Kyiv Formula and String Dualities

Alba Grassi

CERN

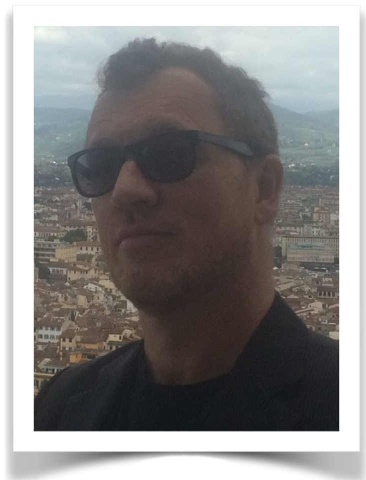
SwissMAP
The Mathematics of Physics
National Centre of Competence in Research

FNSNF
Swiss National Science Foundation

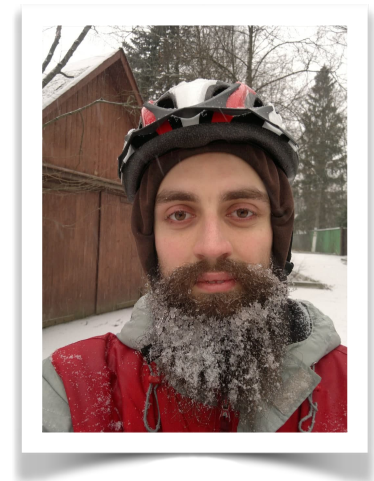
UNIVERSITÉ DE GENÈVE

Mostly based on work (done and in progress) in collaboration with

M.Bershtein



G.Bonelli

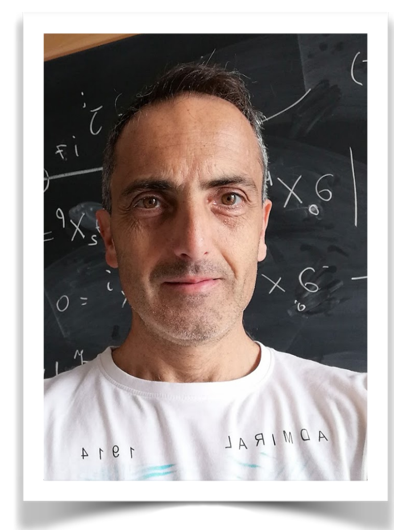


P.Gavrylenko

Q.Hao

Y.Hatsuda

M.Mariño



A.Tanzini

Introduction and Motivation

During the last decades string theory has provided several new results and applications in various fields. These results are often a consequence of dualities:

matrix models

gauge theory

operator theory

⟷

geometry

gravity

string theory

Introduction and Motivation

Example 1: Mirror symmetry [Candelas et al]

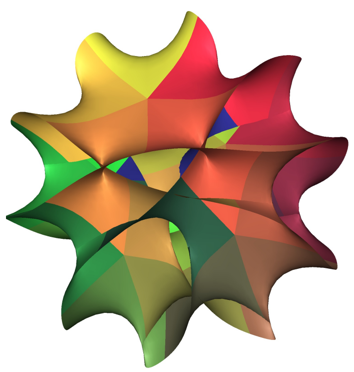
manifold X $\longleftrightarrow$ manifold $\widetilde{X}$

Underlying intuition: string propagation in both spaces is identical.

→ Application: difficult computations on one manifold can be mapped into simpler problems on its mirror partner.

Introduction and Motivation

Another interesting aspect:



geometrical objects ⟷ algebraic objects $\int \mathrm{d}M \mathrm{e}^{-N\mathrm{Tr}(V(M))}$

Example 2: [Witten, Kontsevich]

Intersection theory on moduli space of Riemann surfaces ⇠⇢ Matrix models

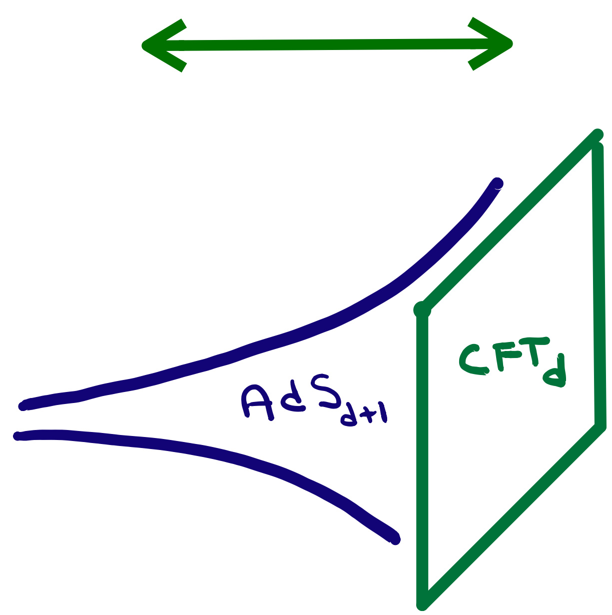
→ Geometry as emergent phenomena: guideline to study quantum modifications to classical geometrical structures

Introduction and Motivation

Example 3: String/Gauge dualities ['t Hooft].

A well known example is the AdS/CFT correspondence [Maldacena]

String Theory on
AdS background



Conformal Field Theory
on lower dimensional
background

→ (dual) non-perturbative definition of string theory

Today's Talk: Kyiv Formula and String Dualities

String Model: Topological String Theory on Toric CY_3

Duality: Topological String / Spectral Theory of Quantum Curves

We will see:

(1) Kyiv formula can be used to prove some aspects of this duality

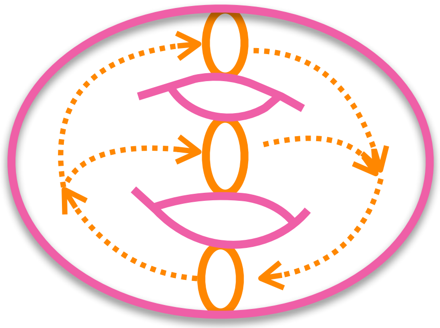
(2) The interplays between these topics lead to new (conjectural but well tested) results in the context of q-difference Painlevé equations

Topological String Theory

In string theory point particles are replaced by strings. Formally this is modelled by considering maps from Riemann surfaces into a target manifold X.

