Isomorphism of sets $f: X \to Y$:

- If $x \neq y$ then $f(x) \neq f(y)$ — Injectivity
- Every $y \in Y$ is equal to $f(x)$ for some $x \in X$. — Surjectivity

Isomorphism of groups $f: X \to Y$:

- f is an isomorphism of sets
- $f(x * y) = f(x) * f(y)$ — "Respects the operation"

12. When two structures are secretly equal.

1. Show that there is only one group of cardinality 3, up to isomorphism.

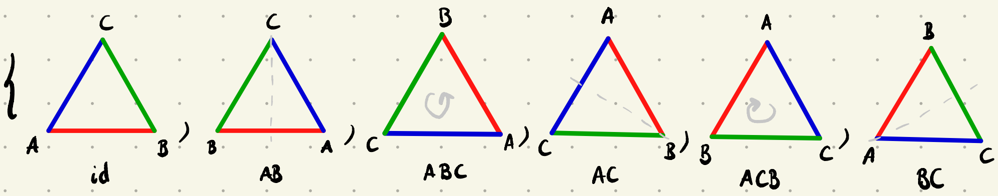
2. Let G and H be two groups and assume:

   1) There is an isomorphism (of groups) $f: G \longrightarrow H$

   2) G is abelian, that is, for each pair of elements $x, y \in G$ we have $x * y = y * x$.

   Prove that H is abelian.

3. Consider the group of symmetries of an equilateral triangle:

$\mathrm{Sym}(\triangle) = \{$ 

$\}$

   - Compute the order of each element
   - Find two elements x and y such that $xy \neq yx$.
   - Use 2 to deduce that $\mathrm{Sym}(\triangle) \not\cong C_6$

4. (Optional) Let $f: G \to H$ be an isomorphism of groups

   - Show that if $e_G \in G$ is the identity element of G, then $f(e_G) = e_H$, the identity element of H.
   - Show that for all $g \in G$, $\mathrm{ord}(g) = \mathrm{ord}(f(g))$. Deduce a second proof of $\mathrm{Sym}(\triangle) \not\cong C_6$.

5. (Optional): Find an isomorphism $\mathrm{Sym}(\triangle) \longrightarrow S_3$. Show that the only groups of cardinality 6 are C_6 and S_3.

6. (Optional) Use 6 in the previous section to prove that there exists a unique group of cardinality p, a prime number.