mathrm{Ai}(v+r)\mathrm{d}r$$

can be re-written as an (infinite–dimensional) IIKS integrable kernel

$$K_T^{\mathrm{Ai}}(u,v) = \frac{\displaystyle\int_{\mathbb{R}} \Big[\mathrm{Ai}(u+r)\mathrm{Ai}'(v+r) - \mathrm{Ai}'(u+r)\mathrm{Ai}(v+r)\Big]\sigma'(T^{1/3}r)\mathrm{d}r}{u-v}.$$

1) Amir-Corwin-Quastel used a generalization of the Tracy-Widom procedure to obtain the integro–differential Painlevé II equation (2010).

2) Thomas Bothner found a first operator-valued Lax pair for the equation, using an operator-valued Riemann-Hilbert problem on the interval $[0,\infty)$ (2020).

First approach

(joint collaboration with T. Claeys and G. Ruzza)

A simpler integrable kernel

Proposition (Amir-Corwin-Quastel 2010, Borodin-Gorin, 2016) :

$$Q(s,T) = \det\left(\mathrm{Id} - \chi_{[-s,\infty)} K_T^{\mathrm{Ai}} \chi_{[-s,\infty)}\right) = \det(\mathrm{Id} - K_{\sigma,s,T}^{\mathrm{Ai}})$$

with

$$K_{\sigma,s,T}^{\mathrm{Ai}}(\lambda,\mu) = \sqrt{\sigma(T^{1/3}(\lambda+s))} K^{\mathrm{Ai}}(\lambda,\mu) \sqrt{\sigma(T^{1/3}(\mu+s))}$$

□

In particular, the proposition above establishes a relation between $Q(s,T)$ and multiplicative statistics for the Airy process :

$$Q(s,T) = \mathbb{E}_{\mathrm{Ai}}\left[\prod_{j=1}^{\infty}\left(1 - \sigma(T^{1/3}(\zeta_j + s))\right)\right]$$

Back to finite–dimensional Riemann-Hilbert problems

Remark :

$$K^{\mathrm{Ai}}_{\sigma,s,T}(\lambda,\mu) = \frac{\mathbf{f}^{\top}(\lambda)\mathbf{h}(\mu)}{\lambda-\mu}$$

$$\mathbf{f}(\lambda) := \sqrt{\sigma(T^{1/3}(\lambda+s))}\begin{pmatrix}-\mathrm{i}\mathrm{Ai}'(\lambda)\\ \mathrm{Ai}(\lambda)\end{pmatrix}, \quad \mathbf{h}(\mu) := \sqrt{\sigma(T^{1/3}(\mu+s))}\begin{pmatrix}-\mathrm{i}\mathrm{Ai}(\mu)\\ \mathrm{Ai}'(\mu)\end{pmatrix}.$$

$$\Downarrow$$

A Riemann–Hilbert problem on the (whole) real line, of size 2×2, with jumps

$$J(z) = \mathbf{1} - 2\pi\mathrm{i}\mathbf{f}(z)\mathbf{h}^{\top}(z).$$

The relevant Riemann-Hilbert problem

$$\boxed{x := sT^{-1/6},\ t := T^{-1/2}}$$

RH problem for $\Psi(z) = \Psi(x,t;z)$

a)

$$\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & 1-\sigma(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R},$$

b)

$$\Psi(z) = \left(\mathrm{Id} + \mathcal{O}(z^{-1})\right) z^{\frac{\sigma_3}{4}} A^{-1} \mathrm{e}^{\left(-\frac{2}{3}tz^{3/2}+xz^{1/2}\right)\sigma_3} \begin{cases} \mathrm{Id}, & |\arg z| < \pi - \delta \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \delta < \pm \arg z < \pi \end{cases}$$

Proposition (M.C., T. Claeys, G. Ruzza) :

$$\partial_x \log Q(x,t) = -\frac{1}{2\pi \mathrm{i} t} \int_{\mathbb{R}} \left(\Psi_+^{-1}(z)\Psi_+'(z)\right)_{21} \mathrm{d}\sigma(z).$$

KdV and $Q(x,t)$

Theorem (M.C., T. Claeys, G. Ruzza) :