Periodic trajectories generate genus g Riemann surfaces

$g = 2$



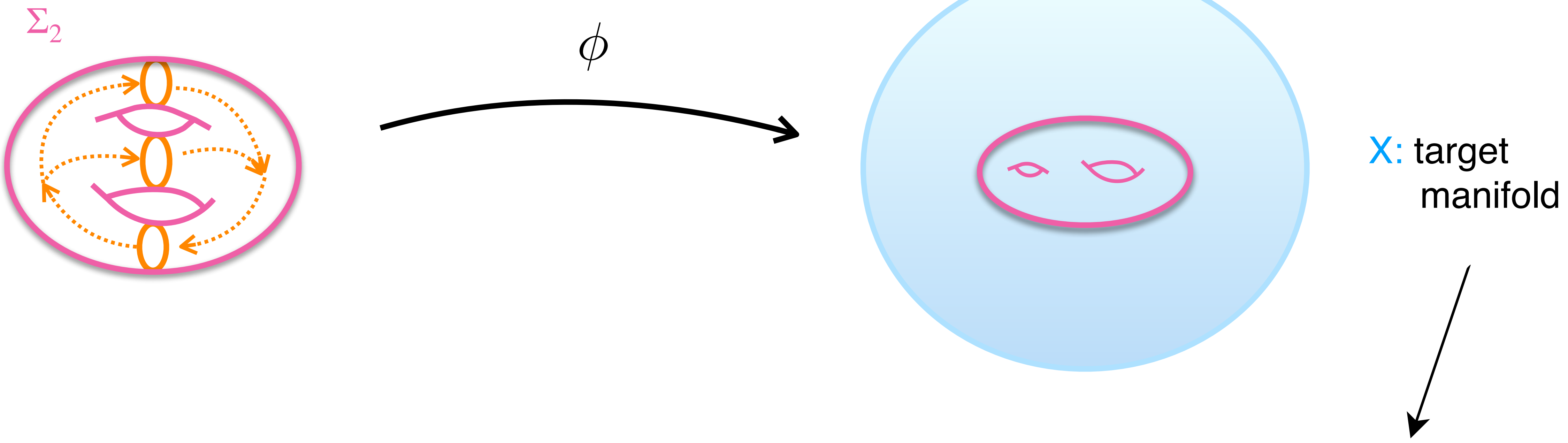
$\sim g_s^2 F_2$

string coupling constant

The details of this process are encoded in the genus g free energies F_g

This is modelled by considering holomorphic maps from Riemann surfaces into a target manifold X.

$$\phi : \quad \Sigma_g \quad \rightarrow \quad X$$

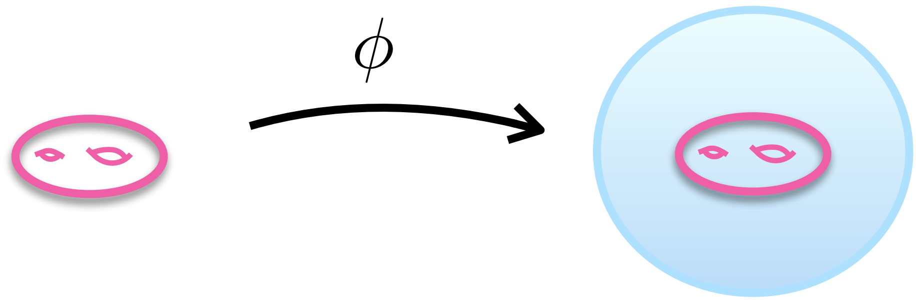


Here X is a 3 dimensional complex manifold (Calabi-Yau manifold)

Topological string theory: the free energies encode the enumerative geometry of the target manifold X

$$F_g(t) = \sum_{d\geq 1} N_g^d \, \mathrm{e}^{-dt}$$

N_g^d are the Gromov-Witten (GW) invariants: “count” holomorphic maps $\phi : \quad \Sigma_g \quad \to \quad X$



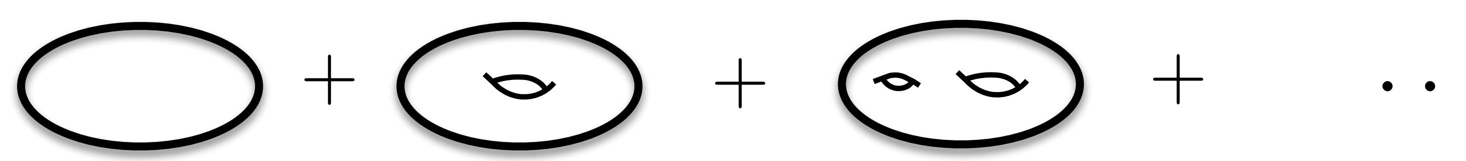
t: Kähler parameter of X

For the geometries X that we will be considering, these have been computed explicitly.

[Aganagic-Klemm-Mariño-Vafa, Bershadsky-Cecotti-Ooguri-Vafa, Bouchard-Klemm-Mariño-Pasquetti, Kontsevich, Pandharipande-Thomas, …]

The (formal) partition function Z is obtained by summing over all genera

$$F = \log Z = \sum_{g \geq 0} g_s^{2g-2} F_g(t) \qquad (1)$$



Problem: $F_g \quad \sim (2g-2)! \qquad g \gg 1$ → zero radius of convergence [Gross- Periwal, Shenker]

→ We are missing some interesting (non-perturbative) phenomena

Question: is there a well-defined function $F = \log Z$ such that (1) is its series expansion?

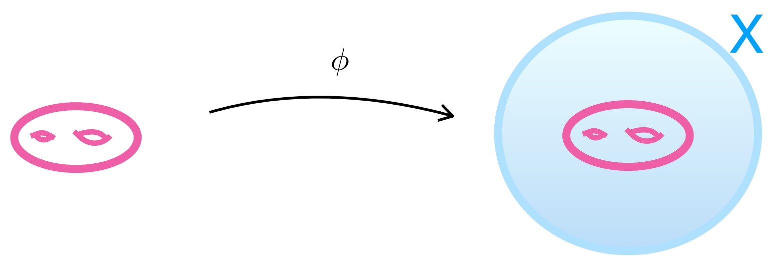
Our answer : Z = spectral traces of suitably constructed quantum mechanical operators on the real line

AG, Hatsuda, Mariño

This gives a new and exact relation between the spectral theory of certain quantum mechanical operators and enumerative geometry/topological string

