Lecture 12

In-class exercises' solutions:

1. Let $A = \begin{pmatrix} -\frac{11}{2} & \frac{3}{2} \\ -18 & 5 \end{pmatrix}$. You're given that $A \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $A \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

   Compute A^{100}, rounding to 3 decimal places.

   We change A to the basis $\mathcal{B}: v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$. So set $S = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}$. Then $S^{-1} = \begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix}$.

   and by last lecture, $A = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix}$. Now $A^{100} = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} (-1)^{100} & 0 \\ 0 & (\frac{1}{2})^{100} \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix}$

   $$\underset{\substack{\text{to 3 dec.} \\ \text{places}}}{=} \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix}$$

   $$= \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 12 & -3 \end{pmatrix}.$$

2. If $A \sim B$, then there exists S_0 invertible s.t. $A = S_0^{-1} B S_0$

   $B \sim C$ " " S_1 " s.t. $B = S_1^{-1} C S_1$

   $$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "S_0"] & \mathbb{R}^n \arrow[l, bend left, "S_0^{-1}"] \arrow[r, bend left, "S_1"] & \mathbb{R}^n \arrow[l, bend left, "S_1^{-1}"] \end{tikzcd}$$

   So $A = S_0^{-1} S_1^{-1} C S_1 S_0$. Now observe that $(S_1 S_0)^{-1} = S_0^{-1} S_1^{-1}$. So $A \sim C$.

Crucial Recap:

- Given a basis $\mathcal{B}$ of $\mathbb{R}^n$ with vectors $v_1, \dots, v_n$, the $\mathcal{B}$-coordinates of $v \in \mathbb{R}^n$ are $[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, where $v = c_1 v_1 + \dots + c_n v_n$.
- The canonical basis is $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, and we denote it $\mathcal{E}$.
- There exist change of basis matrices $S_{\mathcal{B} \to \mathcal{E}} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, $S_{\mathcal{E} \to \mathcal{B}} = S_{\mathcal{B} \to \mathcal{E}}^{-1}$.

  s.t. $[v]_{\mathcal{E}} = S_{\mathcal{B} \to \mathcal{E}} [v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}} = S_{\mathcal{E} \to \mathcal{B}} [v]_{\mathcal{E}}$.
- Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ with matrix A, and a basis $\mathcal{B}$ as above, the matrix of T with respect to $\mathcal{B}$ is $S_{\mathcal{B} \to \mathcal{E}}^{-1} A S_{\mathcal{B} \to \mathcal{E}}$. (Then $[Tv]_{\mathcal{B}} = S^{-1} A S [v]_{\mathcal{B}}$)
  $S = S_{\mathcal{B} \to \mathcal{E}}$
- If there exists an invertible matrix S s.t. $A = S^{-1} B S$, then $A \sim B$.
- If $A \sim D$, where D is diagonal, then A is diagonalizable.

Definition 1: To diagonalize an $n\times n$ matrix is to find an invertible matrix S s.t. $S^{-1}AS$ is diagonal.

Discussion: In-class exercise revisited.

Recall that the exercise gave us two vectors $v_1 = \begin{pmatrix}1\\3\end{pmatrix}$ and $v_2 = \begin{pmatrix}-1\\-4\end{pmatrix}$. These had the property that $Av_1 = -v_1$ and $Av_2 = \frac{1}{2}v_2$. This allowed us to compute A^{100} by hand!

Problem: in real life we're rarely handed such vectors. We have to find them.

Definition 2: Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Then a <u>nonzero</u> vector $v \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$ iff $Av = \lambda v$. Note: "eigen" means "own" in German.

Example 1: $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\-4\end{pmatrix}$ are eigenvectors of $A = \begin{pmatrix}-\frac{11}{2} & \frac{3}{2}\\ -18 & 5\end{pmatrix}$, with eigenvalues -1 and $\frac{1}{2}$ respectively.

Discussion: how to find these in the first place? Brilliant idea: if $Av = \lambda v$ then

$(A-\lambda I_n)v = 0$. (zero vector) This is saying $A-\lambda I_n$ has a nonzero vector in its kernel!

Therefore, $A-\lambda I_n$ is not invertible, so $\det(A-\lambda I_n) = 0$!

