The Edge of the XXZ

Columbia University
Isomonodromic Deformations,
Painleve Equations, and
Integrable Systems

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Overview

1) The Heisenberg-Ising (XXZ) spin-½ Chain

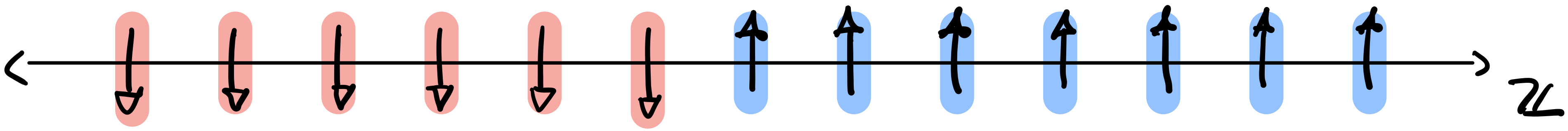
2) The Results

3) The Proof

4) Final Remarks

The Heisenberg-Ising XXZ spin-1/2 Chain

Heisenberg-Ising Spin-½ Chain (XXZ)



- $e_x = |\cdots \underset{x_1}{\uparrow} \cdots \underset{x_2}{\uparrow} \cdots\cdots \underset{x_N}{\uparrow} \cdots \rangle \in \ell^2(\mathcal{X}_N)$

w/ $N \to \infty$

$$h_{i,i+1} = \frac{1}{2}\left(\sigma_j^1 \sigma_{j+1}^1 + \sigma_j^2 \sigma_{j+1}^2 + \Delta\left(\sigma_j^3 \sigma_{j+1}^3 - 1\right)\right)$$

- $h_{i,i+1} : \begin{cases} |\uparrow\uparrow\rangle \mapsto 0, & |\uparrow\downarrow\rangle \mapsto -\Delta|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \to 0, & |\downarrow\uparrow\rangle \mapsto -\Delta|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \end{cases}$

anisotropy
↓
$\Delta \in \mathbb{R}$

Schrödinger Equation

- $\Psi(t) = \sum_{x \in \mathcal{X}_N} \psi(x;t)\, e_x$, wave function

w/ $e_x = |\cdots \underset{x_1}{\uparrow} \cdots \underset{x_2}{\uparrow} \cdots\cdots \underset{x_N}{\uparrow} \cdots\rangle \in \ell^2(\mathcal{X}_N)$

- $\mathcal{H} = \sum_{i \in \mathbb{Z}} h_{i,i+1}$, Hamiltonian

w/ $h_{i,i+1}$: $\begin{cases} |\uparrow\uparrow\rangle \mapsto 0, & |\uparrow\downarrow\rangle \mapsto -\Delta|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \mapsto 0, & |\downarrow\uparrow\rangle \mapsto -\Delta|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \end{cases}$

$$\Rightarrow \quad \boxed{i \frac{d}{dt}\Psi(t) = \mathcal{H}\,\Psi(t)}$$

Schrödinger Equation

Probabily Function

- $\Psi(t) = \sum_{x \in \mathcal{X}_N} \psi(x;t)\, e_x$, wave function

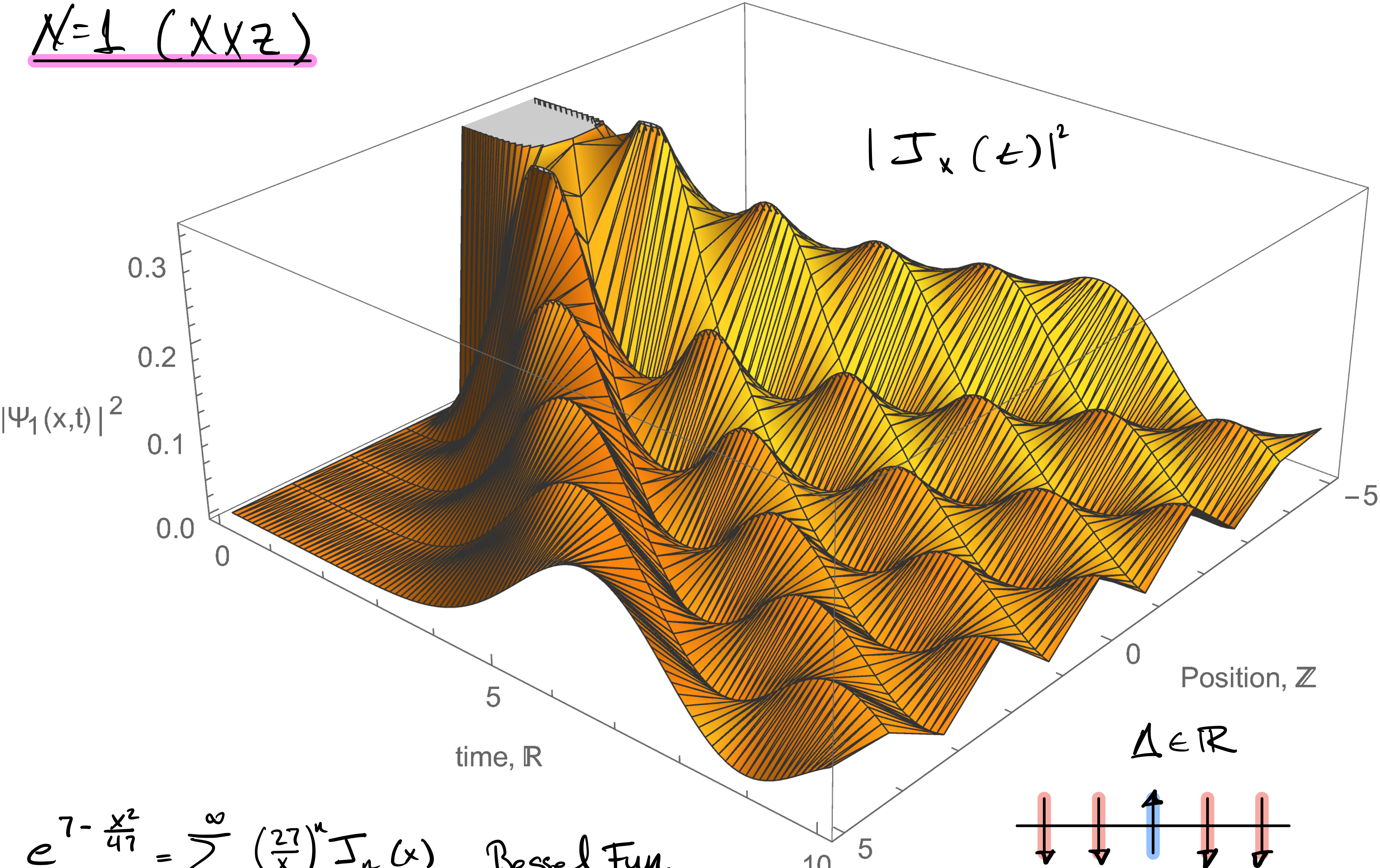
w/ $e_x = |\cdots \underset{x_1}{\uparrow} \cdots \underset{x_2}{\uparrow} \cdots \underset{x_N}{\uparrow} \cdots\rangle \in \ell^2(\mathcal{X}_N)$

$$\Rightarrow \mathbb{P}(X(t)=x) = \langle \Psi(t) | e_x \rangle \langle e_x | \Psi(t) \rangle$$