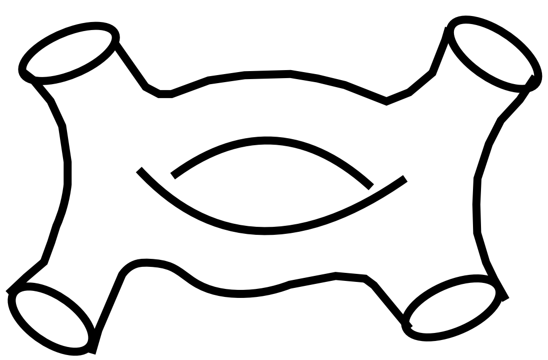
→ Topological String / Spectral Theory duality

Example:



Consider the target geometry X to be the canonical bundle over $\mathbb{CP}_1 \times \mathbb{CP}_1$ also known as local $\mathbb{P}_1 \times \mathbb{P}_1$

Using the mirror symmetry we can relate such geometry to [Batyrev, Hori-Vafa, Katz-Klemm-Vafa, Dijkgraaf et al, . . .].



$$m e^{x} + e^{p} + e^{-p} + e^{-x} + \kappa = 0$$

(mirror curve to local $\mathbb{P}_1 \times \mathbb{P}_1$)

↑

This is the classical version of the operator

$$\mathcal{O}(\hat{x}, \hat{p}) = m e^{\hat{x}} + e^{-\hat{x}} + e^{\hat{p}} + e^{-\hat{p}} \qquad [\hat{x}, \hat{p}] = i\hbar$$

Terminology: $\mathcal{O}$ is the quantum mirror curve to local $\mathbb{P}_1 \times \mathbb{P}_1$

Theorem: The operator $\rho = \mathscr{O}^{-1}$ has a discrete spectrum $\{E_n^{-1}\}_{n\geq 0}$ and it is of trace class on $L^2(\mathbb{R})$

[AG-Hatsuda-Mariño
Kashaev-Mariño
Laptev-Schwimmer-Takhtajan]

$$\mathrm{Tr}\rho^N = \sum_{n\geq 0} E_n^{-N} < \infty$$

The kernel of the operator ρ is $\rho(x,y) = \dfrac{\mathrm{e}^{-u(x,m,\hbar)-u(y,m,\hbar)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$

where $u(x,m,\hbar)$ is determined by the Faddeev quantum dilogarithm ϕ_b

[Kashaev-Mariño-Zakany]

$$u(x,m,\hbar) = \pi x \mathrm{b}/2 + \log\left|\frac{\phi_{\mathrm{b}}(x - \frac{1}{4\pi b}\log m + ib/4)}{\phi_{\mathrm{b}}(x + \frac{1}{4\pi b}\log m - ib/4)}\right| + \frac{1}{8}\log m \qquad \hbar = \pi b^2$$

Some definitions:

Fredholm determinant: $$\det\left(1+\kappa\rho\right) = \prod_{n\geq 0}\left(1+\frac{\kappa}{E_n}\right)$$

Fermionic spectral traces: $$Z(N,\hbar) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\mathrm{sgn}(\sigma)}\int_{\mathbb{R}^N}\mathrm{d}x_1\cdots\mathrm{d}x_N\prod_{i=1}^{N}\rho(x_i, x_{\sigma(i)})$$

S_N : permutation of N elements

Example: $Z(1,\hbar) = \mathrm{Tr}\rho$ or $Z(2,\hbar) = \frac{1}{2}\left((\mathrm{Tr}\rho)^2 - \mathrm{Tr}\rho^2\right)$

We have: $$\det\left(1+\kappa\rho\right) = \sum_{N\geq 0} Z(N,\hbar)\kappa^N$$

X= canonical bundle over $\mathbb{CP}_1 \times \mathbb{CP}_1$ —quantization of mirror curve→ quantum mechanical operator ρ

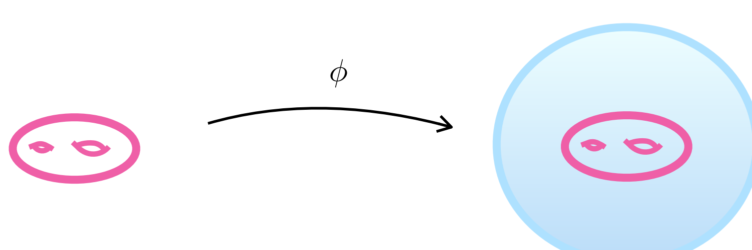
$$Z(N,\hbar) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\mathrm{sgn}(\sigma)}\int_{\mathbb{R}^N}\mathrm{d}x_1\cdots\mathrm{d}x_N\prod_{i=1}^{N}\rho(x_i, x_{\sigma(i)})$$

Claim: [AG-Hatsuda-Mariño]

$$\log Z(N,\hbar) = \sum_{g\geq 0}\hbar^{2-2g}F_g(t) + \mathcal{O}(\mathrm{e}^{-\hbar})$$

when $\hbar, N \to \infty$ with $t = \frac{N}{\hbar}$ fixed

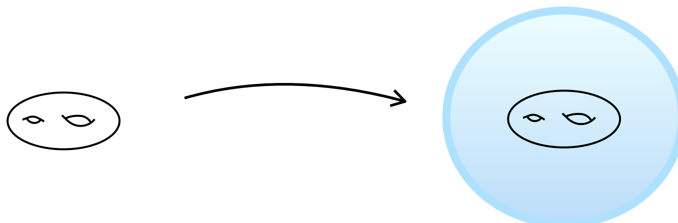
Enumerative geometry / Topological string amplitudes on the target geometry X = canonical bundle over $\mathbb{CP}_1 \times \mathbb{CP}_1$



Note: $\hbar = g_s^{-1}$

Topological string/
Enumerative geometry

ϕ



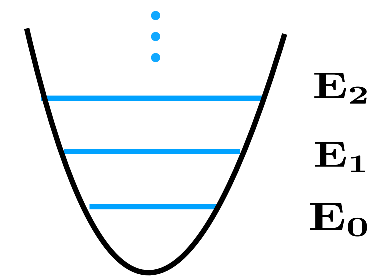
g_s

[AG-Hatsuda-Mariño]
[AG-Codesido-Mariño]

Spectral theory of a class of quantum mechanical operators called quantum mirror curves

ρ

$\hbar^{-1}$



string perturbation theory
$\equiv$ g_s small

non-pert effects in quantum mechanics $\equiv$ $\hbar$ large

non-pert effects
$\equiv$ g_s large

WKB method in quantum mechanics $\equiv$ $\hbar$ small

→ Exact analytic solution for spectral theory of difference equations (relativistic integrale systems)

To make contact with Painlevé equations and Kyiv construction it is useful to formulate our duality at the level of the Fredholm determinant.