The function

$$v(x,t) := \partial_x^2 \log Q(x,t) + \frac{x}{2t}$$

solves the KdV equation

$$v_t + 2vv_x + \frac{1}{6}v_{xxx} = 0.$$

Moreover, it can be expressed as

$$v(x,t) = -\frac{1}{t}\int_{\mathbb{R}} \phi^2(r;x,t)\mathrm{d}\sigma(r) + \frac{x}{2t}$$

where ϕ solves the Schrödinger equation

$$\partial_x^2 \phi(z;x,t) = (z - 2v(x,t))\phi(z;x,t)$$

and has asymptotic behavior

$$\phi(z;x,t) \sim t^{1/6}\mathrm{Ai}\left(t^{2/3}z - xt^{-1/3}\right), \quad z \to \infty, \ |\arg(z)| < \pi - \delta.$$

From KdV to the integro–differential PII equation

Corollary (M.C., T. Claeys, G. Ruzza) :

From

$$\partial_x^2 \phi(z;x,t) = (z - 2v(x,t))\phi(z;x,t)$$

and [1]

$$v(x,t) = -\frac{1}{t}\int_{\mathbb{R}} \phi^2(r;x,t)\mathrm{d}\sigma(r) + \frac{x}{2t}$$

we immediately get

$$\partial_x^2 \phi(z;x,t) = \left(z - \frac{x}{t} + \frac{2}{t}\int_{\mathbb{R}} \phi^2(r;x,t)\mathrm{d}\sigma(r) \right) \phi(z;x,t).$$

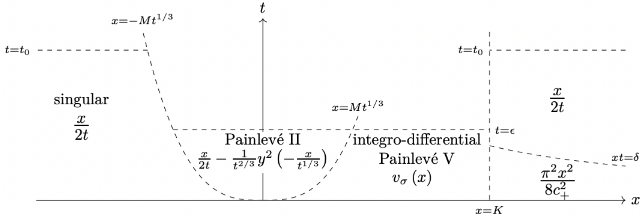
and recover the "Tracy–Widom" formula of Amir-Corwin-Quastel.

1. See also the so–called "Trace Formula" in Deift-Trubowitz, 1979.

Remarks

1) These results extend to the case of a general weight σ, possibly with discontinuities, producing a whole family of unbounded solutions of KdV.

2) The behavior of $u(x,t)$ is singular as $t \to 0$. It gives informations about the asymptotics of the tails of the narrow wedge solution of KPZ.
(Picture taken from Charlier-Claeys-Ruzza, '21)



The finite-temperature discrete Bessel process (work in progress)

$$K_u^{\mathrm{Be}}(a,b) := \sum_{\ell \in \mathbb{Z}'} \sigma(\ell) J_{a+\ell}(2L) J_{b+\ell}(2L); \quad \sigma(\ell) := (1+u^{\ell})^{-1}.$$

Borodin 2007, Betea-Bouttier 2019 : This kernel defines the point process associated to a certain deformation of the (Poissonized) Plancherel measure, the cylindrical Plancherel measure. It is a discrete integrable kernel, and as such it is associated to a discrete RH problem.

$$F_u(L;s) := \det(\mathrm{I} - K_u^{\mathrm{Be}} \chi_{(s,s+1,\cdots)}).$$

Theorem (M.C., G. Ruzza) :

$$\frac{\partial^2}{\partial L^2} \log F_u(L,s) + \frac{1}{L}\frac{\partial}{\partial L} \log F_u(L,s) + 4 = 4\frac{F_u(L,s+1)F_u(L,s-1)}{F_u(L,s)^2}.$$

(Reduction of 2D-Toda equation)

$$\log F_u(L,s) = \log\left(\prod_{i=1}^{\infty} 1 - \sigma(-i-s)\right) - \frac{\sigma(-s) - \sigma(-s-1)}{1-\sigma(-s-1)} L^2 + \mathcal{O}(L^4), \quad L \mapsto 0.$$

Continuous limit to KdV

$$\epsilon := (1-u) \to 0; \quad L = L(x,t;\epsilon) := \frac{1}{\epsilon^3 t^2}, \quad s = s(x,t;\epsilon) = \frac{2}{\epsilon^3 t^2} - \frac{x}{\epsilon t}.$$