$$\mathbb{P}(X(t)=x) = \overline{\psi}(x;t)\, \psi(x;t)$$

$$\boxed{\mathbb{P}(X(t)=x) = |\psi(x;t)|^2}$$

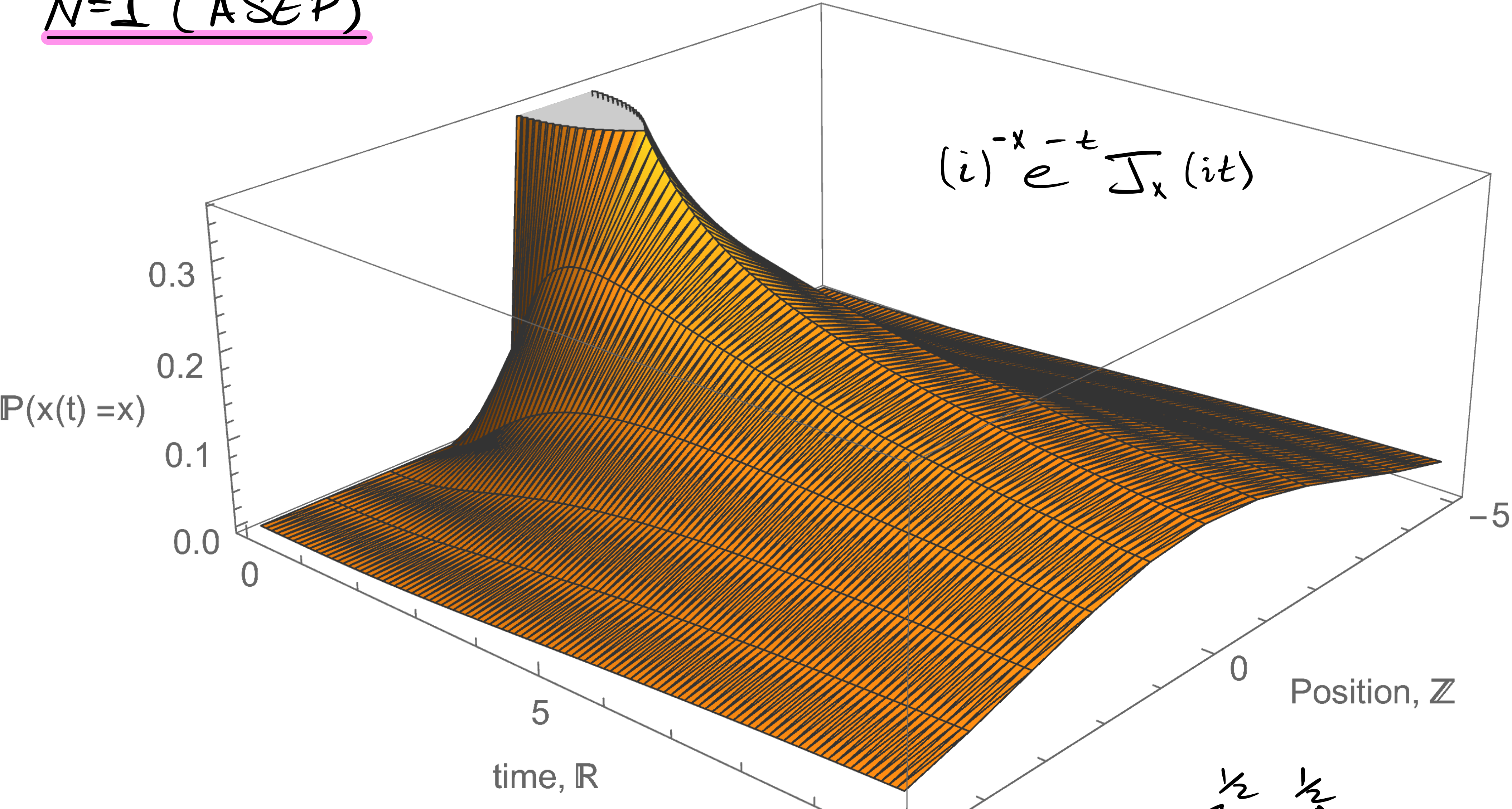
N=1 (XXZ)



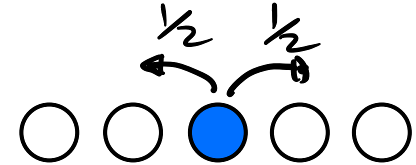
Δ ∈ ℝ

$$e^{t-\frac{x^2}{4t}} = \sum_{n=-\infty}^{\infty} \left(\frac{2t}{x}\right)^n J_n(x), \text{ Bessel Fun}$$

N=1 (ASEP)



$$e^{\tau - \frac{x^2}{4\tau}} = \sum_{n=-\infty}^{\infty} \left(\frac{2\tau}{x}\right)^n J_n(x), \quad \text{Bessel Fun}$$

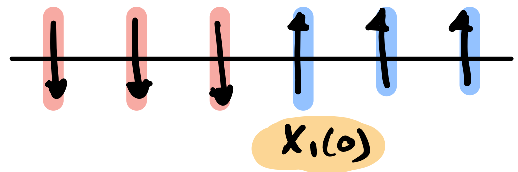


The Results

Thm (S-Tracy-Widom)

Physics (Eisler-Rácz '12, Stephan '17, Viti-Stephan-Dubail '16)

For $\Delta = 0$, let $x_1(t)$ be the position of the left-most up-spin w/ domain wall IC



Then,

$$\mathbb{P}\left(\frac{x_1(t) + 2t}{t^{1/3}} \geq -s \right) = F_{TW}^{GUE}(s)$$

$$= \det(1 - K_{Ai})_{L^2(s,\infty)}$$

w/ $$K_{Ai}(x,y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$$

Conjecture (S-Tracy-Widom '22) $\Delta \neq 0$

$$\lim_{t \ll N \to \infty} \mathbb{P}\left(\frac{X_1(t) + 2t}{t^{1/3}} \geq -s \right) =$$

$$\lim_{t \ll N \to \infty} \sum_{\sigma \in S_N} (-1)^{\sigma} \sum_{S \subset \{1,\dots,N\}} (-1)^{|S|} t^{-|S|/3} F(\sigma, S) \prod_{j \in S} K_{Ai}(s + V_j, s + V_{\sigma(j)})$$

- $V_j t^{1/3} = \gamma_j + 1$, $\gamma_j = \delta$
- $F(\sigma, S)$ depends $\{\gamma_j\}_{j=1}^{\infty}$, indep of t
- K_{Ai} is the Airy Kernel

Coefficients

$\sigma \in S_N$

$S \subset \{1, \dots, N\}$

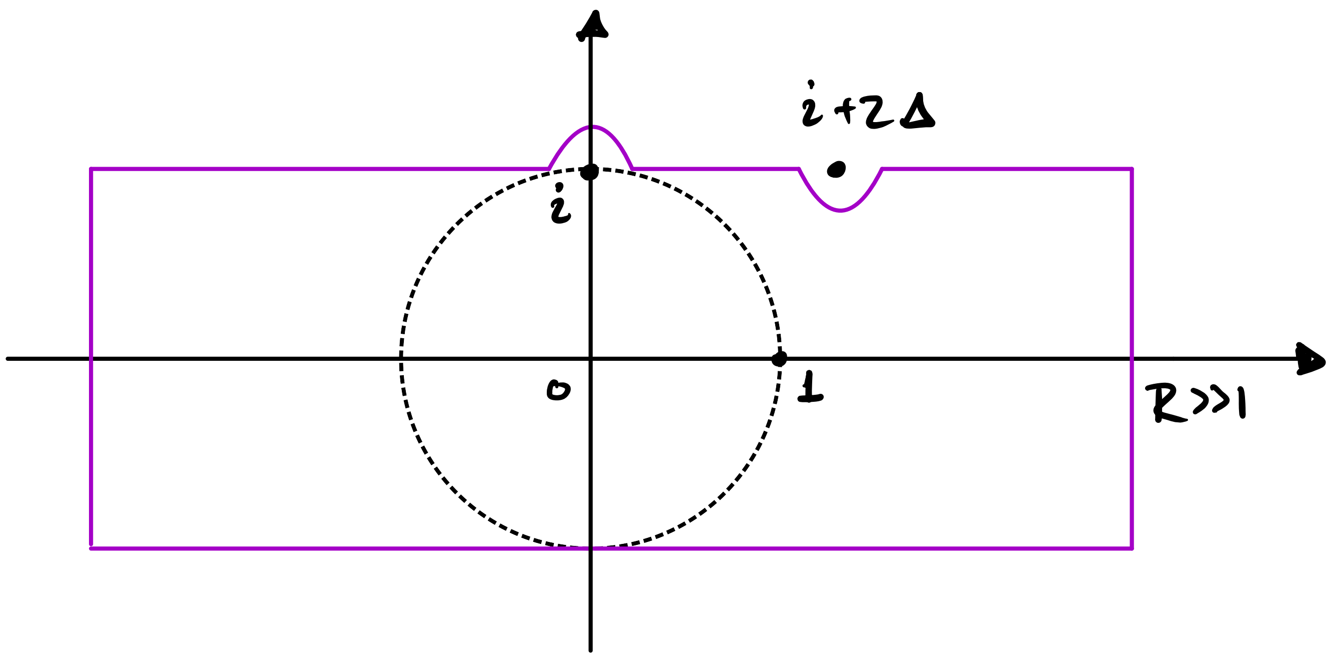
$$F(\sigma, S) =$$

$j - \sigma(j) + \#\{k \in S^c \mid j<k, \sigma(j)>\sigma(k)\} - \#\{k \in S^c \mid k<j, \sigma(k)>\sigma(j)\}$

$$(i)^{|S^c|} \underbrace{\oint_{\hat\Gamma} \cdots \oint_{\hat\Gamma}}_{N} B(\tau;\sigma,S) \prod_{j \in S^c} \left(\frac{\tau_{\sigma(j)} - (2\Delta + i)}{(2\Delta i + 1)\tau_{\sigma(j)} - i} \right)^{\nu(\sigma,S)} \prod_{j \in S^c} (i\tau_{\sigma(j)})^{y_j - y_{\sigma(j)} - 1} d^{\sigma(S^c)} \}$$

- $$B(\tau;\sigma,S) = \prod_{\substack{j<k \\ \sigma(j)>\sigma(k)}} \left(\frac{1 + \tau_{\sigma(k)}\tau_{\sigma(j)} - 2\Delta\tau_{\sigma(j)}}{1 + \tau_{\sigma(k)}\tau_{\sigma(j)} - 2\Delta\tau_{\sigma(k)}} \right) \quad ; \quad j,k \in S^c$$

- $\hat\Gamma =$



$\Delta = 0$

$F(\sigma, S) =$

$$(i)^{|S^c|} \oint_{\hat{\Gamma}} \cdots \oint_{\hat{\Gamma}} B(\tau;\sigma,S) \prod_{j\in S^c} \left(\frac{\tau_{\sigma(j)} - (2\Delta + i)}{(2\Delta i + 1)\tau_{\sigma(j)} - i} \right)^{\nu} \prod_{j\in S^c} (i\tau_{\sigma(j)})^{y_j - y_{\sigma(j)} - 1} d^{\sigma(S^c)}\tau$$