Claim: [AG-Hatsuda-Mariño]

We can compute the Fredholm determinant of ρ using topological string/enumerative geometry

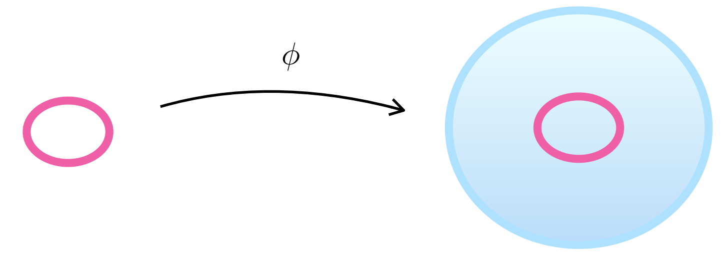
Example: consider $\rho(x,y) = \frac{e^{-u(x,m,\hbar)-u(y,m,\hbar)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$ and set $\hbar = 2\pi,\ m = 1$. Then we have

$$\det\left(1+\kappa\rho\right) \sim \theta_3\left(\xi - \frac{1}{12}, \tau\right)$$

where $\xi = \frac{1}{2\pi^2}\left(t\partial_t^2 F_0 - \partial_t F_0\right)$ and $\tau = \frac{2i}{\pi}\partial_t^2 F_0$ with $t = t(\kappa) = (\text{quantum})\ \text{mirror map}$

F_0 : genus zero GW

invariants on local $\mathbb{P}_1 \times \mathbb{P}_1$



More generically the expression has the following form

$$\det\left(1+\kappa\rho\right) = \sum_{n\in\mathbb{N}} \exp\left[J(\mu+2\pi \mathrm{i} n, \hbar, m)\right], \qquad \kappa = \mathrm{e}^{\mu}$$

J : topological string grand potential

This is a particular combination of topological string free energy **and** "refined" topological string free energy in the Nekrasov-Shatashvili limit

Next: it exists a particular limit where our duality makes contact with well known statements in theory of Painlevé equations —> proof in this particular limit.

[Bonelli-AG-Tanzini]

Painlevé equations

VI : $$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt} + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{q^2}+\frac{\gamma(t-1)}{(q-1)^2}+\frac{\delta t(t-1)}{(q-t)^2}\right)$$

V : $$\frac{d^2q}{dt^2} = \left(\frac{1}{2q}+\frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{(q-1)^2}{t^2}\left(\alpha q+\frac{\beta}{q}\right) + \frac{\gamma q}{t} - \frac{1}{2}\frac{q(q+1)}{q-1}$$

IV : $$\frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2-\alpha)q + \frac{\beta}{q}$$

III_1 : $$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{q^2\left(\alpha+4q\right)}{4t^2} + \frac{\beta}{4t} - \frac{1}{q}$$

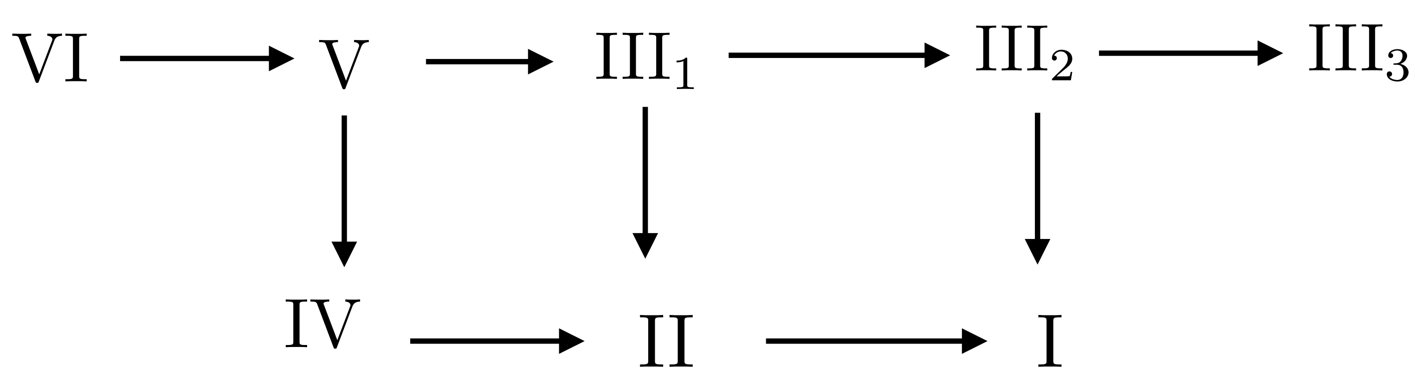
III_2 : $$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} + \frac{\alpha}{4t} - \frac{1}{q}$$

III_3 : $$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

II : $$\frac{d^2q}{dt^2} = 2q^3 + tq+\alpha$$

I : $$\frac{d^2q}{dt^2} = 6q^2 + t$$

Painlevé equations can be organised into a confluence diagram



Example:

$$\mathrm{III}_2: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} + \frac{\alpha}{4t} - \frac{1}{q}$$