(Betea-Bouttier)

$$F(x,t) := F_u\left(L(x,t;\epsilon), s(x,t;\epsilon)\right)$$

Formally expanding the 2D Toda equation, we find at order ϵ^4

$$\frac{\partial^2}{\partial t \partial x} \log F(x,t) + \frac{x}{t}\frac{\partial^2}{\partial x^2} \log F(x,t) + \left(\frac{\partial^2}{\partial x^2} \log F(x,t)\right)^2 + \frac{1}{6}\frac{\partial^4}{\partial x^4} \log F(x,t) = 0,$$

which is equivalent to the KdV equation for $\frac{\partial^2}{\partial x^2} \log F(x,t) + \frac{x}{2t}$.

Second approach

(joint collaboration with T. Bothner and S. Tarricone)

A n–parametric family of kernels

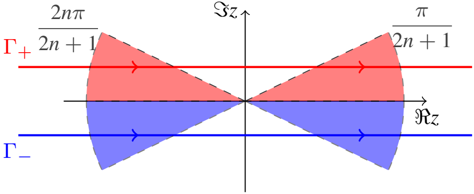
$$\psi_n(\lambda, x) := \frac{\lambda^{2n+1}}{2n+1} + x\lambda,$$

$$\mathrm{Ai}_n(x) := \frac{1}{2\pi}\int_{\Gamma_+} \mathrm{e}^{i\psi_n(\lambda,x)}\mathrm{d}\lambda := \frac{1}{2\pi}\int_{\Gamma_-} \mathrm{e}^{-i\psi_n(\mu,x)}\mathrm{d}\mu,$$

$$K^{\mathrm{Ai}_n}_{s,T}(u,v) := \int_{-\infty}^{\infty} \sigma(\tilde{T}r)\mathrm{Ai}_n(u+s+r)\mathrm{Ai}_n(v+s+r)\mathrm{d}r; \quad \sigma(r) := \frac{1}{1+\exp(-r)}, \quad \tilde{T} := T^{1/(2n+1)};$$

$$D_n(s,T) := \det\left(\mathrm{Id} - \chi_{[0,\infty)} K^{\mathrm{Ai}_n}_{s,T} \chi_{[0,\infty)}\right)$$

$[D_1(s,T) = Q(-s,T)]$



A physical interpretation (Le Doussal-Majumdar-Schehr, 2019)

Consider a fermionic gas of N particles and one-particle hamiltonian

$$H_q := -\frac{d^2}{dq^2} + q^{2n}$$

Denote

$$\Psi_{\mathbf{k}}(q_1, \ldots, q_N) := \frac{1}{\sqrt{N!}} \det \left(\psi_{k_i}(q_j) \right)_{i,j=1}^{N}, \quad \text{where} \quad H_q \psi_j(q) = \lambda_j \psi_j(q).$$

The pdf of the particles, according to the Boltzmann-Gibbs distribution, is given by

$$f_\beta(q_1, \ldots, q_N) := \frac{1}{Z_N(\beta)} \sum_{\mathbf{k} \in \mathbb{N}^N : k_1 < \ldots < k_N} |\Psi_{\mathbf{k}}(q_1, \ldots, q_N)|^2$$

with

$$Z_N(\beta) = \sum_{\mathbf{k} \in \mathbb{N}^N : k_1 < \ldots < k_N} \mathrm{e}^{-\beta \sum \lambda_k}.$$

Theorem (Dean-Le Doussal-Majumdar-Schehr 2016, Lietchy-Wang 2020)

$$\lim_{N \to \infty} \mathbb{P} \left(\frac{q_{\max} - \sqrt{2N}}{2^{-1/2} N^{-1/6}} \leq s \,\middle|\, n = 1, \beta = \left(\frac{T}{N} \right)^{1/3}, T > 0 \right) = D_1(s, T).$$

A physical interpretation (Le Doussal-Majumdar-Schehr, 2019) II

In the moment representation $H_p = p^2 + (-1)^n \frac{d^{2n}}{dp^{2n}}$, it is physically argued that there exists constants a_n, b_n such that

$$p_{\text{max}}(N) \sim a_n N^{\frac{n}{n+1}} =: p_{\text{edge}}(N)$$

and the density $\rho_N(p)$ behaves like

$$\rho_N(p) \sim b_n N^{\frac{1}{2(n+1)}} \left(p_{\text{edge}}(N) - p\right)^{\frac{1}{2n}}.$$