(crossed out: $B(\tau;\sigma,S) \to 1$, fraction $\to 1$)

$$= \prod_{j \in S^c} \mathbb{1}(\sigma(j) = j)$$

$S = \phi \quad \rightarrow \quad \nu = 0$

$$\sum_{\sigma\in S_N} F(\sigma,\phi) = \sum_{\sigma\in S_N} (-1)^{\sigma} \oint_{\Gamma} \cdots \oint_{\Gamma} \prod_{\substack{j<k \\ \sigma(j)>\sigma(k)}} \left(\frac{1 + \tau_{\sigma(k)}\tau_{\sigma(j)} - 2\Delta\tau_{\sigma(j)}}{1 + \tau_{\sigma(k)}\tau_{\sigma(j)} - 2\Delta\tau_{\sigma(k)}} \right) \prod_{j=1}^{N} \tau_{\sigma(j)}^{y_j - y_{\sigma(j)} - 1} d^N \tau$$

$$(\text{TW}'07) = 1$$

$\Delta = 0$

$$\mathbb{P}\left(\frac{X_1(t) + 2t}{t^{1/3}} \geq -s \right) =$$

$$\sum_{\sigma \in S_N} (-1)^{\sigma} \sum_{S \subset \{1,\dots,N\}} (-1)^{|S|} t^{-|S|/3} F(\sigma, S) \prod_{j \in S} K_{Ai}(s + v_j, s + v_{\sigma(j)})$$

$$= \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{k=1}^{N} \left(\mathbb{1}(\sigma(k) = k) - t^{-1/3} K_{Ai}(s + v_k, s + v_{\sigma(k)}) \right)$$

$$= \det\left(Id - t^{-1/3} K_{Ai} \right)_{\ell^2\{s, s+t^{1/3}, \dots, s+Nt^{1/3}\}}$$

$$\to \det\left(Id - K_{Ai} \right)_{L^2(s,\infty)} = F_{TW}^{GUE}(s)$$

as $t \ll N \to \infty$

The Proof

Thm (S-Tracy-Widom '22) Take $IC = \{y_i\}_{i=1}^{N}$. Then,

$$\Psi(x,t) = \frac{1}{(2\pi i)^N} \oint_{r/R} \frac{dz_1}{z_1} \cdots \oint_{r/R} \frac{dz_N}{z_N} \left(\sum_{\sigma \in S_N} A_\sigma \prod_{j=1} z_{\sigma(j)}^{x_j - y_{\sigma(j)}} \right) e^{-itE(z)}$$

w/ $A_\sigma = \prod_{\substack{i<j \\ \sigma(i)>\sigma(j)}} - \frac{1 + z_i z_j - 2\Delta z_i}{1 + z_i z_j - 2\Delta z_j}$; $E(z) = \sum_{j=1}^{N} (z_j^{-1} + z_j - 2\Delta)$

Rem: Result follows from ASEP result (TW '07)

$$\mathcal{H}^{XXZ} = T\, \mathcal{H}^{ASEP}\, T^{-1}$$

Marginal Distribution

$R r < 1$

$$\mathrm{Prob}^N(X_1(t) \geq s) = \sum_{s \leq x_1 < \cdots < x_N} |\Psi(x_1, \ldots, x_N; t)|^2$$

$$= \frac{1}{(2\pi i)^{2N}} \oint_R \frac{d\zeta^N}{\zeta^N} \oint_r \frac{d\xi^N}{\xi^N} \left(\sum_{\mu,\sigma \in S_N} \frac{A_\sigma A_\mu \, \zeta_{\sigma(2)} \xi_{\sigma(2)} \zeta^2_{\sigma(3)} \xi^2_{\sigma(3)} \cdots \zeta^{N-1}_{\sigma(N)} \xi^{N-1}_{\sigma(N)}}{(1-\zeta_{\sigma(2)}\xi_{\sigma(2)} \cdots \zeta_{\sigma(N)}\xi_{\sigma(N)}) \cdots (1-\zeta_{\sigma(N)}\xi_{\sigma(N)})} \right)$$

Cantini, Colomo, Pronko '20
Wheeler, Zinn-Justin '16
Petrov '21

$$\times \prod_{j=1}^N (\zeta_j \xi_j)^{s-y_j} e^{-it[E(\zeta_j) - E(\xi_j)]}$$

$$= \frac{1}{(2\pi i)^{2N}} \oint_R \frac{d\zeta^N}{\zeta^N} \oint_r \frac{d\xi^N}{\xi^N} \frac{\prod_{i,j=1}^N (\zeta_i + \xi_j - 2\Delta \zeta_i \xi_j)}{\prod_{i<j} (1+\zeta_i\zeta_j - 2\Delta\zeta_i)(1+\xi_i\xi_j - 2\Delta\xi_i)}$$

$$\times \det\left(\frac{1}{(1-\zeta_i\xi_j)(\zeta_i + \xi_j - 2\Delta\zeta_i\xi_j)} \right)_{i,j=1}^N \prod_{j=1}^N (\zeta_j \xi_j)^{s-y_j} e^{-it[E(\zeta_j) - E(\xi_j)]}$$

↑ Izergin-Korepin determinant

Steepest Descent Contour

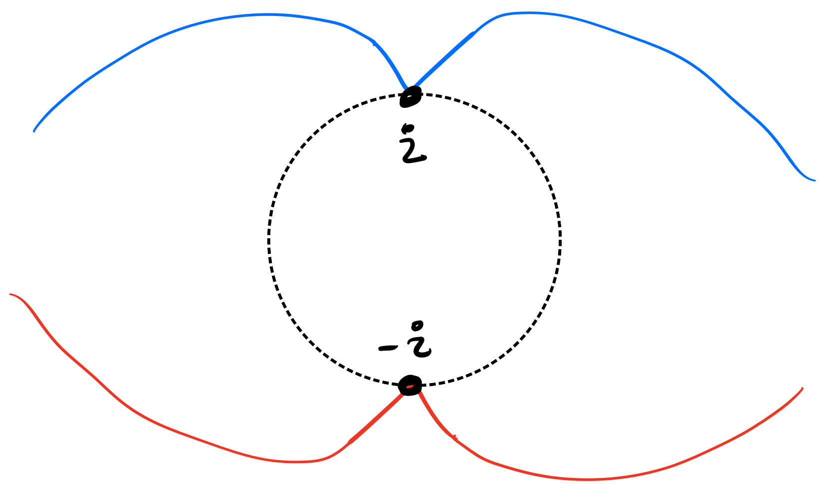
- $(\tau\varsigma)^s e^{-it(\tau+\tau^{-1}-\varsigma-\varsigma^{-1})} = \exp\left(G(\tau;t,s) - H(\varsigma;t,s)\right)$

w/ $G = s\log(\tau) - it(\tau+\tau^{-1})$

$H = s\log(\varsigma) + it(\varsigma+\varsigma^{-1})$

$\to \quad G'(i;t,-2t) = G''(i;t,-2t) = 0$

$H'(-i;t,-2t) = H''(-i;t,-2t) = 0$



Contour Deformation I ($\zeta \in C_r \to C_{R'}$)

$$\mathrm{Prob}^N(X_1(t) \le s)$$

$$= \frac{1}{(2\pi i)^{2N}} \oint_R \frac{d\eta^N}{\eta^N} \oint_r \frac{d\zeta^N}{\zeta^N} \frac{\prod_{i,j=1}^{N} (\eta_i + \zeta_j - 2\Delta \eta_i \zeta_j)}{\prod_{i<j} (1+\eta_i\eta_j - 2\Delta\eta_i)(1+\zeta_i\zeta_j - 2\Delta\zeta_i)}$$

$$\times \det\left(\frac{1}{(1-\eta_i\zeta_j)(\eta_i+\zeta_j-2\Delta\eta_i\zeta_j)}\right)_{i,j=1}^{N} \prod_{j=1}^{N} (\eta_j \zeta_j)^{s-y_j} e^{-it[E(\eta_j)-E(\zeta_j)]}$$

- Poles:

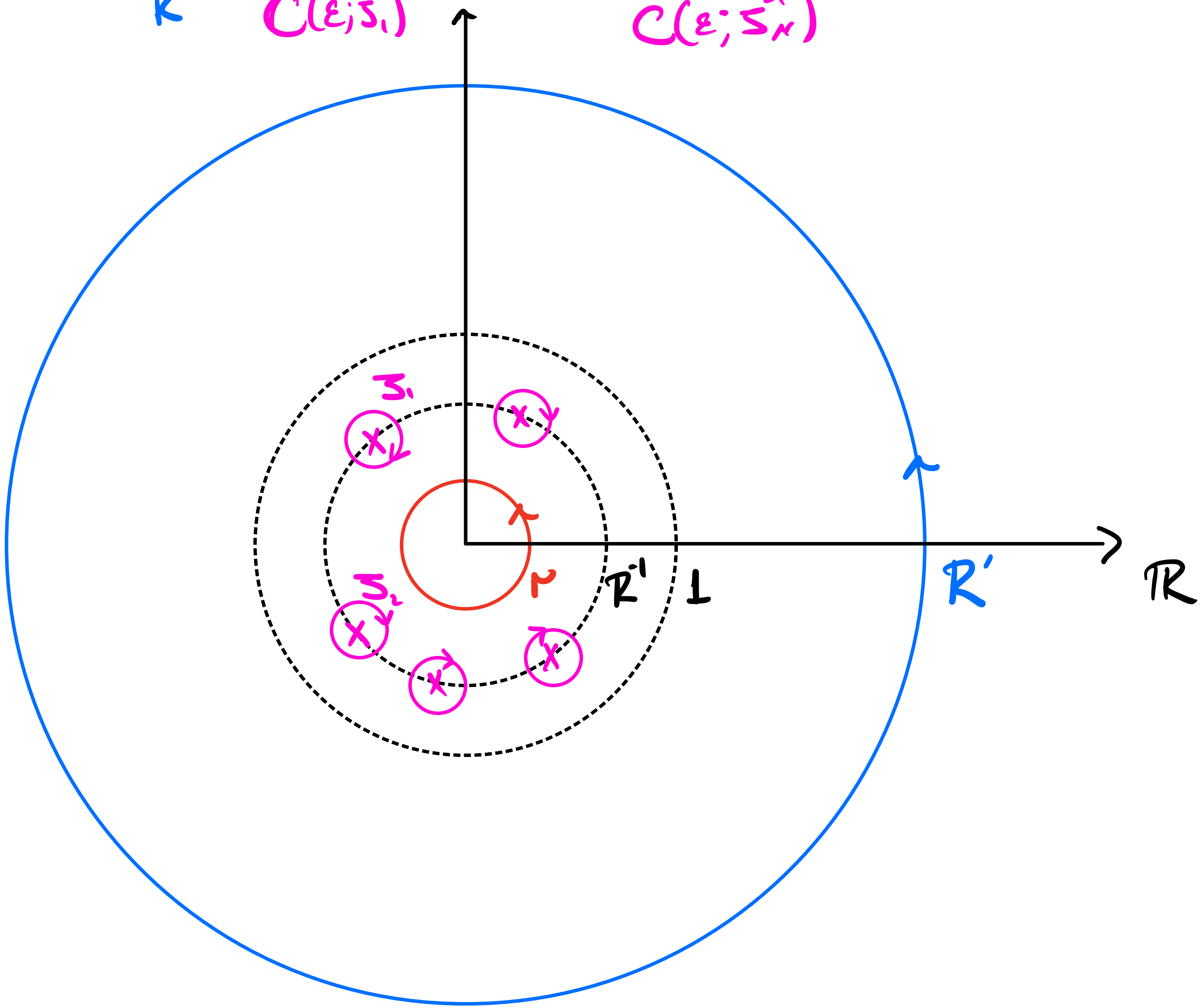
(a) $\zeta_i = \eta_j^{-1}$

(b) $\zeta_j = 2\Delta - \zeta_i^{-1}$ → Residue vanishes

(c) $\zeta_i = (2\Delta - \zeta_j)^{-1}$ → Residue vanishes

Contour Deformation I

- $\oint_{r} d\zeta = \left(\oint_{R'} - \oint_{C(\varepsilon;\zeta_1^{-1})} - \cdots - \oint_{C(\varepsilon;\zeta_N^{-1})} \right) d\zeta$



Contour Deformation I

$$\to \oint_R d\tau^N \oint_r d\varsigma^N I_N(\tau,\varsigma)$$

$$= \oint_R d\tau^N \Big(\oint_R - \oint_{C(\varepsilon;\varsigma_1^{-1})} - \cdots - \oint_{C(\varepsilon;\varsigma_N^{-1})} \Big)^N d\varsigma^N I_N(\tau,\varsigma)$$

$$= \sum_{\tau} \oint_R d\varsigma^N \oint_{C(\tau(1))} d\tau_1 \cdots \oint_{C(\tau(N))} d\tau_N \, I_N(\tau,\varsigma)$$

w/ $\tau : \{1,\dots,N\} \to \{0,1,\dots,N\}$

& $C(k) = \begin{cases} C(R',0) & \text{if } k=0 \\ -C(\varepsilon,\varsigma_k^{-1}) & \text{if } k\neq 0 \end{cases}$

$k \to \tau(k) \neq 0$

$\to \quad \varsigma_k = \tau^{-1}_{\tau(k)}$

$k \to \tau(k) = 0$

$\to C_{R'}$

Residue Computation ($\varsigma_k = \eta^{-1}_{\tau(k)}$)

$$\operatorname*{Res}_{\varsigma_k = \eta^{-1}_{\tau(k)}} I_N(\eta, \varsigma) =$$

$$\left(\frac{\prod_{i,j} (\eta_i + \varsigma_j - 2\Delta \eta_i \varsigma_j) \prod_{j=1}^{N} \left[(\eta_j \varsigma_j)^{s - \gamma_j} e^{-it(E(\eta_j) - E(\varsigma_j))} \right]}{\prod_{i<j} (1 + \eta_i \eta_j - 2\Delta \eta_i)(1 + \varsigma_i \varsigma_j - 2\Delta \varsigma_i)} \right) \Bigg|_{\varsigma_k = \eta^{-1}_{\tau(k)}}$$

$$\times \operatorname*{Res}_{\varsigma_k = \eta^{-1}_{\tau(k)}} \det \left(\frac{1}{(1 - \eta_i \varsigma_j)(\eta_i + \varsigma_j - 2\Delta)} \right)_{i,j=1}^{N}$$

Residue Computation

$$\operatorname*{Res}_{\zeta_k = \eta_{\tau(k)}^{-1}} \det\left(\frac{1}{(1-\eta_i\zeta_j)(\eta_i+\zeta_j-2\Delta)}\right)_{i,j=1}^{N}$$

$$= \operatorname*{Res}_{\zeta_k = \eta_{\tau(k)}^{-1}} \begin{vmatrix} D(\eta_1,\zeta_1) & \cdots & \overset{k}{\cdots} & \cdots & D(\eta_1,\zeta_N) \\ \vdots & & & & \\ & & D(\eta_{\tau(k)},\zeta_k) & & \\ \vdots & & & & \\ D(\eta_N,\zeta_1) & \cdots & \cdots & \cdots & D(\eta_N,\zeta_N) \end{vmatrix} \quad \tau(k)$$

$$= (-1)^{\tau(k)-k} \left(1+\eta_{\tau(k)}^2 - 2\Delta\eta_{\tau(k)}\right)^{-1} \det\left(D(\eta_i,\zeta_j)\right)_{\substack{i\neq\tau(k)\\ j\neq k}}$$

Residue Computation

$$\left(\frac{\prod_{i,j} (\eta_i + \xi_j - 2\Delta \eta_i \xi_j) \prod_{j=1}^{N} \left[(\eta_j \xi_j)^{s-\gamma_j - 1} e^{-it(E(\eta_j)-E(\xi_j))} \right]}{\prod_{i<j} (1+\eta_i\eta_j - 2\Delta\eta_i)(1+\xi_i\xi_j - 2\Delta\xi_i)} \right) \Bigg|_{\xi_k = \eta^{-1}_{\tau(k)}}$$

$$= \frac{\prod_{\substack{i\neq\tau(k)\\ j\neq k}} (\eta_i + \xi_j - 2\Delta \eta_i \xi_j) \prod_{j\neq\tau(k)}^{N} \eta_j^{s-\gamma_j} e^{-itE(\eta_j)} \prod_{j\neq k}^{N} \xi_j^{s-\gamma_j} e^{itE(\xi_j)}}{\prod_{\substack{i<j\\ i,j\neq\tau(k)}} (1+\eta_i\eta_j-2\Delta\eta_i) \prod_{\substack{i<j\\ i,j\neq k}} (1+\xi_i\xi_j-2\Delta\xi_i)}$$

$$\times \prod_{\tau(k)<l} \left(\frac{1+\eta_{\tau(k)}\eta_l - 2\Delta\eta_l}{1+\eta_{\tau(k)}\eta_l-2\Delta\eta_{\tau(k)}}\right) \prod_{k<m}\left(\frac{\eta_{\tau(k)}+\xi_m-2\Delta\eta_{\tau(k)}\xi_m}{\eta_{\tau(k)}+\xi_m-2\Delta}\right) \eta_{\tau(k)}^{\gamma_{\tau(k)}-\gamma_k-1}$$