$$t = s\epsilon$$

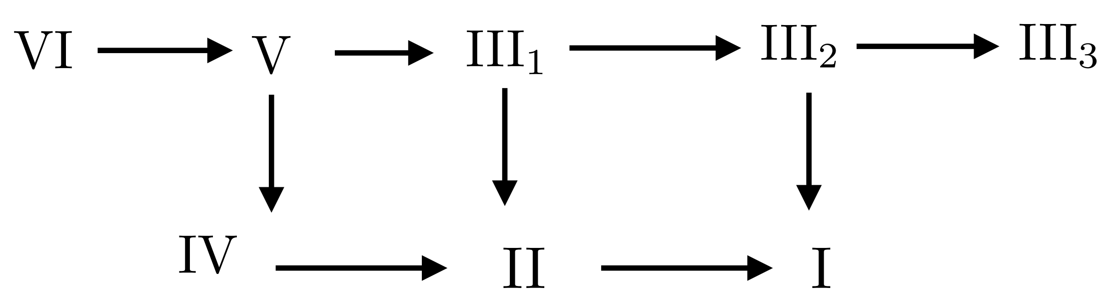
$$\alpha = -4/\epsilon$$

$$\epsilon \to 0$$

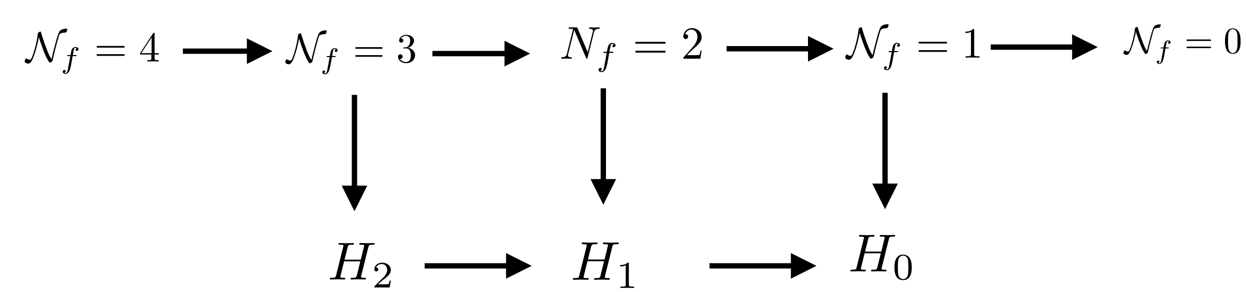
$$\mathrm{III}_3: \quad \frac{d^2q}{ds^2} = \frac{1}{q}\left(\frac{dq}{ds}\right)^2 - \frac{1}{s}\frac{dq}{ds} + \frac{2q^2}{s^2} - \frac{2}{s}$$

Recently there has been an important progress in constructing generic solutions to such equations in an explicit form by using the Nekrasov partition function of a corresponding Seiberg-Witten theory [Gamayun-Iorgov-Lisovyy]

Painlevé equations



four dimensional Seiberg-Witten theory



Painlevé free parameters $\sim$ masses of hypermultiplets/mass deformations

time $\sim$ gauge coupling e^{-1/g_{YM}^2}

See Y. Yamada's talk

Kyiv Formula: an example PIII_3

Theorem: [Gamayun,Iorgov,Lisovyy - Its, Lisovyy, Tykhyy- Iorgov, Lisovyy, Teschner- Bershtein,Shchechkin - Gavrylenko,Lisovyy]

$$q(t,\sigma,\eta) = \sqrt{t}\,\mathrm{e}^{-2\pi i\eta}\left(\frac{\tau^{\mathrm{GIL}}(t,\sigma,\eta)}{\tau^{\mathrm{GIL}}(t,\sigma+\frac{1}{2},\eta)}\right)^2$$

solves Painlevé III_3

$$\mathrm{III}_3: \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}$$

with initial conditions specified by σ and η.

Kyiv Formula: an example PIII_3

$$\tau^{\mathrm{GIL}}(t,\sigma,\eta) = \sum_{n\in\mathbb{N}} \mathrm{e}^{2\pi \mathrm{i} n\eta} Z(\sigma+n,t)$$

where $$Z(\sigma,t) = t^{\sigma^2}\frac{\mathscr{B}(\sigma,t)}{G(1+2\sigma)G(1-2\sigma)}$$

with $\mathscr{B}(\sigma,t) =$ Nekrasov instanton function for the pure 4 dim $SU(2)$ $\mathscr{N}=2$ SYM theory (in the self-dual phase $\epsilon_1 = -\epsilon_2 = \epsilon$)— also called $N_f = 0$ theory

$$\mathscr{B}(\sigma,t) = 1 + \sum_{n\geq 1} c_n(\sigma)t^n = 1 + \frac{t}{2\sigma^2} + \frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)}t^2 + \cdots$$

gauge theory language: $\sigma = a/\epsilon$: vev of scalars in vector multiplet

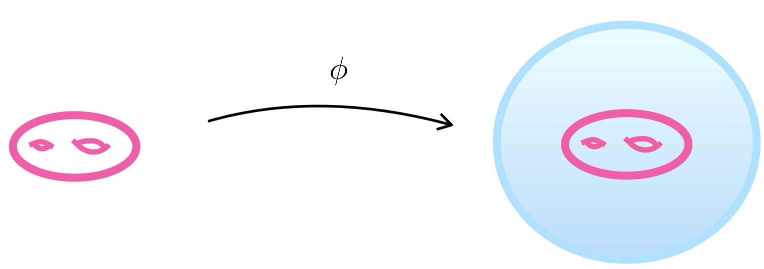
$t = \Lambda^4/\epsilon^4$: instanton counting parameter

What does this have to do with topological string and spectral theory?

Reminder:

enumerative geometry / GW invariants

spectral theory of quantum mechanical operators on $L^2(\mathbb{R})$



$$\rho(x, y) = \frac{e^{-u(x,b,m)-u(y,b,m)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$$

J : topological string grand potential (GW invariants)

$$\sum_{n\in\mathbb{N}} \exp\left[J(\mu + 2\pi i n, b, m)\right] = \det\left(1 + \kappa\rho\right), \qquad \kappa = e^{\mu}$$

On the spectral theory side:

$$\rho(x,y) = \frac{e^{-u(x,b,m)-u(y,b,m)}}{4\pi \cosh\left(\frac{x-y}{2}\right)} \qquad u(x,b,m) = -\frac{xb^2}{4} - \log\left|\frac{\phi_b\left(\frac{bx}{2\pi} - \frac{1}{4\pi b}\log m + ib/4\right)}{\phi_b\left(\frac{bx}{2\pi} + \frac{1}{4\pi b}\log m - ib/4\right)}\right| + \frac{1}{8}\log m$$

Set $\log m = \frac{i\sigma}{2\pi} + b^2 \log(b^2/t)$, $\log \kappa = \frac{b^2}{2}\log(b^2/t) + \log(1 + e^{\frac{i\sigma}{2\pi}})$ $\qquad \hbar = \pi b^2$

Take $b \to \infty$

$$\det\left(1 + \kappa\rho\right) \xrightarrow{b\to\infty} \det\left(1 + \cos(\sigma)\rho_{\rm III}\right)$$

On the spectral theory side:

$$\rho(x,y) = \frac{e^{-u(x,b,m)-u(y,b,m)}}{4\pi\cosh\left(\frac{x-y}{2}\right)} \qquad u(x,b,m) = -\frac{xb^2}{4} - \log\left|\frac{\phi_b\left(\frac{bx}{2\pi} - \frac{1}{4\pi b}\log m + ib/4\right)}{\phi_b\left(\frac{bx}{2\pi} + \frac{1}{4\pi b}\log m - ib/4\right)}\right| + \frac{1}{8}\log m$$