Main Formula :

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{p_{\text{max}}(N) - p_{\text{edge}}(N)}{c_n N^{-e_n}} \leq s \,\middle|\, \beta = \frac{\tilde{T}}{d_n N^{f_n}}, \tilde{T} > 0 \right) = D_n(s, \tilde{T})$$

$$e_n := \frac{n}{(n+1)(2n+1)}, \qquad f_n := \frac{2n^2}{(n+1)(2n+1)}.$$

The main result

Theorem (T. Bothner, M.C., S. Tarricone)

$$D_n(s,T) = \exp\left[-\int_s^{\infty}(t-s)\left(\int_{\mathbb{R}} u^2(t|x)\sigma'(\tilde{T}x)\mathrm{d}x\right)\mathrm{d}t\right],$$

where $u(s|x) \equiv u(s|x;n,T)$ solves the n-th member of the integro-differential Painlevé II hierarchy* and it is such that

$$u(s|x) \sim \mathrm{Ai}_n(s+x)$$

as $s \to +\infty$, pointwise in $x \in \mathbb{R}$.

i) $n = 1$: Amir–Corwin–Quastel (2010).

ii) $n = 2$: Krajenbrink (2020).

*The Painlevé II equation is a self–similar reduction of the modified KdV equation. The Painlevé II hierarchy is the set of equations obtained as self–similar reductions of the equations of the modified KdV hierarchy.

(Flashka-Newell 1980, H. Ayrault 1979, Clarkson-Joshi-Mazzocco 2006)

$K_{s,T}^{\mathrm{Ai}_n}$ and an operator–valued Flashka-Newell Lax pair

Lemma :

$$K_{s,T}^{\mathrm{Ai}_n}(u,v) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}}\left[\int_{\Gamma_+}\int_{\Gamma_-} e^{\mathrm{i}\psi_n(\lambda,u+s+r)-\mathrm{i}\psi_n(\mu,v+s+r)}\frac{\mathrm{d}\mu\mathrm{d}\lambda}{\lambda-\mu}\right]\mathrm{d}\sigma(\tilde{T}r)$$

□

The path to the Flashka–Newell Riemann-Hilbert Problem

Let $\Sigma := \Gamma_+ \cup \Gamma_-$. Using a conjugation by Fourier transform prove that

$$D_n(s,T) = \det(1 - C_{n,s,T}),$$

where $C_{n,s,T}$ is an operator on $L^2(\Sigma)$ of integrable (IIKS) type.

Previous results for $T \to \infty$:

- M. Bertola - M.C. for $n = 1$, 2012.
- M.C. - T. Claeys - M. Girotti for n arbitrary, 2019.

The relevant integrable kernel

$$k_1(\lambda|z) := \frac{1}{2\pi} \mathrm{e}^{\frac{i}{2}\psi_n(\lambda,2s+2z)} \chi_{\Gamma_+}(\lambda), \quad k_2(\lambda|z) := \frac{1}{2\pi} \mathrm{e}^{-\frac{i}{2}\psi_n(\lambda,0)} \chi_{\Gamma_-}(\lambda),$$

$$m_1(\lambda|z) := \mathrm{e}^{-\frac{i}{2}\psi_n(\lambda,2s+2z)} \chi_{\Gamma_-}(\lambda), \quad m_2(\lambda|z) := \mathrm{e}^{\frac{i}{2}\psi_n(\lambda,0)} \chi_{\Gamma_+}(\lambda).$$

$$C_{n,s,T}(\lambda,\mu) = \frac{\int_{\mathbb{R}} \Big(k_1(\lambda|z) m_1(\mu|z) + k_2(\lambda|z) m_2(\mu|z) \Big) \sigma'(\tilde{T}z) \mathrm{d}z}{\lambda - \mu}.$$

Notations :

a) $\mathcal{H}_p := \bigoplus_{j=1}^{p} L^2\Big(\mathbb{R}, \mathrm{d}\sigma\Big).$

b) $\mathcal{I}(\mathcal{H}_p)$ the space of Hilbert-Schmidt integral operators on $\mathcal{H}_p$ with kernel in $L^2(\mathbb{R}^2, \mathrm{d}\sigma \otimes \mathrm{d}\sigma; \mathbb{C}^{p\times p})$.

