Bounds on stalks of perverse sheaves

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Everything is over an algebraically closed field k (which can be loosened to an infinite perfect field, I think). We will consider sheaves of $\mathbb{F}$-modules, where $\mathbb{F}$ is a finite field of characteristic ℓ invertible in k.

1 Main theorem

To state the main theorem, we need the following definition.

Definition 1. Let X be a smooth variety of dimension n, and let C be a conical cycle in T^*X of dimension n. Let x be a closed point of X. For $0 \leqslant i \leqslant n$, we define the i**th polar multiplicity** of C at x (denoted $\gamma_C^i(x)$) as follows:

- If $0 \leqslant i \leqslant n-1$: Let V be a rank $i+1$ subbundle of T^*X defined over a neighborhood of x such that the fiber V_x is a generic point of the Grassmannian of $(i+1)$-dimensional subspaces of T_x^*X. Then $\gamma_C^i(x)$ is defined as the multiplicity of the pushforward cycle $\pi_*(\mathbb{P}(C) \cap \mathbb{P}(V))$ at x, where $\pi : \mathbb{P}(T^*X) \to X$ is the projection. (Note that this pushforward is an i-dimensional cycle.)
- If $i = n$: $\gamma_C^n(x)$ is the multiplicity of the zero section T_X^*X in C.

Lemma 1. The 0th polar multiplicity of C at x is the multiplicity of T_x^*X in C.

Proof. $\mathbb{P}(V)$ is a generic section of $\mathbb{P}(T^*X)$ defined near x. Thus, over x, it can only intersect the components of $\mathbb{P}(C)$ with maximal dimension, i.e. the copies of $\mathbb{P}(T_x^*X)$. We get the lemma from the fact that the intersection of $\mathbb{P}(T_x^*X)$ with a generic section is a point of multiplicity 1 above x. $\square$

Of course, it has to be shown that polar multiplicities are well-defined, which I won't do for time's sake. Here's the main theorem:

Theorem 1. Let K be a perverse Λ-sheaf on X smooth projective. Then for all i, $\dim_{\mathbb{F}} \mathcal{H}^{-i}K_x \leqslant \gamma_{CC(K)}^i(x)$.

Corollary 1. For a conical closed set C in T^*X, let C_x denote the fiber of C above x. If $i < n - \dim SS(K)_x$, then $\mathcal{H}^{-i}K_x \cong 0$.

2 Alternate definition of polar multiplicities

By definition, the multiplicity of a cycle at a point $x \in X$ is the local intersection number of that cycle with a sufficiently general smooth subvariety through x. In particular, polar multiplicities are local intersection numbers on X. Using the projection formula, we can alternatively define polar multiplicities as local intersection numbers on $P^*(T^*X)$.

Definition 2. Let Y be a smooth variety with a map $f : Y \to X$. Let C_1 and C_2 be cycles on Y with total dimension $\dim Y$ such that $C_1 \cap C_2 \cap f^{-1}(x)$ is proper. Suppose that all connected components of $C_1 \cap C_2$ are either contained in $f^{-1}(x)$ or disjoint from $f^{-1}(x)$. We define the **local intersection number** $(C_1, C_2)_{Y,x}$ as the degree of the part of the refined intersection $C_1 \cdot C_2$ supported on $f^{-1}(x)$.

Definition 3 (Alternate)**.** We present an alternate definition of the ith polar multiplicity of a conical cycle C at a point $x \in X$.

- If $0 \leqslant i \leqslant n-1$: Let Y be a sufficiently general (locally defined) smooth codimension i subvariety of X through x, and let V be a sufficiently general (locally defined) rank $i+1$ subbundle of the restriction $T^*X|_Y$. Define $\gamma^i_C(x)$ as the local intersection number $(\mathbb{P}(C), \mathbb{P}(V))_{\mathbb{P}(T^*X),x}$.
- If $i = n$: Same as before (multiplicity of the zero section).