So if we want to find λ, a good starting point is solving the equation (in λ):

$$\det(A-\lambda I_n) = 0.$$

Definition 3: Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Then the characteristic polynomial of T (or of A) is $\det(A-\lambda I_n)$. (This is a polynomial in λ)

Example 1 (ctd'): Suppose we don't know the eigenvalues -1 and $\frac{1}{2}$ of A.

The equation $\det(A-\lambda I_n) = 0$ says: $\det\left(\begin{pmatrix}-\frac{11}{2} & \frac{3}{2}\\ -18 & 5\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) = 0$

$$\Leftrightarrow$$

$$\det\left(\begin{pmatrix}-\frac{11}{2}-\lambda & \frac{3}{2}\\ -18 & 5-\lambda\end{pmatrix}\right) = 0$$

$$\Leftrightarrow$$

$$\left(-\frac{11}{2}-\lambda\right)\cdot(5-\lambda) + \frac{3}{2}\cdot 18 = 0$$

$$\Leftrightarrow$$

$$\lambda^2 - 5\lambda + \frac{11}{2}\lambda - \frac{55}{2} + 27 = 0$$

$$\Leftrightarrow$$

$2.25 = 1.5^2$

$$\lambda^2 + \frac{1}{2}\lambda - \frac{1}{2} = 0 \quad \rightsquigarrow \quad \lambda = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4\cdot\left(-\frac{1}{2}\right)}}{2} = \begin{cases}\lambda = \frac{1}{2}\\ \lambda = -1\end{cases}$$

Great! We found the eigenvalues. What about the eigenvectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\-4\end{pmatrix}$?

These are linear systems!

- $\lambda = \frac{1}{2}$: $\quad Av = \frac{1}{2}v \iff v \in \operatorname{Ker}(A - \frac{1}{2}I_n)$. We know how to compute these!

$$A - \tfrac{1}{2}I_n = \begin{pmatrix} -\frac{11}{2}-\frac{1}{2} & \frac{3}{2} \\ -18 & 5-\frac{1}{2}\end{pmatrix} = \begin{pmatrix} -6 & \frac{3}{2} \\ -18 & \frac{9}{2}\end{pmatrix}$$

$$(A - \tfrac{1}{2}I_n)\,v = \begin{pmatrix}0\\0\end{pmatrix} : \quad \left(\begin{array}{cc|c} -6 & \frac{3}{2} & 0 \\ -18 & \frac{9}{2} & 0 \end{array}\right) \xrightarrow[\mathrm{II} \to \frac{1}{-18}\mathrm{II}]{\mathrm{I} \to \frac{1}{-6}\mathrm{I}} \left(\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 1 & -\frac{1}{4} & 0 \end{array}\right) \xrightarrow{\mathrm{II} \to \mathrm{II} - \mathrm{I}} \left(\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \operatorname{Ker}(A - \tfrac{1}{2}I_n) = \left\{ \begin{pmatrix} \frac{1}{4}t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$\Rightarrow \begin{pmatrix}\frac{1}{4}\\1\end{pmatrix}$ is an eigenvector of A with eigenvalue $\frac{1}{2}$. $\qquad$ Basis: $\begin{pmatrix}\frac{1}{4}\\1\end{pmatrix}$

Notice: we can scale the basis to e.g. $-4 \cdot \begin{pmatrix}\frac{1}{4}\\1\end{pmatrix} = \begin{pmatrix}-1\\-4\end{pmatrix}$ but we don't need to.

- $\lambda = -1$:

$$A - \lambda I_n = \begin{pmatrix} -\frac{11}{2}+1 & \frac{3}{2} \\ -18 & 5+1\end{pmatrix} = \begin{pmatrix} -\frac{9}{2} & \frac{3}{2} \\ -18 & 6\end{pmatrix}$$

$$\operatorname{Ker}(A - \lambda I_n): \quad \begin{pmatrix} -\frac{9}{2} & \frac{3}{2} \\ -18 & 6 \end{pmatrix} \xrightarrow[\mathrm{II} \to \frac{1}{-18}\mathrm{II}]{\mathrm{I} \to \frac{-2}{9}\mathrm{I}} \begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & -\frac{1}{3} \end{pmatrix} \xrightarrow{\mathrm{II} \to \mathrm{II} - \mathrm{I}} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}$$

$$\operatorname{Ker}(A - \lambda I_n) = \left\{ \begin{pmatrix} \frac{1}{3}t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix}\frac{1}{3}\\1\end{pmatrix}\right) \rightsquigarrow \text{Basis: } \begin{pmatrix}\frac{1}{3}\\1\end{pmatrix}.$$

$\Rightarrow \begin{pmatrix}\frac{1}{3}\\1\end{pmatrix}$ is an eigenvector for A with eigenvalue -1.