c) $(M_i(\lambda) \otimes K_j(\lambda)) \in \mathcal{I}(\mathcal{H}_1)$ a λ-family ($\lambda \in \Sigma$) of rank 1 integral operators with kernels

$$(M_i(\lambda) \otimes K_j(\lambda))(x,y) = m_i(\lambda|x) k_j(\lambda|y).$$

The operator-valued RH problem

$\mathbf{X}(\lambda) = \mathbf{X}(\lambda; n, s, T) \in \mathcal{I}(\mathcal{H}_2)$ such that

(1) $\mathbf{X}(\lambda) = \mathbb{I}_2 + \mathbf{X}_0(\lambda)$ and $\mathbf{X}_0(\lambda) \in \mathcal{I}(\mathcal{H}_2)$ with kernel $\mathbf{X}_0(\lambda|x, y)$ analytic in $\mathbb{C} \setminus \Sigma$.

(2) $\mathbf{X}(\lambda)$ admits continuous boundary values $\mathbf{X}_\pm(\lambda) \in \mathcal{I}(\mathcal{H}_2)$ on Σ, which satisfy

$$\mathbf{X}_+(\lambda) = \mathbf{X}_-(\lambda) \begin{pmatrix} \mathbb{I} & 2\pi \mathrm{i}\Big(M_1(\lambda) \otimes K_2(\lambda)\Big) \\ 0 & \mathbb{I} \end{pmatrix}, \quad \lambda \in \Gamma_-.$$

$$\mathbf{X}_+(\lambda) = \mathbf{X}_-(\lambda) \begin{pmatrix} \mathbb{I} & 0 \\ 2\pi \mathrm{i}\Big(M_2(\lambda) \otimes K_1(\lambda)\Big) & \mathbb{I} \end{pmatrix}, \quad \lambda \in \Gamma_+.$$

(3) $\mathbf{X}(\lambda) \sim \mathbb{I}_2$ for $|\lambda| \to \infty$ with a particular condition on the operator norm of $\mathbf{X}_0$.

The associated Lax pair

$$\mathbf{M}(\lambda) := \begin{pmatrix} M_1(\lambda) \\ M_2(\lambda) \end{pmatrix}; \quad M_{1/2} \text{ operators on } \mathcal{H}_1 \text{ of multiplication by } m_{1/2}(\lambda|x).$$

Proposition (T. Bothner, M.C., S. Tarricone) :

$\mathbf{N}(\lambda) := \mathbf{X}(\lambda)\mathbf{M}(\lambda)$ satisfies the following Lax pair :

$$\begin{cases} \frac{\partial \mathbf{N}}{\partial \lambda}(\lambda) = \mathbf{A}(\lambda)\mathbf{N}(\lambda), \quad \mathbf{A}(\lambda) = \sum_{k=0}^{2n} \mathbf{A}_k \lambda^{2n-k} + \hat{\mathbf{A}}_{2n} \\ \frac{\partial \mathbf{N}}{\partial s}(\lambda) = \mathbf{B}(\lambda)\mathbf{N}(\lambda), \quad \mathbf{B}(\lambda) = \lambda \mathbf{B}_0 + \mathbf{B}_1 \end{cases},$$

where

$$\mathbf{B}_1(x,y) = \begin{pmatrix} 0 & -\mathrm{i}U(x,y) \\ \mathrm{i}V(x,y) & 0 \end{pmatrix}, \quad \mathbf{B}_0(x,y) = \frac{\delta(x-y)}{\sigma'(\tilde{T}y)} \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_0(x,y) = \frac{\delta(x-y)}{2\sigma'(\tilde{T}y)} \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \hat{\mathbf{A}}_{2n}(x,y) = \frac{\delta(x-y)}{\sigma'(\tilde{T}y)} \begin{pmatrix} -\mathrm{i}(s+x) & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, $\partial_s^2 \log D_n(s,T) = -\mathrm{Tr}_{\mathcal{H}_1}(UV)$