Here, "sufficiently general" means that $T_xY \subset T_xX$ and $V_x \subset T^*_xX$ are generic subspaces. This helps ensure that $(\mathbb{P}(C), \mathbb{P}(V))_{\mathbb{P}(T^*X),x}$ makes sense (the intersection $\mathbb{P}(C) \cap \mathbb{P}(V)$ is 0-dimensional) and is independent of choice of Y and x.

Lemma 2. The definitions agree.

Proof. As mentioned above, the multiplicity of $\pi_*(\mathbb{P}(C) \cap \mathbb{P}(V))$ (for a subbundle $V \subset T^*X$ on X, not Y) is defined as $(\pi_*(\mathbb{P}(C) \cap \mathbb{P}(V)), Y)_{X,x}$ for sufficiently general Y. We have

$$\begin{aligned}(\pi_*(\mathbb{P}(C) \cap \mathbb{P}(V)), Y)_{X,x} &= (\mathbb{P}(C) \cap \mathbb{P}(V), \pi^*Y)_{\mathbb{P}(T^*X),x} \\ &= (\mathbb{P}(C), \mathbb{P}(V) \cap \pi^*Y)_{\mathbb{P}(T^*X),x}.\end{aligned}$$

Here, the first equality is the projection formula, and the second equality is associativity of refined intersections. Note that $\mathbb{P}(V) \cap \pi^*Y$ is simply $\mathbb{P}(V|_Y)$. It remains to check that the pair $(Y, V|_Y)$ is sufficiently general if V_x is sufficiently general and Y is sufficiently general depending on V_x. This check is done by counting dimensions. □

3 Proof of the main theorem

3.1 Proper C-transversality

The proof of the main theorem makes use of the functoriality of characteristic cycles under certain pullbacks. To state this functoriality result, we'll need to define proper C-transversality.

Definition 4. Let W be smooth of dimension m, and let $h : W \to X$ be a map. We say that h is **properly C-transversal** if it is C-transversal and each irreducible component of h^*C has dimension m.

Definition 5. Let A be a conical cycle of dimension n on T^*X, and let C be its support. Let $h : W \to X$ be properly C-transversal. We define the conical cycle $h^!A := (-1)^{n-m} dh_*(h^{-1}(A))$ on T^*W, where m is the dimension of W and the maps are $dh : W \times_X T^*X \to T^*W$ and $h : W \times_X T^*X \to T^*X$.

Theorem 2. Let $h : W \to X$ be properly C-transversal, and let L be a complex micro-supported on C. Then $CC(h^*L) = h^!CC(L)$.

There's also a functoriality result for proper pushforwards, but I'm too lazy to state it precisely, so we'll see the result when we need it.

3.2 The proof modulo some lemmas

Theorem 3. Let K be a perverse Λ-sheaf on X smooth projective. Then for all i, $\dim_{\mathbb{F}} \mathcal{H}^{-i}K_x \leqslant \gamma^i_{CC(K)}(x)$.

Proof. The proof is done by induction on i. Fix a projective embedding of X, and take a generic pencil of conics $\overline{X} \subset X \times \mathbb{P}^1$. Let $p : \overline{X} \to X$ and $q : \overline{X} \to \mathbb{P}^1$ be the projections. Since the pencil is generic, we can choose it so that x does not lie on the base locus, so that p is an isomorphism above a neighborhood of x and q makes sense as a map from a neighborhood of x to $\mathbb{P}^1$. Thus, we will sometimes use $\overline{X}$ and X interchangeably.

To induct, we use the nearby/vanishing cycles distinguished triangle for q:

$$p^*K \to R\Psi_q p^*K \to R\Phi_q p^*K.$$

This triangle gets us an exact sequence of cohomology sheaves:

$$(R^{-i-1}\Phi_q p^*K)_x \to \mathcal{H}^{-i}(K)_x \to (R^{-i}\Psi_q p^*K)_x \to (R^{-i}\Phi_q p^*K)_x.$$

Since the pencil is generic, we may assume that p is an isomorphism above a neighborhood of x. Thus, p^*K is perverse near $p^{-1}(x)$ (which we'll just denote x for convenience). By the perverse t-exactness of nearby cycles, $R\Psi_q p^*K[-1]$ is perverse near x as well.

