Non-commutative cluster varieties and some applications

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Based on joint work with Maxim Kontsevich

G: split s.s. alg gr$/\mathbb{Q}$.

Σ: Riemann surf with punctures of top. type S

$$\mathcal{M}_{DR}(G,\Sigma,\beta) = \left\{ \begin{array}{l} G\text{-bundles with merom} \\ \text{conn on } \Sigma, \text{ of irr. type } \beta \end{array} \right\}$$

$$\downarrow$$

$\mathcal{P}_{G,S,\beta}$ - wild moduli space

$$\Gamma_{G,S,\beta} = \pi_1(\mathcal{P}_{G,S,\beta})$$

$$\downarrow$$

$\mathcal{M}_{g,n}$

Moduli space of Stokes data

$$\begin{tikzcd} \mathcal{M}_{DR}(G,\Sigma,\beta) \arrow[rr, "\widetilde{RH}"] \arrow[dr, "\pi_{DR}"'] & & \widetilde{\mathcal{M}}_B(G,S,\beta) \arrow[dl, "\pi_B"] \\ & \mathcal{P}_{G,S,\beta} & \end{tikzcd}$$

locally constant

$\Gamma_{G,S,\beta}$

Th (Linhui Shen - AG, AG - M. Kontsevich - arXiv 2108 04268)

$\widetilde{\mathcal{M}}_B(G,S,\beta)$ has $\Gamma_{G,S,\beta}$- equivariant cluster Poisson structure

Bertola-Korotkin, Nekrasov

τ-funct: Compare 2 kinds of Poisson coordinate systems

$G = GL_n(R)$ $\quad R = Mat_N(\mathbb{C})$

R: any noncomm- field

× × ×

1) Basic example:

V: vect space $/R$ $\quad rk_R V = m$

$\mathcal{F}$: flag

$$V = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^m = 0$$

codim $\mathcal{F}^i = i$

$Conf_n$ (flags) $\quad (A, B)$

Def A pair of flags is generic if

$a+b=m$

$$V \xrightarrow{\sim} V/_{A^a} \oplus V/_{B^b} \quad \forall a,b$$

$\mathcal{B}$

$$\left\{\begin{array}{c}\text{Generic pairs} \\ \text{of flags in } V\end{array}\right\} \xleftrightarrow{1:1} \{V = L^1 \oplus \dots \oplus L^m\}$$

A

$$L^b = A^{a-1} \cap B^{b-1} \qquad a+b = m+1$$

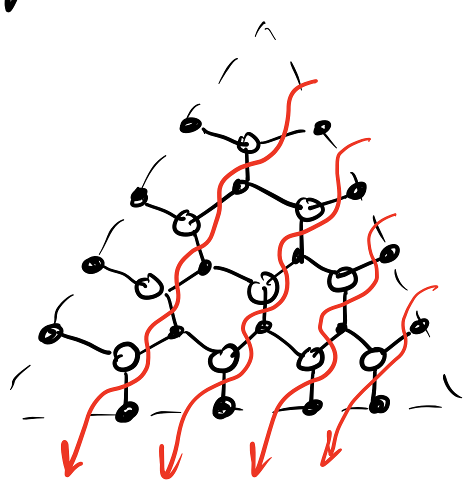
Def Generic triple (A, B, C):

$$\forall\ a+b+c = m \qquad V \xrightarrow{\sim} V/A^a \oplus V/B^b + V/C^c$$

Theorem $\exists$ canonical equivalence of groupoids

$$\left\{\begin{array}{c}\text{Generic triples} \\ \text{of flags in } V\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}R\text{-line Bundles} \\ \text{with connections} \\ \text{on Bipartite graph} \end{array}\right\}$$

Γ_m

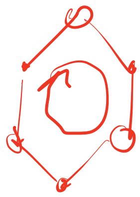


$m=4$

① $R = \mathbb{C} \Rightarrow \text{RHS} = (\mathbb{C}^\times)^{\binom{m-1}{2}}$

Cluster Poisson coordinates

R: non-comm $\Rightarrow$ No longer have coordinates



$R^\times / \text{conj by } R^\times$

Construction

$\circ$ - vert of $\Gamma_m = \{a+b+c = m-1\}$

$\bullet$ - vertices —//— $= \{a+b+c = m-2\}$

$$L^\circ_{a,b,c} = A^a \cap B^b \cap C^c$$

$$P_{a,b,c} = \text{—//—//—}$$

$$\begin{tikzcd} & L^\circ_{a+1,b,c} \arrow[d] & \\ & P_{a,b,c} & \\ L^\circ_{a,b,c+1} \arrow[ur] & & L^\circ_{a,b+1,c} \arrow[ul] \end{tikzcd}$$