We have: $\det\left(1+\kappa\rho\right) \xrightarrow{b\to\infty} \det\left(1+\cos(\sigma)\rho_{\mathrm{III}}\right)$

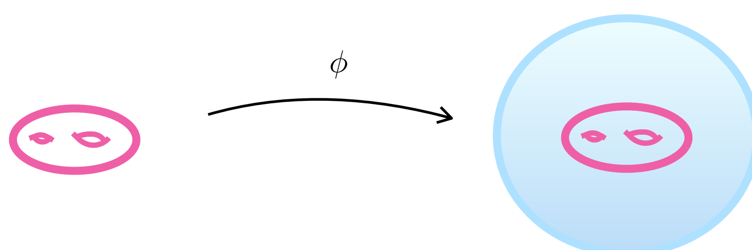
where $\rho_{\mathrm{III}}(x,y) = \dfrac{e^{-t^{1/4}\cosh x - t^{1/4}\cosh y}}{4\pi\cosh\left(\frac{x-y}{2}\right)}$

It was proven by [McCoy et al, Widom, …] that $\det\left(1+\cos(\sigma)\rho_{\mathrm{III}}\right)$ solves Painlevé III_3 with a particular choice of initial conditions.

What does this have to do with topological string and spectral theory?

Reminder:

enumerative geometry / GW invariants ⟷ spectral theory of quantum mechanical operators on $L^2(\mathbb{R})$



$$\rho(x,y) = \frac{e^{-u(x,b,m)-u(y,b,m)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$$

J : topological string grand potential (GW invariants)

$$\sum_{n\in\mathbb{N}} \exp\left[J(\mu + 2\pi i n, b, m)\right] = \det\left(1+\kappa\rho\right), \qquad \kappa = e^{\mu}$$

On the enumerative geometry side:

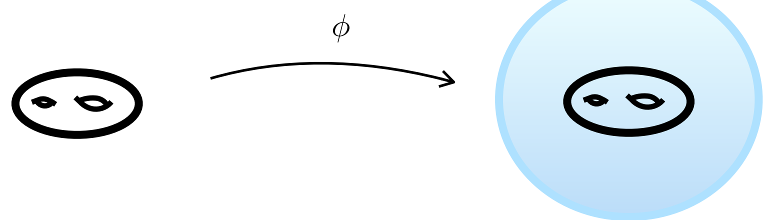
$$\sum_{n\in\mathbb{N}} \exp\left[J(\mu + 2\pi \mathrm{i} n, b, m)\right] \quad \rightarrow \quad \tau^{\mathrm{GIL}}(t, \sigma, \eta = 0)$$

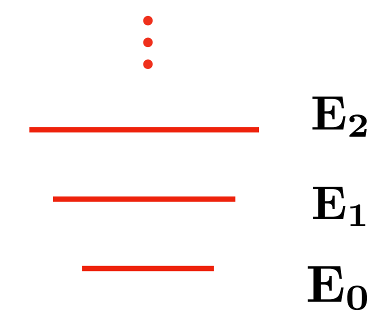
By using recent results of [Lisovyy et al, Its et al, Bershtein et al] it follows that τ solves Painlevé III_3 with same initial conditions.

topological string on target geometry X= canonical bundle over $\mathbb{CP}_1 \times \mathbb{CP}_1$

spectral theory of

$$\rho(x,y) = \frac{\mathrm{e}^{-u(x,m,b)-u(y,m,b)}}{4\pi \cosh\left(\frac{x-y}{2}\right)}$$

