Week 7

11. Lightning introduction to group theory

1. Decide whether the following sets and operations form groups:

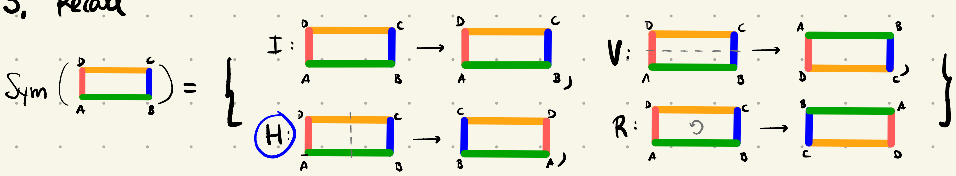
   a) $(\mathbb{R}, +)$
   b) $([-1,1], +)$
   c) $((0,\infty), \cdot)$
   d) $([1,\infty), \cdot)$
   e) 
   f) 

Group Axioms:

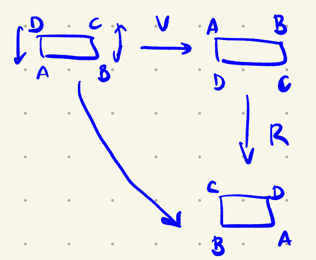
- Closure: for all $x, y \in G$, $x * y \in G$
- Associativity: for all $x, y, z \in G$, $(x*y)*z = x*(y*z)$
- Identity element: there is $e \in G$ s.t. $ex = x$ and $xe = x$ for all $x \in G$
- Inverses: for all $x \in G$ there is $x^{-1} \in G$ s.t. $xx^{-1} = e$ and $x^{-1}x = e$

2. Let $C_2 \times C_2$ be the group with set $\{(0,0), (0,1), (1,0), (1,1)\}$ and operation given by addition modulo 2, so for example $(1,0) + (1,1) = (2,1) = (0,1)$. Write down the operation table and prove that $C_2 \times C_2$ is indeed a group.

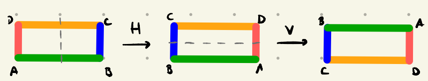
3. Recall



$R * V = H$



with multiplication given by successive application of the maps, for instance



(Notice that we apply the transformations from right to left)

therefore $V * H = R$

Write down the operation table for Sym(▭). Do you see any similarities with that in 2?

4. Recall that $C_6 = \{0,1,2,3,4,5\}$ with operation addition modulo 6. Compute the order of each element and find two different subgroups

5. (Optional): Prove that the unit element in a group is unique.
   (Hint: suppose there are two, say e and f, and show that in fact $e = f$.)

6. (Challenge) Let G be a finite group and let $H \leq G$ be a subgroup. A (left) coset of G by H is a set of the form $gH = \{gh \mid h \in H\}$. Prove the following statements.

   - All cosets have the same number of elements.
   - If two cosets have an element in common, then they are equal.
   - Every $g \in G$ lies in some coset.

Deduce that the number of elements of H divides the number of elements of G.