$$L^\bullet_{a,b,c} = \operatorname{Ker}\left(L^\circ \oplus L^\circ \oplus L^\circ \longrightarrow P\right)$$

$$
\begin{tikzcd}
 & L_i^0 & \\
 & L_{a,b,c} \arrow[u] \arrow[dl] \arrow[dr] & \\
L^0 & & L_1^0
\end{tikzcd}
$$

Decorated flags

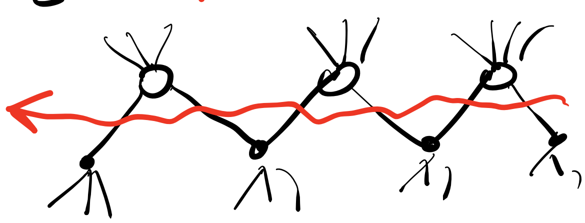
A flag $\mathcal{F}$ + $f_i \in \dfrac{\mathcal{F}^{i-1}}{\mathcal{F}^i}$, $f_i \neq 0$

Theorem $\exists$ canonical equiv

$$\{\text{Generic triples of decorated flags}\}$$

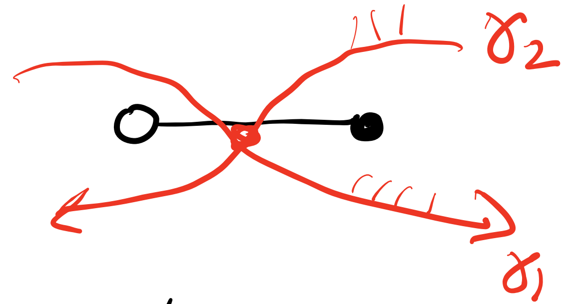
$$\Updownarrow$$

$$\left\{\begin{array}{l}\mathbb{R}\text{-line bundles with connection on } \Gamma_m \\ + \text{trivialization on every zig-zag}\end{array}\right\}$$



Key difference: $\exists$ canonical $\mathcal{A}$-coordinates

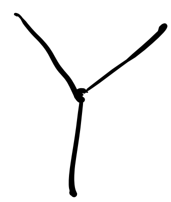
$$S_{\gamma_1} = \Delta_E S_{\gamma_2}$$