Residue Computation

Lemma: 2N 2N−|K_2|

$$\oint_{\mathcal{C}_R}\cdots\oint_{\mathcal{C}_R}\oint_{\mathcal{C}^{\tau(1)}}\cdots\oint_{\mathcal{C}^{\tau(N)}} I_N(\xi,\zeta)d^N\zeta d^N\xi = \oint_{\mathcal{C}_R}\cdots\oint_{\mathcal{C}_R}\oint_{\mathcal{C}_{R'}}\cdots\oint_{\mathcal{C}_{R'}} I_N(\xi,\zeta;\tau)f(\xi,\zeta;\tau)\left(\prod_{k\in K_1} d\zeta_j\right)d^N\xi$$

with the integrands given as follows

$$I_N(\xi,\zeta;\tau) = \frac{\prod_{j\in J_1,k\in K_1}(\xi_j+\zeta_k-2\Delta\xi_j\zeta_k)D_N(\xi,\zeta;\tau)}{\prod_{\substack{j<k\\ j,k\in J_1}}(1+\xi_j\xi_k-2\Delta\xi_j)\prod_{\substack{j<k\\ j,k\in K_1}}(1+\zeta_j\zeta_k-2\Delta\zeta_j)}\prod_{j\in J_1}\xi_j^{x-y_j-1}e^{-\mathrm{i}t\epsilon(\xi_j)}\prod_{k\in K_1}\zeta_k^{x-y_k-1}e^{\mathrm{i}t\epsilon(\zeta_k)}$$

$$f(\xi,\zeta;\tau) = \prod_{\ell=1}^{M}\left(\prod_{\substack{\tau_\ell<k\\ k\neq\tau_{\ell+1},\dots,\tau_M}}\left(\frac{1+\xi_{\tau_\ell}\xi_k-2\Delta\xi_k}{1+\xi_{\tau_\ell}\xi_k-2\Delta\xi_{\tau_\ell}}\right)\prod_{\substack{k_\ell<k\\ k\neq k_{\ell+1},\dots,k_M}}\left(\frac{\xi_{\tau_\ell}+\zeta_k-2\Delta\xi_{\tau_\ell}\zeta_k}{\xi_{\tau_\ell}+\zeta_k-2\Delta}\right)\right)\prod_{\ell=1}^{M}\xi_{\tau_\ell}^{y_{k_\ell}-y_{\tau_\ell}-1},$$

$$D_N(\xi,\zeta;\tau) = (-1)^{\sum_{\ell=1}^M\tau_\ell-k_\ell}\det\left(d(\xi_j,\zeta_k)\right)_{j\in J_1,k\in K_1} = (-1)^{\sum_{\ell=1}^M\tau_\ell-k_\ell}\sum_{\gamma:K_1\to J_1}\prod_{k\in K_1}(-1)^{\gamma(k)-k}d(\xi_{\gamma(k)},\zeta_k)$$

so that

$$K_1 := \tau^{-1}(0),\quad K_2 := K_1^c = \{k_1<\cdots<k_M\}.$$ ← ζ-vars

$$J_2 = \tau(K_2) = \{\tau_1=\tau(k_1),\dots,\tau_M=\tau(k_M)\},\quad J_1 = J_2^c.$$ ← ξ-vars

Residue Computation (Fix τ)

- $I_N(\xi, \zeta; \tau) = I_N(\xi^{J_1}, \zeta^{K_1})$
- $f(\xi, \zeta; \tau) = f(\xi^{J_1}, \xi^{J_2}, \zeta^{K_1})$

$$\to \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \oint_{\mathcal{C}_{R'}} \cdots \oint_{\mathcal{C}_{R'}} I_N(\xi, \zeta; \tau) f(\xi, \zeta; \tau) \left(\prod_{k \in K_1} d\zeta_j \right) d^N \xi$$

$$= \underbrace{\oint_{C_R} \cdots \oint_{C_R} \oint_{C_{R'}} \cdots \oint_{C_{R'}}}_{2|K_1|} I_N(\xi^{J_1}, \zeta^{K_1}) \tilde{F}(\xi^{J_1}, \zeta^{K_1}) \, d^{K_1}\zeta \, d^{J_1}\xi$$

with

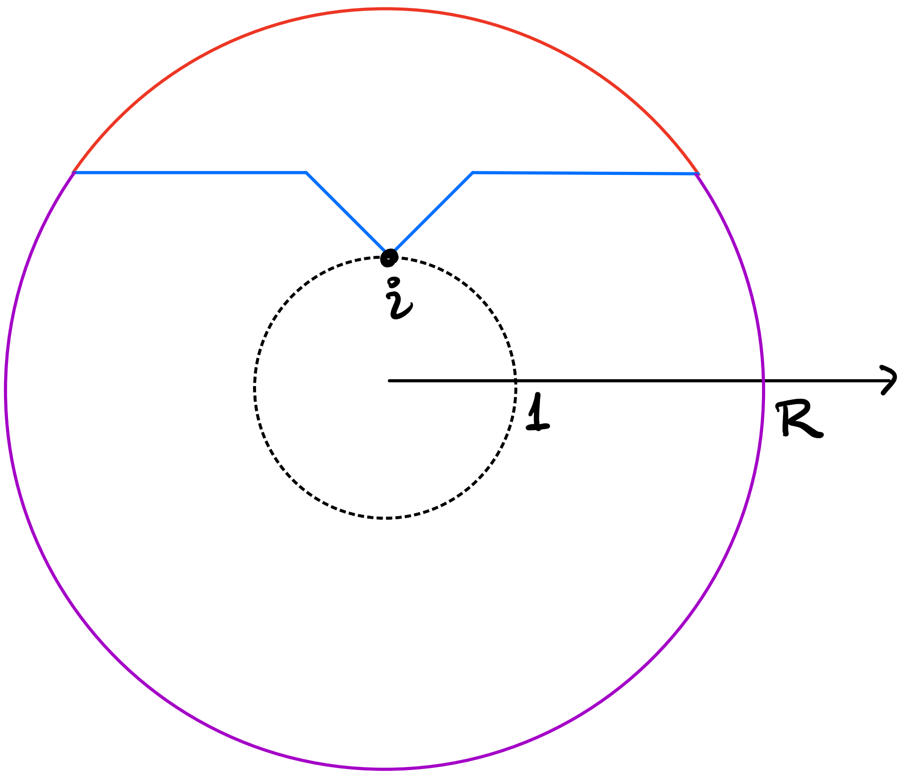
(independent of t)

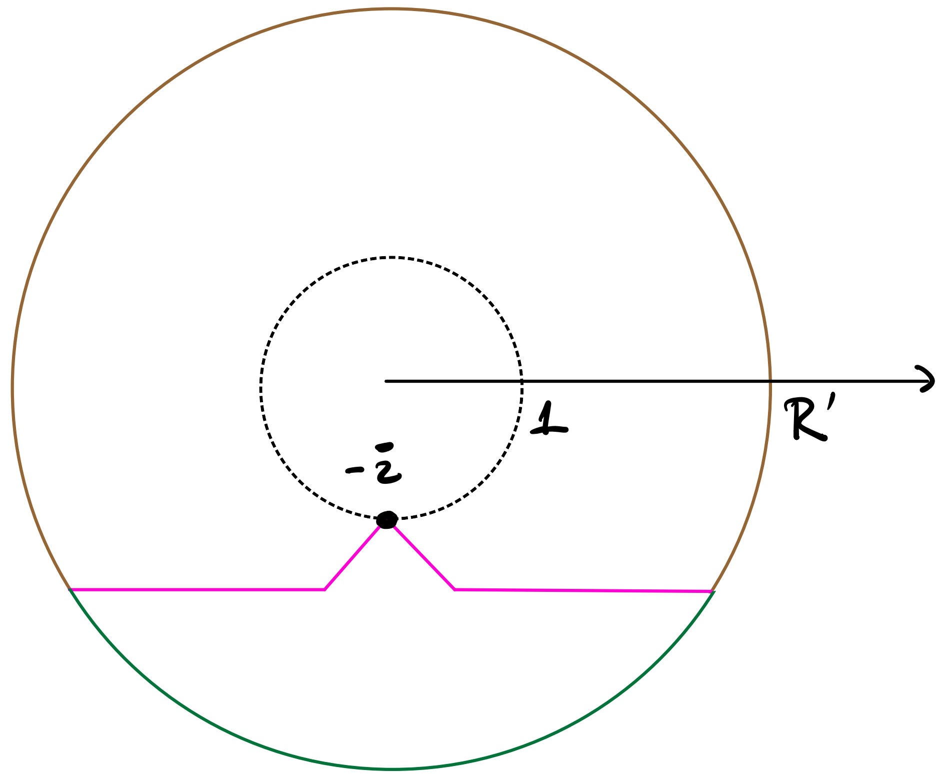
$$\tilde{F}(\xi^{J_1}, \zeta^{K_1}) := \oint_{C_{R'}} \cdots \oint_{C_{R'}} f(\xi^{J_1}, \xi^{J_2}, \zeta^{K_1}) \, d^{J_2}\xi$$

Contour Deformation II

$$\oint_{C_R}\cdots\oint_{C_R}\oint_{C_{R'}}\cdots\oint_{C_{R'}} \mathcal{I}_N(\tau^{J_1},\varsigma^{K_1})\,\tilde{F}(\tau^{J_1},\varsigma^{K_1})\,d^{K_1}\varsigma\, d^{J_1}\tau$$

$$= \oint_{\Gamma_+}\cdots\oint_{\Gamma_+}\oint_{\Gamma_-}\cdots\oint_{\Gamma_-} \mathcal{I}_N(\tau^{J_1},\varsigma^{K_1})\,\tilde{F}(\tau^{J_1},\varsigma^{K_1})\,d^{K_1}\varsigma\, d^{J_1}\tau$$

