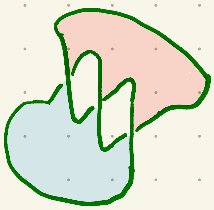
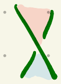
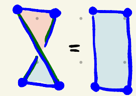
10. Surfaces attached to Knots

1. Applying the Seifert algorithm to a certain diagram for the trefoil, we get the following surface:

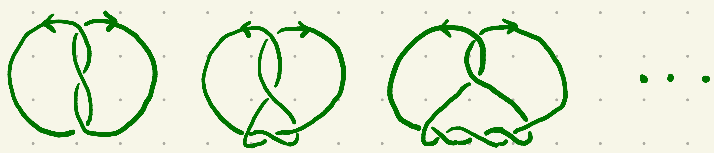


- Find a mesh in order to compute its Euler characteristic (Hint: for the vertical bands , a possible mesh is )
- Use this to prove that $g(\text{trefoil}) = 1$.

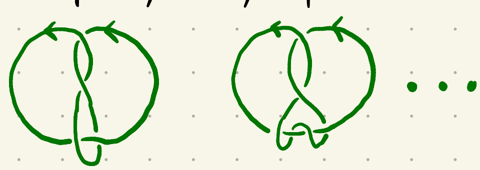
2. Use the method you used in 1 to prove that the Euler characteristic of the Seifert surface obtained from applying the Seifert algorithm with s Seifert circles and c crossings is given by $\chi = s - c$. If the link is in fact a knot, give a formula for the genus in terms of s and c.

3. Use the additivity of the genus $(g(K_1 \# K_2) = g(K_1) + g(K_2))$ to prove that knots with genus 1 are prime. [This constitutes our first general method to prove primality of knots].

4. Use your result in 2 to show that the genus of each of the following knots is 1:



Using 3, conclude that there exist infinitely many prime knots.

5. (Optional): Prove that the knots  also have genus 1.

Use this and Tait's first conjecture* to prove that for every $m \geq 3$, there exists a prime knot with crossing number m.

*Any reduced alternating link diagram has the smallest number of crossings