Th Δ_E are ratios of Gelfand-Retakh quasideterminants

Monomial relations

$$\boxed{\Delta_{E_1}\cdot\Delta_{E_2}\cdot\Delta_{E_3} = -1}$$

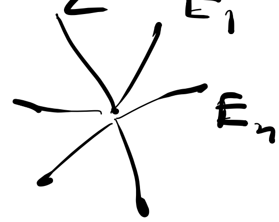


No more relation

Def Γ: a bipartite ribbon graph

A-coord on Γ: $\{\Delta_E \in \mathbb{R}^\times\}$ E - oriented

- $\Delta_E \, \Delta_{\bar{E}} = -1$



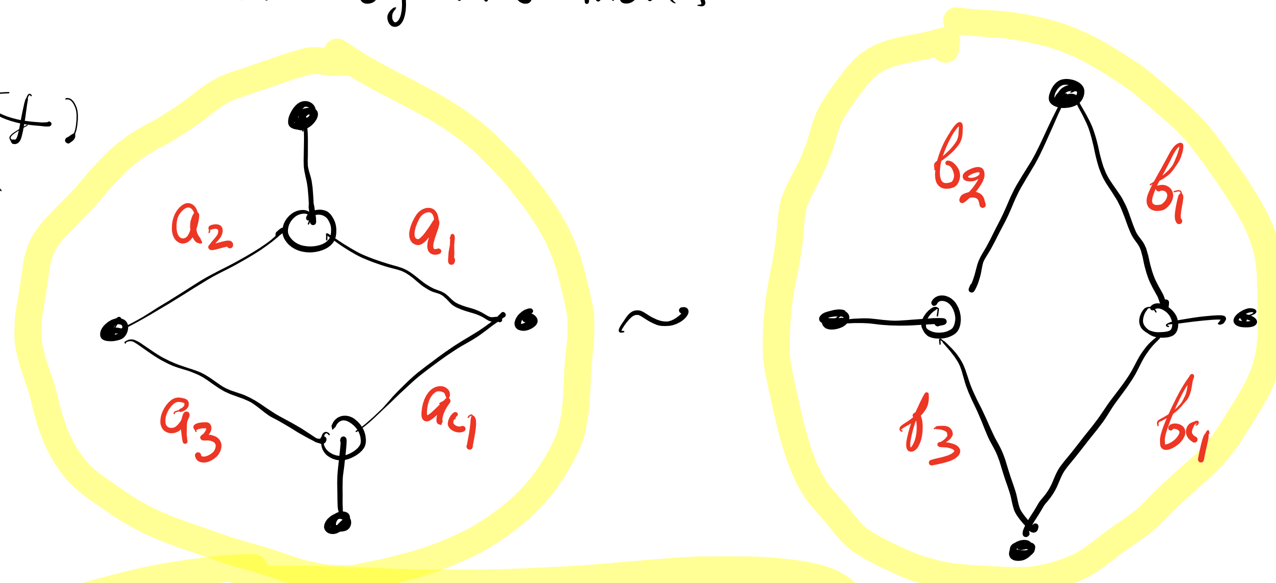
- $\cdot\cdot$ $\Delta_{E_1} \cdots \Delta_{E_n} = -1$

Lemma $\{$A-coord on $\Gamma\}$ $\overset{1:1}{\longleftrightarrow}$ $\mathbb{R}$-line bundle with conn on Γ

trivialized on each zig-zag

Two by two moves

(*)



$$b_1 = (1 + A_3^{-1})\, a_3 \qquad A_3 = a_3 a_4 a_1 a_2$$

$$b_2 = (1 + A_4)\, a_4$$

<u>Claim</u> Pentagon rel'n

Non-comm cluster var: $\mathcal{A}_{(\Gamma)}$

Glue all non-comm cluster $\mathcal{X}$-dom by iso's as do 2x2 moves

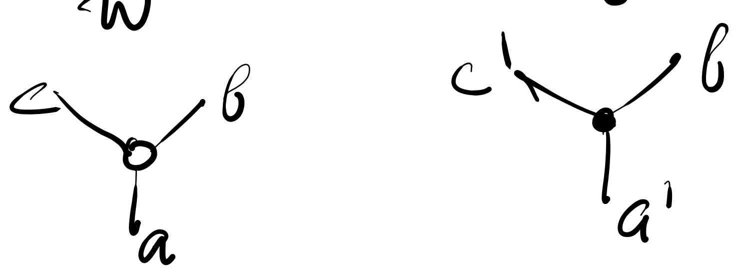
Non-commutative 2-form Ω on $\mathcal{A}_{(\Gamma)}$

$$\{a, b\} = da\, db\, b^{-1} a^{-1}$$

($a \sim A \in Mat_n$ $\quad \frac{1}{N} Tr \left(dA\, dB\, B^{-1} A^{-1} \right)$
$b \sim B \in Mat_n$)

<u>**theorem**</u> (Assume Γ : 3-valent)

$$\Omega^2_\Gamma := \sum_{w}{}' \{a, b\} - \sum_{b}{}' \{a', b'\}$$



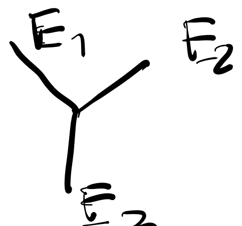
- invariant under dramid (x)

× × x

$$d \Omega^2_\Gamma = \sum_{\substack{\text{external} \\ E \text{ of } \Gamma}}{}' \omega^3_E$$

CS

$$\omega^3_E = \left(a_E^{-1}\, d a_E \right)^3$$



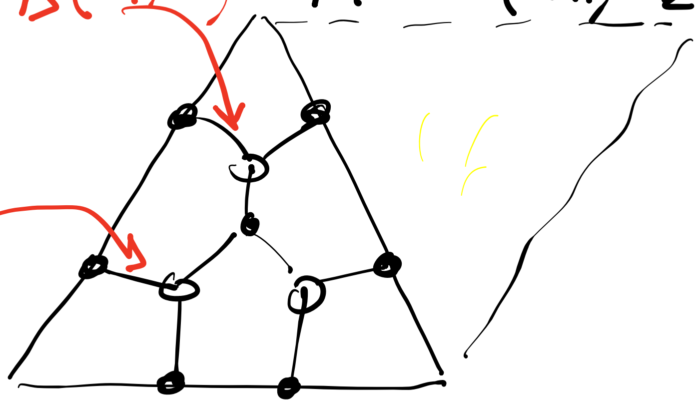
$$\omega^3_{E_1} + \omega^3_{E_2} + \omega^3_{E_3} = d\{a, b\}$$

$$H^n(BG, \Omega^{\geq n}) \ni c_n \qquad c_2$$

(the second Chern class $= (\{;\cdot\}, w^3)$)