The integro–differential Painlevé II hierarchy

The compatibility condition gives, for $u(s|x) := U(x,x)$, the n-th member of the integro-differential Painlevé II hierarchy.

$$n=1: \quad (s+x)u = u'' - 2u\langle u,u\rangle,$$

$$\begin{aligned} n=2: \quad -(s+x)u = u'''' &- 4u''\langle u,u\rangle - 8u'\langle u',u\rangle - 6u\langle u,u''\rangle \\ &- 2u\langle u',u'\rangle + 6u\langle u,u\rangle^2, \end{aligned}$$

$$\begin{aligned} n=3: \quad (s+x)u = u'''''' &- 6u''''\langle u,u\rangle - 8u\langle u'''',u\rangle - 24u'''\langle u',u\rangle \\ &- 19u'\langle u,u'''\rangle - 13u\langle u''',u'\rangle - 31u''\langle u'',u\rangle \\ &- 11u\langle u'',u''\rangle - 25u''\langle u',u'\rangle - 45u'\langle u'',u'\rangle \\ &+ 15u''\langle u,u\rangle^2 + 55u\langle u,u\rangle\langle u'',u\rangle + 60u'\langle u',u\rangle\langle u,u\rangle \\ &+ 25u\langle u',u'\rangle\langle u,u\rangle + 55u\langle u',u\rangle^2 - 20u\langle u,u\rangle^3. \end{aligned}$$

Here $'$ states for the derivation w.r.t. the real parameter s and

$$\langle f,g\rangle := \int_{\mathbb{R}} f(s|x)g(s|x)\sigma'(\tilde{T}x)\mathrm{d}x$$

The general form of the equations of the hierarchy

$$\boxed{(s+x)u(s|x) = -\big((\mathcal{L}^u_+\mathcal{L}^u_-)^n u\big)(s|x)}$$

Given a function $\mathbb{R}^2 \ni (s,x) \mapsto f(s|x)$, we define

$$(\mathcal{L}^u_+ f)(s|x) := i(D_s f)(s|x) - i\big\langle (D_s^{-1}\{u,f\})(s|x,\cdot), u\big\rangle - 2i\big(D_s^{-1}\langle u,f\rangle\big)u(s|x),$$
$$(\mathcal{L}^u_- f)(s|x) := i(D_s f)(s|x) + i\big\langle (D_s^{-1}[u,f])(s|x,\cdot), u\big\rangle,$$

where

$[\alpha,\beta] := \alpha\otimes\beta - \beta\otimes\alpha$ is intended as rank two integral operator with kernel

$$[\alpha,\beta](s|x,y) = \alpha(s|x)\beta(s|y) - \beta(s|x)\alpha(s|y),$$

$\{\alpha,\beta\} := \alpha\otimes\beta + \beta\otimes\alpha$ the same but with kernel

$$\{\alpha,\beta\}(s|x,y) = \alpha(s|x)\beta(s|y) + \beta(s|x)\alpha(s|y),$$

In the “classical” case ($T\to\infty$) this recursion was introduced by H. Airault in 1979.

An integro–differential modified KdV hierarchy

$$t_1 = sT^{\frac{1}{2n+1}}, \quad t_{2n+1} := \frac{T}{2n+1}.$$

Proposition (T. Bothner, M.C., S. Tarricone)

$$v(t_1, t_{2n+1}|x) := \frac{1}{T^{\frac{1}{2n+1}}} u\left(s|\frac{x}{T^{\frac{1}{2n+1}}}\right)$$

is a solution of the integro–differential modified KdV hierarchy

$$\frac{\partial v}{\partial t_{2n+1}}(t_1, t_{2n+1}|x) = \left((\mathcal{L}_-^v \mathcal{L}_+^v)^n \frac{\partial v}{\partial t_1}\right)(t_1, t_{2n+1}|x).$$

$\square$

$$n = 1 : \frac{\partial v}{\partial t_3} = -\frac{\partial^3 v}{\partial t_1^3} + 3\frac{\partial v}{\partial t_1} < v, v > +3v\left\langle \frac{\partial v}{\partial t_1}, v\right\rangle,$$

where now

$$\langle f, g\rangle := \int_{\mathbb{R}} f(s|x)g(s|x)\sigma'(x)\mathrm{d}x.$$

Thanks !