Our base case is $i = 0$. We need the following lemmas.

Lemma 3. Let $L \in D^b_c(X, \mathbb{F})$ be a complex. Let $CC'(L)$ (resp. $SS'(L)$) be $CC(L)$ (resp. $SS(L)$) with any occurrence of $T^*_x X$ removed. As above, let $\overline{X} \subset X \times \mathbb{P}^1$ be a generic pencil of conics with projections $p : \overline{X} \to X$ and $q : \overline{X} \to \mathbb{P}^1$. Let $y = q(x)$, and let $i : q^{-1} \hookrightarrow \overline{X}$ be the inclusion. Then we have the following:

(1) $CC(R\Psi_q p^*L) = i^!p^!CC'(L)$.

(2) In a neighborhood of x, $R\Psi_q p^*L$ is supported at x.

(3) The multiplicity of $T^*_x X$ in $CC(K)$ is $-\operatorname{dimtot}(R\Phi_q p^*L)_x$.

(4) If L is perverse, then $(R\Phi_q p^*L)_x$ is concentrated in degree -1.

Lemma 4. Let C be a conical cycle, and let C' be C with all copies of $T^*_x X$ removed. Let $\widetilde{X}$ be a generic conic section of X through x, let $i : \widetilde{X} \to X$ be the inclusion, and let $\widetilde{C} := -i^!C'$.

Then for $i > 0$, the ith polar multiplicity of C at x equals the $(i-1)$st polar multiplicity of $\widetilde{C}$ at x.

Since $R\Psi_q p^*K[-1]$ is perverse, $R^0\Psi_q p^*K \cong 0$. Thus,

$$\dim \mathcal{H}^0(K)_x \leqslant \dim(R^{-1}\Psi_q p^*K)_x \leqslant \operatorname{dimtot}(R^{-1}\Psi_q p^*K)_x = -\operatorname{dimtot}(R\Phi_q p^*K)_x,$$

where the last equality follows from Lemma 3. By the Milnor formula, $-\operatorname{dimtot}(R\Phi_q p^*K)_x = (CC(K), dq^*(\omega))_{T^*X,x}$ (the RHS should actually be an intersection number in $T^*\overline{X}$, but since $\overline{X}$

and X are the same in a neighborhood of x, we are okay). It can be shown that q is $SS'(K)$ in a neighborhood of x, so that the only component of $SS(K)$ that intersects $dq^*(\omega)$ is $T^*_x X$. Thus, $(CC(K), dq^*(\omega))_{T^*X,x}$ is the multiplicity of $T^*_x X$ in $CC(K)$, as $T^*_x X$ is the only conical cycle with high enough fiber dimension at x. This multiplicity is the 0th polar multiplicity of $CC(K)$ at x, so we have proven the theorem for $i = 0$.

By Lemma 3(4), $(R^{-i-1}\Phi_q p^* K)_x \cong 0$ for all $i > 0$, so the map $\mathcal{H}^{-i}(K)_x \to (R^{-i}\Psi_q p^* K)_x$ is injective for all $i > 0$. Thus, $\dim \mathcal{H}^{-i}(K)_x \leqslant \dim(R^{-i}\Psi_q p^* K)_x$ for all $i > 0$. Because $R\Psi_q p^* K[-1]$ is perverse, we can apply the inductive hypothesis to deduce that $\dim \mathcal{H}^{-i}(K)_x$ is at most the $(i-1)$st polar multiplicity of $CC(R\Psi_q p^* K[-1])$ at x. By Lemma 3, $CC(R\Psi_q p^* K[-1]) = -i^! CC'(K)$, and by Lemma 4, the $(i-1)$st polar multiplicity of $-i^! CC'(K)$ at x is the same as the ith polar multiplicity of $CC(K)$. Thus, we conclude that $\dim \mathcal{H}^{-i}(K)_x$ is at most the ith polar multiplicity of $CC(K)$ at x. □