Upshot: "Algorithm" to diagonalize a matrix:

1. Solve the polynomial equation $\det(A - \lambda I_n) = 0$. $\rightsquigarrow$ Get eigenvalues $\lambda_1, \dots, \lambda_k$.
2. Find nonzero vectors in $\operatorname{Ker}(A - \lambda_1 I_n), \dots, \operatorname{Ker}(A - \lambda_k I_n)$ forming a basis $\mathcal{B} = v_1, \dots, v_n$. "an eigenbasis"

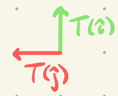
   (! Warning: Step 2 may fail!)
3. Let $S = S_{\mathcal{B} \to \mathcal{E}} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. Then $D = S^{-1}AS$ is diagonal with entries $\lambda_1, \dots, \lambda_k$

(some eigenvalues may be repeated).

Exploring where our "Algorithm" can fail

Definition 3: Let $\lambda \in \mathbb{R}$, $T: \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation with matrix A. Then the λ-eigenspace of T is $E_\lambda = \mathrm{Ker}(A - \lambda I_n)$. Equivalently, E_λ is the subspace of all the vectors $v \in \mathbb{R}^n$ such that $Av = \lambda v$.

Example 2: Consider the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This represents a $90°$ rotation:

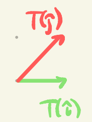


Intuitively, this has no eigenvectors. Let's perform the "algorithm".

1. Char poly: $\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$. This has no real roots. Thus A has no eigenvalues

$\to$ There are no eigenvectors. $\Rightarrow$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable.

Example 3: Consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This represents a shear:



Intuitively, the only eigenvectors lie on the x-axis. Let's perform the "algorithm".

1. Char poly: $\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$. The only root is $\lambda = 1$.
2. Let us find bases for the E_λ's. Our only E_λ is $E_1 = \mathrm{Ker}(A - 1 \cdot I_n) = \mathrm{Ker}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Now $\mathrm{Ker}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y = 0 \right\}$

$= \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

Unfortunately, this is a 1-dimensional subspace of $\mathbb{R}^2$, hence any basis for it will only have 1 vector. Since E_1 is the only eigenspace, this means we will not be able to find an eigenbasis.

$\Rightarrow$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Remark: Examples 2 and 3 are the two kinds of things that can go wrong.

Example 4: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

1) Char poly: $\det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda)$ $\Rightarrow$ Eigenvalues: $\lambda = 1, \lambda = 2$.

2) $E_1 = \mathrm{Ker}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$. $\to$ dim $= 1$

$E_2 = \mathrm{Ker}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$ $\to$ dim $= 1$

Cannot get an eigenbasis

$\Rightarrow$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not diagonalizable

The following theorem should not be surprising at this point:

Theorem 1: Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix A. Then the following statements are equivalent:

1) A is diagonalizable.

2) There exists a basis of $\mathbb{R}^n$ consisting of eigenvectors for A.

3) The dimensions of the eigenspaces add up to n.

You may worry that the bases for E_λ and $E_{\lambda'}$ are linearly dependent.

The following theorem says this cannot happen.

Theorem 1: If A is an $n\times n$ matrix, $v, w \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq c_2$ satisfy $Av = c_1 v$ and $Aw = c_2 w$ then v and w are linearly independent.

<u>Proof</u>: Suppose $\lambda v + \mu w = 0$ $(*)$. Then $A(\lambda v + \mu w) = A\cdot 0 \Rightarrow \lambda c_1 v + \mu c_2 w = 0$

Multiplying $(*)$ by c_1, we get $c_1 \lambda v + c_1 \mu w = 0$

Subtracting these get $\mu(c_1 - c_2) w = 0$

Since $c_1 - c_2 \neq 0$, $\mu w = 0 \Rightarrow \mu = 0$. Similarly, $\lambda = 0$. Thus v and w are l.i. $\square$.

In-class exercises:

1. Determine whether the following matrices are diagonalizable and diagonalize them if possible:

1) $\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}$

2) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

3) $\begin{pmatrix} 0 & 1/2 & -1/2 \\ -4 & 3 & -4 \\ -3 & 3/2 & -5/2 \end{pmatrix}$