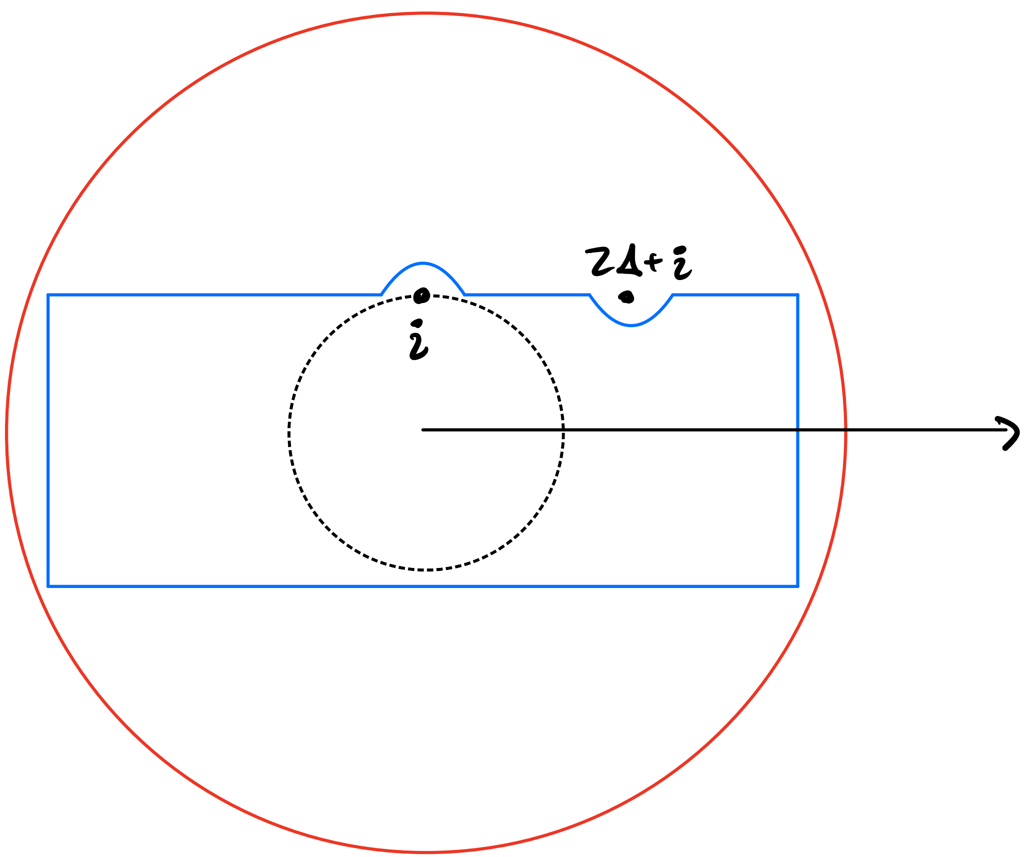




Contour Deformation II

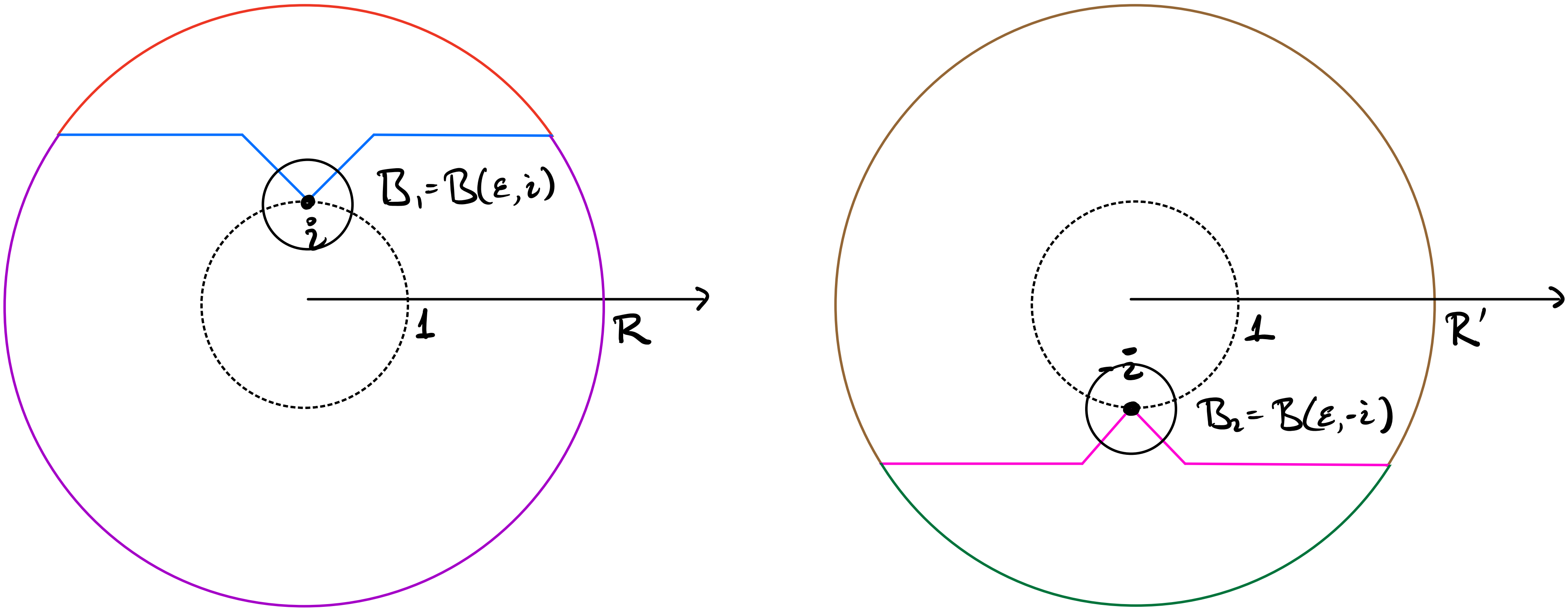
$$\widetilde{F}(\tau^{J_1}, \varsigma^{K_1}) := \oint_{C_R'} \cdots \oint_{C_R'} f(\tau^{J_1}, \tau^{J_2}, \varsigma^{K_1})\, d^{J_2}\tau$$

$$= \oint_{\hat{\Gamma}} \cdots \oint_{\hat{\Gamma}} f(\tau^{J_1}, \tau^{J_2}, \varsigma^{K_1})\, d^{J_2}\tau$$



Steepest Descent (Conjectural)

Take $t \to \infty$ & $\varepsilon = \varepsilon(t) = t^{-1/4}$



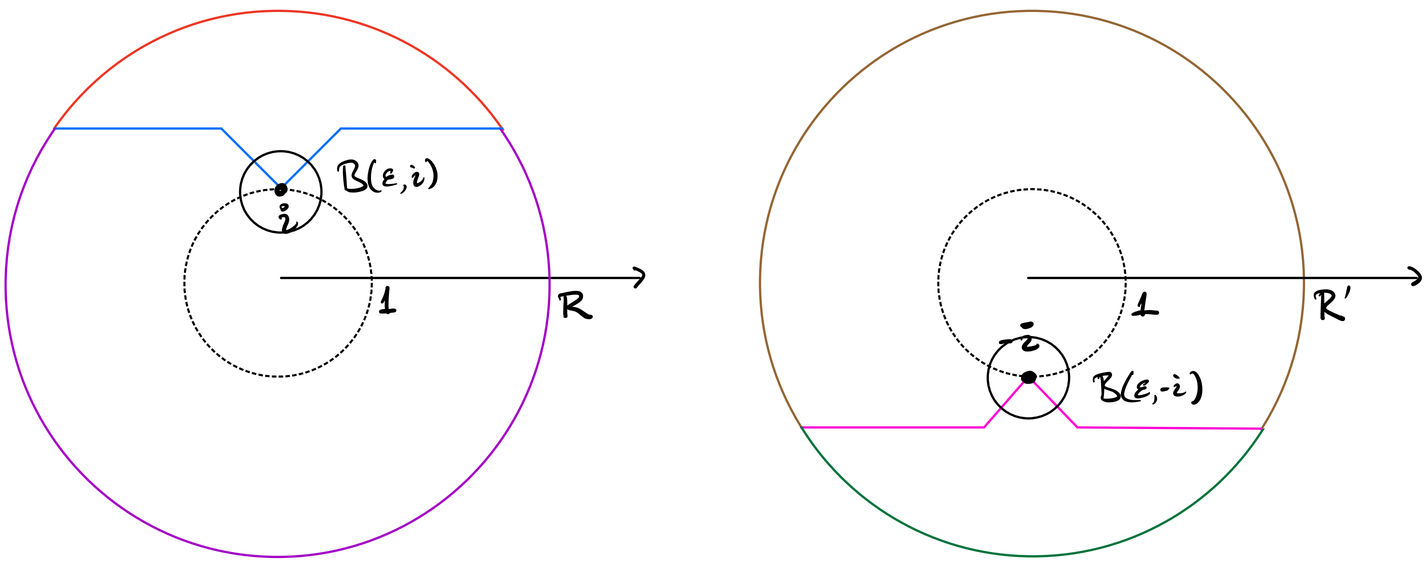
$$\oint_{\Gamma_+} = \int_{\Gamma_+ \cap B_1} + \int_{\Gamma_+ \cap B_1^c} \nearrow 0 \quad \& \quad \oint_{\Gamma_-} = \int_{\Gamma_- \cap B_2} + \int_{\Gamma_- \cap B_2^c} \nearrow 0$$

