Lecture 18

Solutions to the in-class exercises

1. $Q^TQ = I_n \Rightarrow \det(Q^T)\det(Q) = 1 \Rightarrow \det(Q)^2 = 1 \Rightarrow \det(Q) = \pm 1.$

2. Eigenvalues: 1 and 3. Eigenspaces: $E_1 = \mathrm{Span}\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)$, $E_3 = \mathrm{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$

$\quad E_1 \downarrow$ ON basis $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $\quad E_3 \downarrow$ ON basis $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

So $\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$

Today: Assorted topics and questions

Some things we didn't cover but I should mention

- Cross products: you've already done the quiz.

- Determinant of a 3x3 matrix

  If you have seen determinants before, chances are you were taught to compute the determinant of a 3x3 matrix as follows:

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$\det\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}).$$

  Note: Laplacian expansion is equally fast and generalizes better.

- The inverse of a matrix.

  If you have seen inverses before, chances are you were taught to compute the inverse of a matrix as follows: $A^{-1} = \frac{1}{\det(A)}\,\mathrm{Adj}(A)^T$

  Here, the adjoint matrix is obtained by $\begin{pmatrix}m_{11} & -m_{12} & m_{13} & \cdots & m_{1n}\\ -m_{21} & m_{22} & & & \vdots\\ m_{31} & & & & \\ \vdots & & & & \\ (-1)^{n+1}m_{n1} & & \cdots & & m_{nn}\end{pmatrix}$, where $m_{ij} = \det(A_{ij})$

  $A_{ij} \downarrow$ A without ith row and jth col.

  This has theoretical value but it's more prone to mistakes (signs, transposes...) than Gaussian elimination.

- Cramer's rule:

If A is invertible and we have a system $A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$, then $x_i = \dfrac{\det\begin{pmatrix}a_{11} & \cdots & \overset{\text{ith position}}{b_1} & \cdots & a_{n1}\\ \vdots & & \vdots & & \vdots\\ a_{1n} & \cdots & b_n & \cdots & a_{nn}\end{pmatrix}}{\det(A)}$.

Again this has some theoretical value but it's much slower than Gaussian elimination.

- LU decomposition

Just like the Gram-Schmidt algorithm yields a factorization $A = QR$, Gaussian elimination gives a factorization $A = LU$, where L is lower triangular and U is upper triangular. Computers generally solve systems using the LU factorization. An example:

$$A = \begin{pmatrix}2 & 4\\ 1 & -1\end{pmatrix} \xrightarrow[L_1=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix}]{I \to \frac{1}{2}I} \begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix} \xrightarrow[L_2=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}]{II \to II - I} \begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix} \xrightarrow[L_3=\begin{pmatrix}1 & 0\\ 0 & \frac{1}{-3}\end{pmatrix}]{II \to \frac{1}{-3}II} \underbrace{\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}}_{U}$$

Now $L_1^{-1} = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, $L_2^{-1} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $L_3^{-1} = \begin{pmatrix}1 & 0\\ 0 & -3\end{pmatrix}$ so

$$L_3 L_2 L_1 A = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \Rightarrow A = \underbrace{L_1^{-1} L_2^{-1} L_3^{-1}}_{L}\, U = \begin{pmatrix}2 & 0\\ 1 & -3\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}.$$

- Linearity condition: to check that a tranformation is linear, we usually apply the definition: $T(v+w) = T(v) + T(w)$ and $T(\lambda v) = \lambda T(v)$. However, these can be condensed into one: a linear tranformation is linear iff for all $v, w \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$, $T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$. (It's easy to see these are equivalent)

- Inner product spaces

The dot product/orthogonality business has an important generalization: a vector space V is an inner product space if it comes with a "product" $\langle v, w\rangle$ such that:

1. $\langle v, w\rangle = \langle w, v\rangle$
2. $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$
3. $\langle cv, w\rangle = c\langle v, w\rangle$.
4. $\langle v, v\rangle \geq 0$.

Important example which is <u>not</u> $\mathbb{R}^n$: continuous functions $f: [-\pi, \pi] \to \mathbb{R}$. (e.g. $\sin(x)$)

The inner product on this is given by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$.

This space (essentially...) has a basis given by $\sin(nx), \cos(nx)$ for all $n \in \mathbb{Z}$.

The problem is: given a function f (e.g. $f = x^2$), what are the scalars such that

$$x^2 = \lambda_1 \sin(x) + \mu_1 \cos(x) + \lambda_2 \sin(2x) + \mu_2 \cos(2x) + \dots \quad ?$$

Gaussian elimination will <u>not</u> work (infinitely many columns!)

Answer: it turns out this basis is <u>orthonormal</u>, so e.g.
$$\lambda_1 = \langle \sin(x), x^2\rangle$$
$$\mu_1 = \langle \cos(x), x^2\rangle$$
$$\vdots$$

This is the beginning of Fourier series, which has remarkable applications in sound engineering, physics and math. (See 3B1B's videos)

- Differential equations

The differential equation

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t) \\ x(0) = x_0 \end{cases}$$

is easy to solve: $\frac{x'}{x} = 1 \Rightarrow \int \frac{x'}{x}\,dt = t + C \Rightarrow \ln(x) = t + C \rightarrow x = K \cdot e^t$

$x(0) = x_0 \Rightarrow x = e^t \cdot x_0$

Similarly, if $\begin{cases} x' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \\ x(0) = x_0 \end{cases}$ then $x = e^{tA} x_0$, where $e^{tA} = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots$

- Quadratic forms:

A quadratic form is a polynomial in $x_1, \dots, x_n$ of the form $\sum_{i,j=1}^{n} \lambda_{ij} x_i x_j$.

These can be seen as maps $x \mapsto x^T A x$, for some symmetric matrix A.

This allows one to change basis into the more pleasing form $\lambda_1 y_1^2 + \dots + \lambda_n y_n^2$.

Geometrically, these are things like finding the axes of an ellipse.

Not much use outside pure math that I know of.

Assorted questions

TRUE OR FALSE?

1. If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are all unit vectors, then T must be an orthogonal transformation.
2. If A is an invertible matrix, then the equation $(A^T)^{-1} = (A^{-1})^T$ must hold.
3. If matrix A is orthogonal, then matrix A^2 must be orthogonal as well.
4. The equation $(AB)^T = A^T B^T$ holds for all $n \times n$ matrices A and B.
5. If A and B are symmetric $n \times n$ matrices, then $A + B$ must be symmetric as well.
6. If matrices A and S are orthogonal, then $S^{-1}AS$ is orthogonal as well.
7. All nonzero symmetric matrices are invertible.
8. If A is an $n \times n$ matrix such that $AA^T = I_n$, then A must be an orthogonal matrix.
9. If $\vec{u}$ is a unit vector in $\mathbb{R}^n$, and $L = \operatorname{span}(\vec{u})$, then $\operatorname{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

17. If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.
18. If matrices A and B commute, then matrices A^T and B^T must commute as well.
19. There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.
20. Every invertible matrix A can be expressed as the product of an orthogonal matrix and an upper triangular matrix.
21. The determinant of all orthogonal 2×2 matrices is 1.
22. If A is any square matrix, then matrix $\frac{1}{2}(A - A^T)$ is skew-symmetric.
23. The entries of an orthogonal matrix are all less than or equal to 1.
24. Every nonzero subspace of $\mathbb{R}^n$ has an orthonormal basis.
25. $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is an orthogonal matrix.

(Note: solutions can be found in the instructor's manual)

In-class exercises: (whichever exercises we haven't done)