Ex Γ_2 $\quad \Delta(a_1, b_2) \quad A = (\vec{a}_1, \vec{a}_2)$

$\Delta(a_2, b_1)$



$B = (b_1, b_2) \qquad\qquad C = (c_1, c_2)$

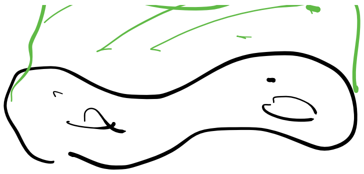
$$(\vec{a}_1, \vec{b}_1, \vec{b}_2) = \begin{pmatrix} a_{11} & b_{11} & b_{21} \\ a_{12} & b_{12} & b_{22} \end{pmatrix}$$

$$\Delta(\vec{a}_1, \vec{b}_2) = (a_{11} - a_{12} b_{12}^{-1} a_{11})^{-1} \cdot (b_{21} - b_{22} b_{12}^{-1} b_{11})$$

$$= (a_1, b_1)_{1,1} \cdot (b_1, b_2)^{-1}_{2,1}$$

Non-comm cluster Lagrangian

(subvariety)



$\mathcal{A}_m(S, \text{decoration at punct})$

Ω_Γ - 2 form

- has a structure of non-comm cluster A-var.

M: 3-fold $\quad \partial M = S$

Consider all m-dim decorated local system on S which can be extended to M

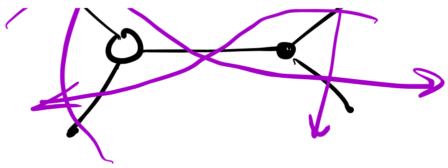
$L_M \subset \mathcal{A}_m(\partial M)$

Lagrangian $\quad$ 2-form

__Goal:__ __Sh__ The same is true in non-commutat + explicit equations defining L

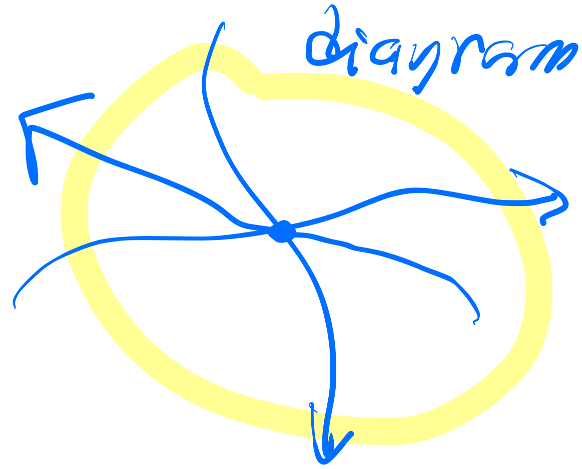
__Wanted:__ 3d analysis of bipartite graphs

D. Thurston:

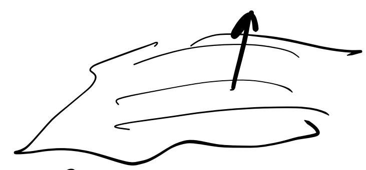


2d

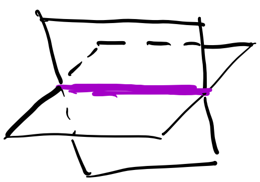
triple crossing diagrams



Def (3d) A *quadruple (Q)-crossing diagram* of cooriented smooth surfaces in 3d mfd M:



- All intersections are either lines



- or quadruple intersection points

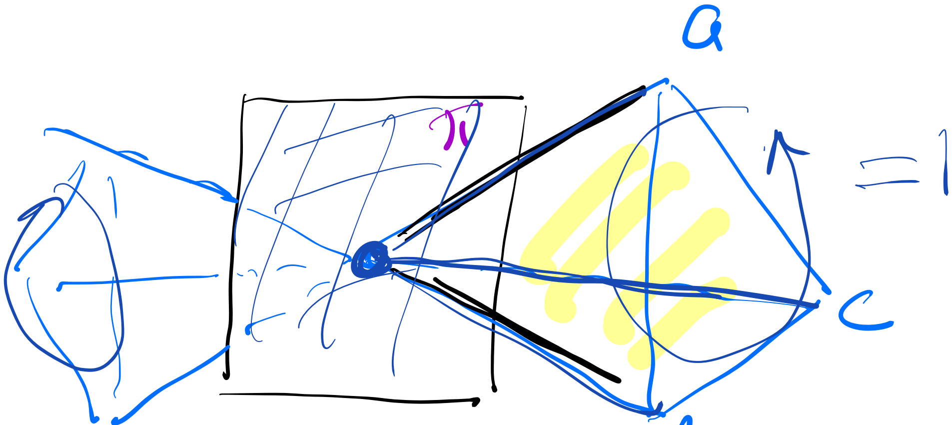
$$\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 + \alpha_4 n_4 = 0$$

$$\alpha_1, \dots, \alpha_4 > 0$$

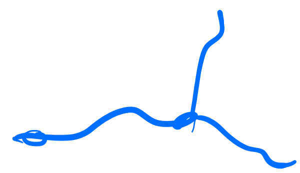
Q - diagram

$Q \subset M$ (3 fold) $\Longrightarrow$ $Q \cap \partial M$ - triple crossings

$$\mathcal{Z}_Q \hookrightarrow \mathcal{A}_{Q\cap\partial M}$$

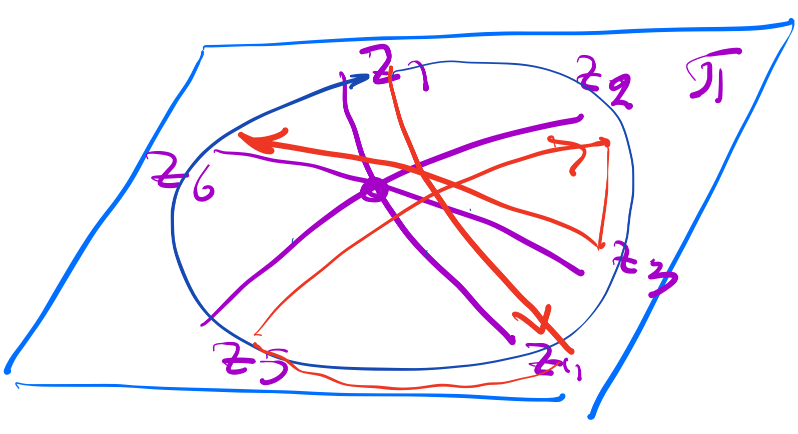


Singularity graph Γ_Q



edge E of Γ_Q $\rightsquigarrow$ $a_E \in R^\times$

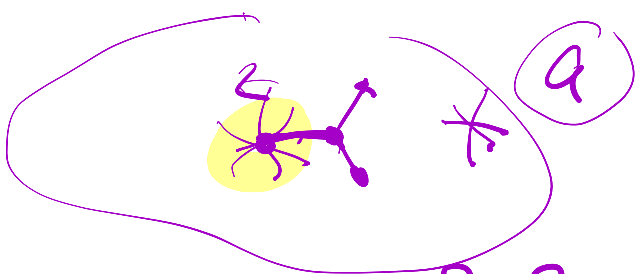
$abc = 1$



$$z_1 z_4 \cdot z_5 z_2 \cdot z_3 z_6 = -1$$

$$\left.\begin{aligned} z_1 z_4 + (z_5 z_2)^{-1} + 1 &= 0 \\ z_5 z_2 + (z_3 z_6)^{-1} + 1 &= 0 \\ z_3 z_6 + (z_1 z_4)^{-1} + 1 &= 0 \end{aligned}\right\} \text{eqns}$$

<u>Claim</u> L_Q is defined by these equations



Isotropic $\Longleftrightarrow$ 2×2 moves preserve the 2-form

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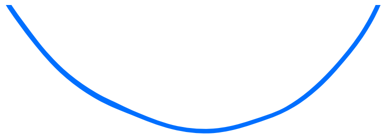

① Triangulate M, induce triang of ∂M

②  $\rightsquigarrow$ $Q_m(T)$

T

When $n=2$ we recover Berenstein–Retakh surface cluster algebra