Steepest Descent

$$\oint_{\Gamma_+} \cdots \oint_{\Gamma_+} \oint_{\Gamma_-} \cdots \oint_{\Gamma_-} \mathcal{I}_N(\tau^{J_1}, \zeta^{K_1}) \tilde{F}(\tau^{J_1}, \zeta^{K_1}) d^{K_1}\zeta \, d^{J_1}\tau$$

$$\approx \int_{\Gamma_+^\varepsilon} \cdots \int_{\Gamma_+^\varepsilon} \int_{\Gamma_-^\varepsilon} \cdots \int_{\Gamma_-^\varepsilon} \mathcal{I}_N(\tau^{J_1}, \zeta^{K_1}) \tilde{F}(\tau^{J_1}, \zeta^{K_1}) d^{K_1}\zeta \, d^{J_1}\tau$$

with $\Gamma_\pm^\varepsilon = \Gamma_\pm \cap B(\varepsilon, \pm i)$



Steepest Descent

Take $\begin{cases} \eta = i + i\eta t^{-1/3} \in \Gamma_+^\varepsilon \\ \zeta = -i + i\zeta t^{-1/3} \in \Gamma_-^\varepsilon \end{cases}$

- $\tilde{F}(\eta^{J_1}, \zeta^{K_1}) \to \tilde{F}(i^{J_1}, -i^{K_1})$

$$= \oint_{C_R'} \cdots \oint_{C_R'} f(i^{J_1}, \eta^{J_2}, -i^{K_1})\, d^{J_2}\eta =: F(\tau)$$

as $t \to \infty$

Steepest Descent

Take $\begin{cases} \eta = i + i\eta t^{-1/3} \in \Gamma_+^\varepsilon \\ \zeta = -i + i\zeta t^{-1/3} \in \Gamma_-^\varepsilon \end{cases}$

- $I_N(\eta^{J_1}, \zeta^{K_1}) \to$

$$(-1)^{\#(\tau)} \sum_{\gamma: K_1 \to J_1} \prod_{k \in K_1} (-1)^{\gamma(k)-k} (i)^{y_{\gamma(k)} - y_k} g(\eta_{\gamma(k)}, \zeta_k; v_{\gamma(k)}, v_k)\, t^{|K_1|/3}$$
$$+ \mathcal{O}\left(t^{(|K_1|-1)/3}\right)$$

w/ $\quad v_j t^{1/3} = y_j + 1$

& $\quad g(\eta, \zeta; x, z) = \dfrac{\exp\left(\frac{1}{3}\eta^3 - \frac{1}{3}\zeta^3 - (s+x)\eta + (s+z)\zeta\right)}{\eta - \zeta}$

Steepest Descent

- $$\mathbb{K}_{Ai}(x,z) = \int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \frac{\exp\left(\frac{1}{3}\tau^3 - \frac{1}{3}\zeta^3 - x\tau + z\zeta\right)}{\tau - \zeta} \, d\tau \, d\zeta$$

Then,

$$\oint_{\Gamma_+} \cdots \oint_{\Gamma_-} I_N(\xi,\zeta;\tau) \left(\oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} f(\xi,\zeta;\tau) d^{J_2}\xi \right) d^{K_1}\zeta \, d^{J_1}\xi$$
$$= t^{-n/3}(-1)^{|J_1| + \sum_{k\in K_2} \tau(k)-k} \sum_{\gamma: K_1 \to J_1} F(\tau) \prod_{k\in K_1} (-1)^{\gamma(k)-k} \mathbf{K}_{Ai}\left(s + v_{\gamma(k)}, s + v_k\right)$$
$$+ \mathcal{O}(t^{(1-n)/3}) + \mathcal{O}(e^{-Ct^{1-3\alpha}}).$$

w/ $n = |K_1|$, $\alpha = 1/4$, & $C > 0$

Conjecture (S - Tracy - Widom '22)

$$\lim_{t \ll N \to \infty} \mathbb{P}\left(\frac{X_1(t) + 2t}{t^{1/3}} \geq -s \right) =$$

$$\lim_{t \ll N \to \infty} \sum_{\sigma \in S_N} (-1)^{\sigma} \sum_{S \subset \{1,\dots,N\}} (-1)^{|S|} t^{-|S|/3} F(\sigma, S) \prod_{j \in S} K_{Ai}(s + V_j, s + V_{\sigma(j)})$$

- $V_j t^{1/3} = \gamma_j + 1$
- $F(\sigma, S)$ depends $\{\gamma_j\}_{j=1}^{\infty}$, indep of t
- K_{Ai} is the Airy Kernel

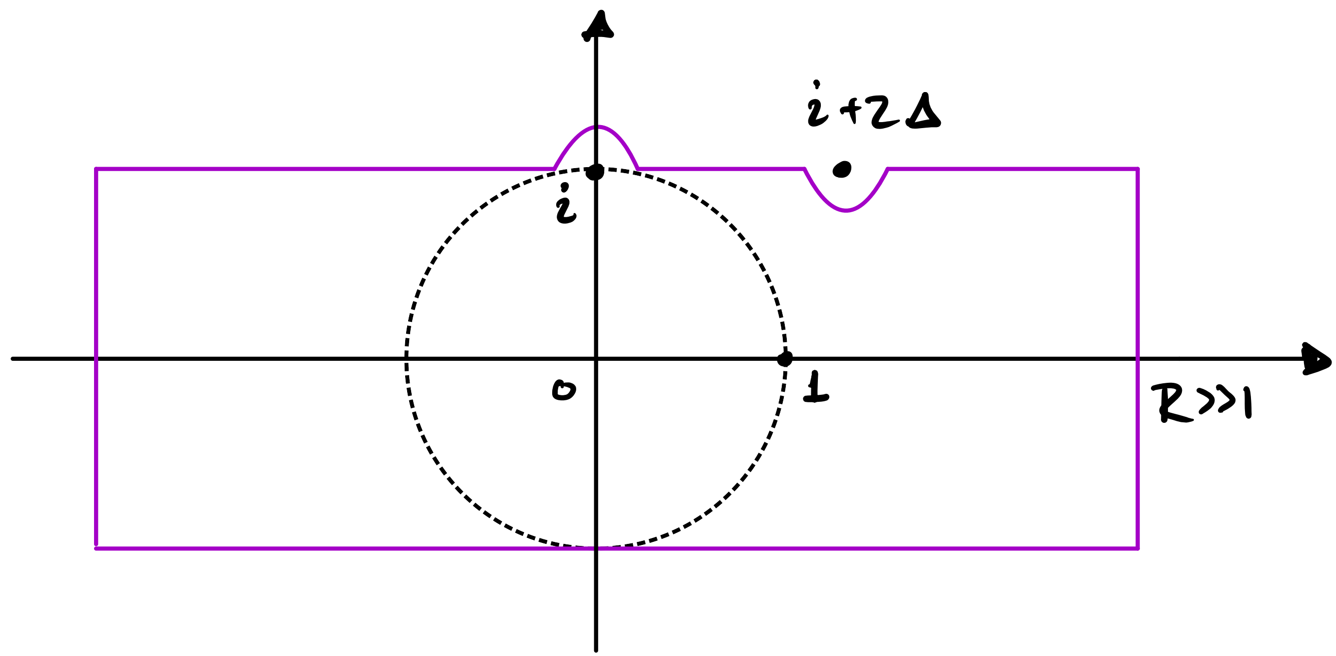
Coefficients

$$\nu(\sigma,S) = j-\sigma(j)+\#\{k\in S^c \mid j<k,\ \sigma(j)>\sigma(k)\} - \#\{k\in S^c \mid k<j,\ \sigma(k)>\sigma(j)\}$$

$$F(\sigma,S) = (i)^{|S^c|} \oint_\Gamma \cdots \oint_\Gamma B(\zeta;\sigma,S) \prod_{j\in S^c}\left(\frac{\zeta_{\sigma(j)}-(2\Delta+i)}{(2\Delta i+1)\zeta_{\sigma(j)}-i}\right)^{\nu(\sigma,S)} \prod_{j\in S^c}\left(i\zeta_{\sigma(j)}\right)^{y_j-y_{\sigma(j)}-1} d^{\sigma(S^c)}\zeta$$

- $$B(\zeta;\sigma,S) = \prod_{\substack{j<k \\ \sigma(j)>\sigma(k)}} \left(\frac{1+\zeta_{\sigma(k)}\zeta_{\sigma(j)}-2\Delta\zeta_{\sigma(j)}}{1+\zeta_{\sigma(k)}\zeta_{\sigma(j)}-2\Delta\zeta_{\sigma(k)}}\right) \; ; \quad j,k\in S^c$$