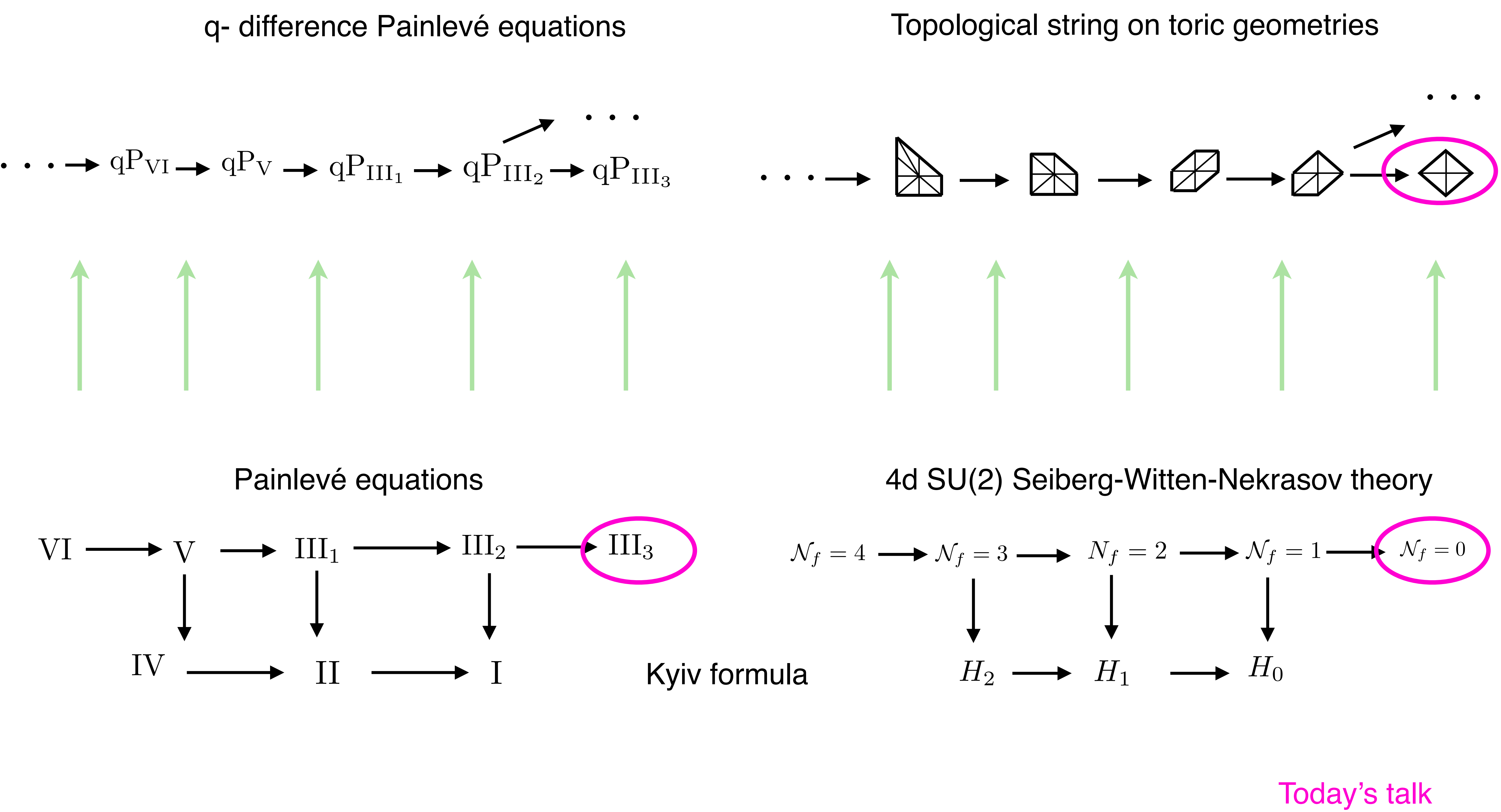




$$\sum_{n\in\mathbb{N}} Z(\sigma+n,t) = \det\left(1+\cos(\sigma)\rho_{\mathrm{III}}\right)$$

Painlevé III_3 equation

This is a small piece of a bigger picture….


q- difference Painlevé equations
Topological string on toric geometries
qP_{VI} → qP_{V} → qP_{III_1} → qP_{III_2} → qP_{III_3}
Painlevé equations
4d SU(2) Seiberg-Witten-Nekrasov theory
VI → V → III_1 → III_2 → III_3
IV → II → I
Kyiv formula
$\mathcal{N}_f = 4$ → $\mathcal{N}_f = 3$ → $\mathcal{N}_f = 2$ → $\mathcal{N}_f = 1$ → $\mathcal{N}_f = 0$
H_2 → H_1 → H_0
Today's talk

Today's Plan

String Model: Topological String Theory on Toric CY_3 fold

Duality: Topological String / Spectral Theory Duality

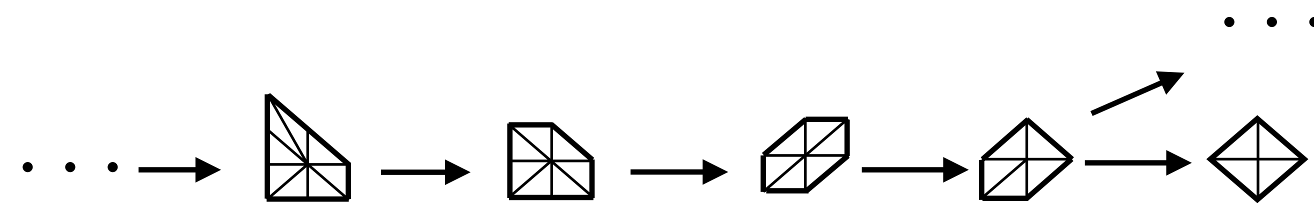
We will see:

(1) Kyiv formula can be used to prove some aspects of this duality

(2) This interplays lead to new (conjectural but well tested) results in the context of q-difference Painlevé equations

$$\cdots \to qP_{\mathrm{VI}} \to qP_{\mathrm{V}} \to qP_{\mathrm{III}_1} \to qP_{\mathrm{III}_2} \to qP_{\mathrm{III}_3}$$

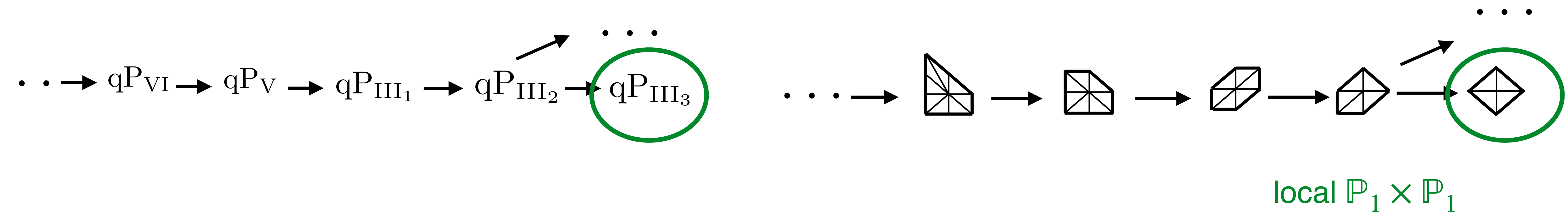
$$qP_{\mathrm{III}_2} \nearrow \cdots$$



Claim: Fredholm determinant of quantum mirror curves to such geometries solves a corresponding q-Painlevé equation

[Bonelli, AG, Tanzini]

Example:



In the example of local P1xP1 the relevant operator ρ is

$$\rho(x,y) = \frac{e^{-u(x,m,\hbar)-u(y,m,\hbar)}}{4\pi \cosh\left(\frac{x-y}{2}\right)} \qquad u(x,b,m) = -\frac{xb^2}{4} - \log\left|\frac{\phi_b\left(\frac{bx}{2\pi} - \frac{1}{4\pi b}\log m + ib/4\right)}{\phi_b\left(\frac{bx}{2\pi} + \frac{1}{4\pi b}\log m - ib/4\right)}\right| + \frac{1}{8}\log m$$

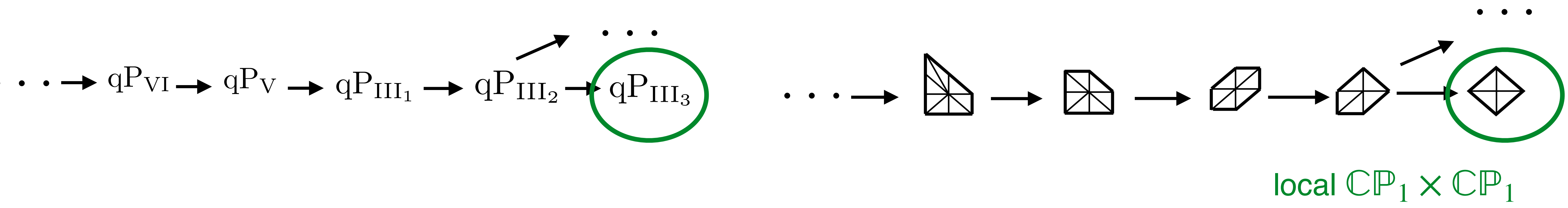
$$\hbar = \pi b^2$$

Its Fredholm determinant

$$\det\left(1 + \kappa\rho\right)$$

solves q-Painlevé III_3

Example:



The Fredholm determinant $\tau_q(\kappa, \xi) \sim \det\left(1 + \kappa\ \rho\right)$

where $q = e^{\frac{4\pi^2}{\hbar}}$, $\xi = \log m$, solves q-Painlevé III_3

$$\tau_q\left(-\kappa, \xi - \frac{4\pi^2 i}{\hbar}\right) \tau_q\left(-\kappa, \xi + \frac{4\pi^2 i}{\hbar}\right)\left(1 + e^{-\xi/2}\right) = \tau_q(\kappa, \xi)^2 + e^{-\xi/2}\tau_q(-\kappa, \xi)^2$$

→ $\det\left(1 + \kappa\rho\right)$ provides a generalisation of the PIII_3 McCoy et at solution for q-Painleve III3

→ Using the interplay between the topological string/spectral theory duality and Kyiv formula we can construct geometrically new Fredholm determinant solutions to q-difference Painlevé equations

Another problem in which this connection is useful is in the study of long-distance expansion of q-Painlevé equations

WIP with P.Gavrylenko and Q. Hao

. . . let us make one step back to Painlevé equation . . .

PIII3 tau function at short-distance (small t)

$$\tau^{\mathrm{GIL}}(t,\sigma,\eta) = \sum_{n\in\mathbb{N}} \mathrm{e}^{2\pi \mathrm{i} n\eta} Z(\sigma+n,t)$$

Gamayun,Iorgov,Lisovyy

$$Z(\sigma,t) = t^{\sigma^2} \frac{\mathscr{B}(\sigma,t)}{G(1+2\sigma)G(1-2\sigma)}$$

$\mathscr{B}(\sigma,t) =$ Nekrasov instanton function for the pure 4 dim $SU(2)$ $\mathcal{N}=2$ SYM theory (in the self-dual phase $\epsilon_1 = -\epsilon_2 = \epsilon$)— also called $N_f = 0$ theory

$$\mathscr{B}(\sigma,t) = 1 + \sum_{n\geq 1} c_n(\sigma) t^n = 1 + \frac{t}{2\sigma^2} + \frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)} t^2 + \cdots$$

This construction was generalised to q-Painleve first by Bershtein and Shchechkin

What about expansion around $t = \infty$?

$$\tau^{\infty}(\rho, \nu, r) = e^{\frac{r^2}{16}} r^{\frac{1}{4}} \sum_{n \in \mathbb{Z}} C(\nu + in) e^{4\pi i n \rho} e^{(\nu+in)r} r^{\frac{1}{2}(\nu+in)^2} \mathscr{B}^{\infty}(\nu + in, r)$$

$$C(\nu) = G(1 + i\nu) 2^{\nu^2} e^{\frac{i\pi\nu^2}{4}} (2\pi)^{-\frac{i\nu}{2}}, \quad t = 2^{-12} r^4$$

$$\mathscr{B}^{\infty}(\nu, r) = 1 + \frac{\nu(2\nu^2 + 1)}{8r} + \frac{\nu^2(4\nu^4 - 16\nu^2 - 11)}{128 r^2} + \dots$$

Its, Lisovyy, Tykhyy- Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini -

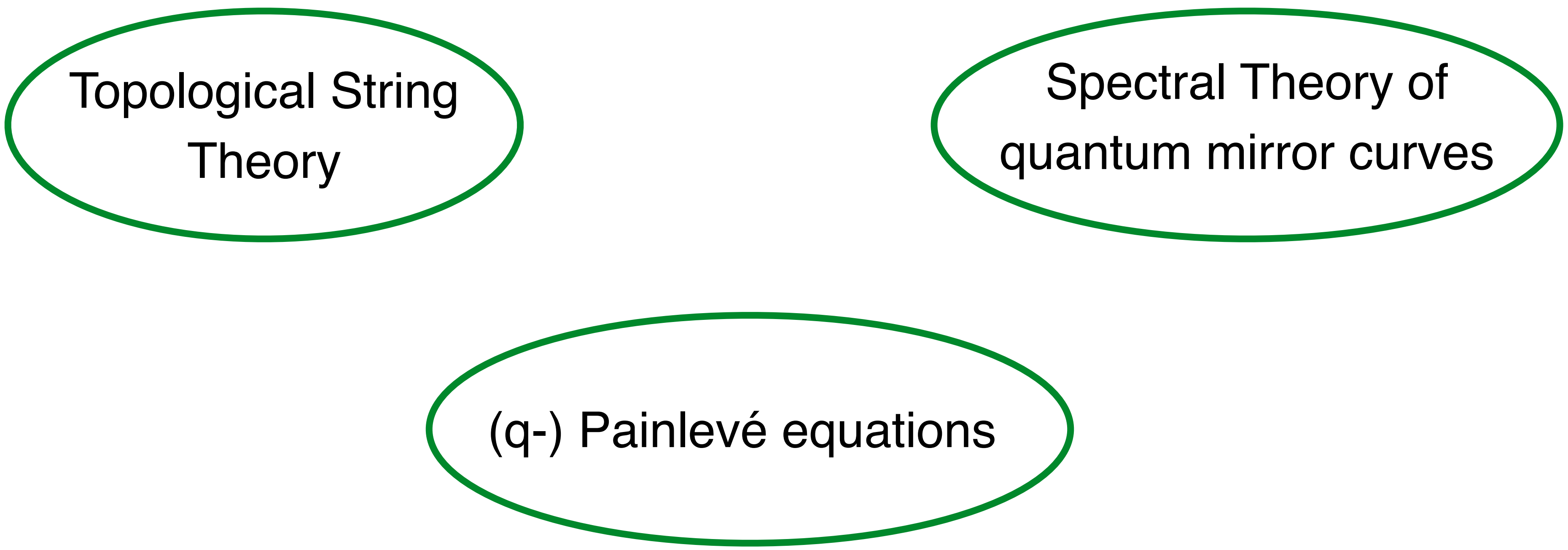
Gavrylenko, Marshakov,Stoyan - …

Can we generalise this to q-Painleve? Yes, on the topological string this is related to the expansion around the conifold point

WIP with P.Gavrylenko and Q. Hao

Summary & Conclusions

We have three main players



… and many connections among them leading to new and interesting results

Thank you!