- $\Gamma =$



Final Remarks

Series Expansion

Take $t \to \infty$ & N = finite

$$\mathbb{P}\left(\frac{X_1(t) + 2t}{t^{1/3}} \geq -s \right) \approx$$

$$\sum_{\sigma \in S_N} (-1)^\sigma \sum_{S \subset \{1,\dots,N\}} (-1)^{|S|} t^{-|S|/3} F(\sigma, S) \prod_{j \in S} \mathbb{K}_{Ai}(s+V_j, s+V_{\sigma(j)})$$

$$= \sum_{\sigma \in S_N} (-1)^\sigma F(\sigma, \phi)$$

$$- \left(\sum_{\sigma \in S_N} (-1)^\sigma \sum_{n=1}^{N} F(\sigma, \{n\}) \mathbb{K}_{Ai}(s+V_n, s+V_{\sigma(n)}) \right) t^{-1/3} + O(t^{-2/3})$$

$$\simeq 1 - \left(\sum_{\sigma \in S_N} (-1)^\sigma \sum_{n=1}^{N} F(\sigma, \{n\}) \right) \mathbb{K}_{Ai}(s,s) t^{-1/3} + O(t^{-2/3})$$

N=2 $\quad F(\sigma, S)\,,\quad \sigma \in S_N\,,\quad S \subset \{1,\dots,N\}\,,\quad |S| = 1$

- $F((1,2),\{1\}) = 1 \quad , \quad F((1,2),\{2\}) = 1$

$$F((2,1),\{1\}) = \frac{-4\Delta^2}{(2\Delta - i)^2} \quad , \quad F((2,1),\{2\}) = \frac{-4\Delta^2}{(2\Delta + i)^2}$$

$$\rightarrow \sum_{\sigma \in S_N} (-1)^\sigma \sum_{n=1}^{N} F(\sigma, \{n\})$$

$$= 1 + 1 + \frac{4\Delta^2}{(2\Delta - i)^2} + \frac{4\Delta^2}{(2\Delta + i)^2}$$

$$= 2 - \frac{8\Delta^2 - 32\Delta^4}{(4\Delta^2 + 1)^2}$$

$N=3$ $\quad F(\sigma, S)$, $\sigma \in S_N$, $S \subset \{1, \ldots, N\}$, $|S| = 1$

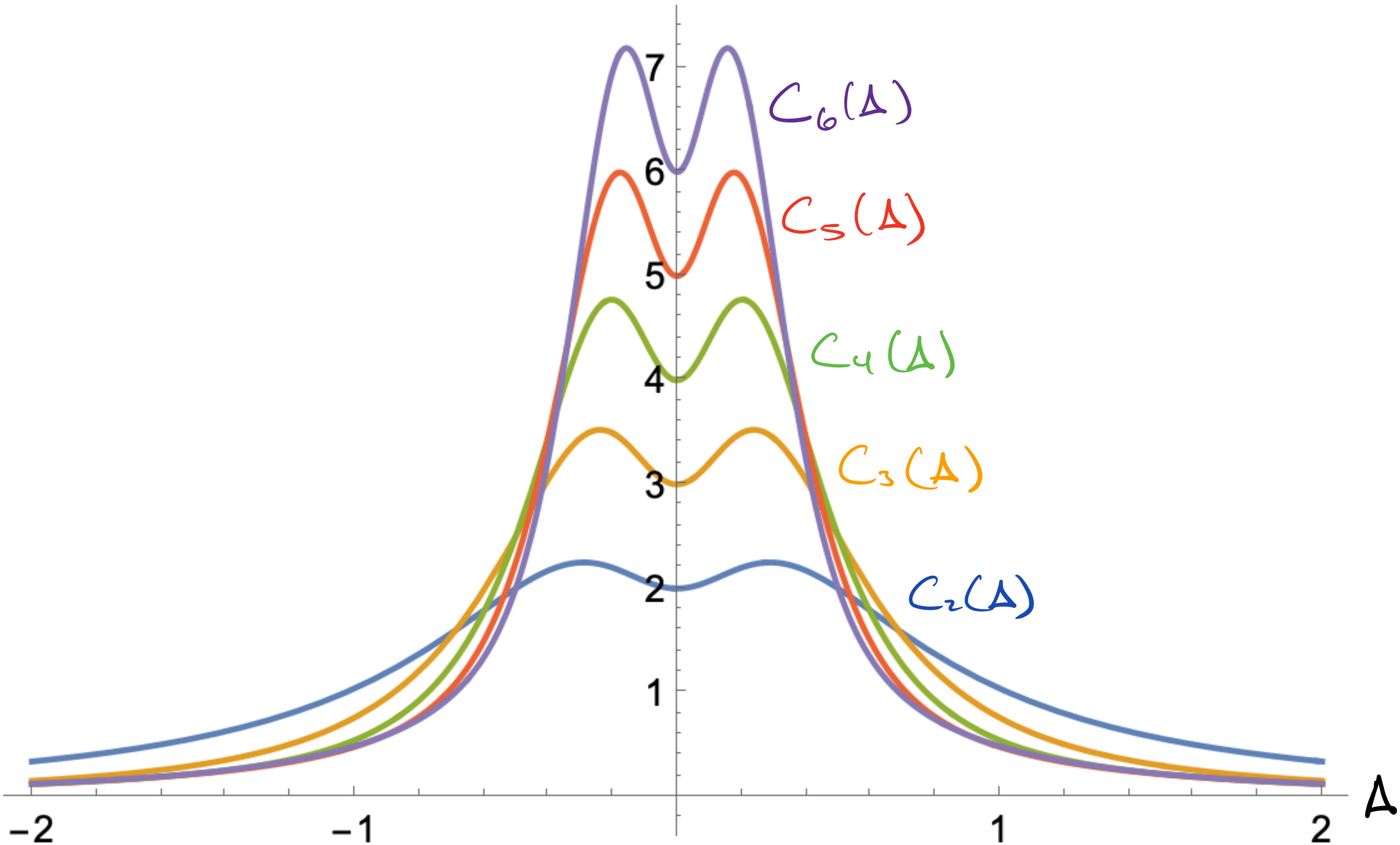
Table for $F(\sigma, S)$.

$S \downarrow, \sigma \rightarrow$	$(1,2,3)$	$(1,3,2)$	$(2,1,3)$	$(2,3,1)$	$(3,1,2)$	$(3,2,1)$
$\emptyset$	1	0	0	0	0	0
$\{1\}$	1	0	$-\frac{4\Delta^2}{(2\Delta-\mathrm{i})^2}$	0	$\frac{16\Delta^4}{(2\Delta-\mathrm{i})^4}$	$\frac{4\Delta^2(4\Delta^2-2\mathrm{i}\Delta-1)}{(2\Delta-\mathrm{i})^4}$
$\{2\}$	1	$-\frac{4\Delta^2}{(2\Delta-\mathrm{i})^2}$	$-\frac{4\Delta^2}{(2\Delta+\mathrm{i})^2}$	0	$\frac{16\Delta^4}{(1-2\mathrm{i}\Delta)^3}$	$\frac{32\Delta^5(-3\mathrm{i}+4\Delta+4\mathrm{i}\Delta^2)}{(4\Delta^2+1)^3}$
$\{3\}$	1	$-\frac{4\Delta^2}{(2\Delta+\mathrm{i})^2}$	0	$\frac{16\Delta^2}{(2\Delta+\mathrm{i})^4}$	$\frac{16\Delta^4}{(1-2\mathrm{i}\Delta)^3}$	$\frac{4\mathrm{i}\Delta^2(\mathrm{i}+2\Delta+8\Delta^3)}{(2\Delta+\mathrm{i})^4}$
$\{1,2\}$	1	$-\frac{4\Delta^2}{(2\Delta-\mathrm{i})^2}$	1	$\frac{8\Delta^2(2\Delta^2-1)}{(2\Delta-\mathrm{i})^4}$	$-\frac{4\Delta^2}{(2\Delta-\mathrm{i})^2}$	$\frac{8\Delta^2(2\Delta^2-1)}{(2\Delta-\mathrm{i})^4}$
$\{1,3\}$	1	$-\frac{4\Delta^2}{(2\Delta+\mathrm{i})^2}$	$-\frac{4\Delta^2}{(2\Delta-\mathrm{i})^2}$	$-\frac{4\Delta^2}{(2\Delta+\mathrm{i})^2}$	$-\frac{4\Delta^2}{(2\Delta-\mathrm{i})^2}$	1
$\{2,3\}$	1	1	$-\frac{4\Delta^2}{(2\Delta+\mathrm{i})^2}$	$-\frac{4\Delta^2}{(2\Delta+\mathrm{i})^2}$	$\frac{8\Delta^2(2\Delta^2-1)}{(2\Delta+\mathrm{i})^4}$	$\frac{8\Delta^2(2\Delta^2-1)}{(2\Delta+\mathrm{i})^4}$
$\{1,2,3\}$	1	1	1	1	1	1

$$\rightarrow \sum_{\sigma \in S_N} (-1)^\sigma \sum_{n=1}^{N} F(\sigma, \{n\}) = \frac{3 + 60\Delta^2 + 32\Delta^4}{(1+4\Delta^2)^3}$$

Graphs ($N = 2, 3, \ldots, 6$)

- $$\sum_{\sigma \in S_N} (-1)^{\sigma} \sum_{n=1}^{N} F(\sigma, \{n\}) =: C_N(\Delta)$$



¿El Fin?
Thank You!