13. The fundamental group of a Knot.

Isomorphism of sets $f: X \to Y$:

- If $x \neq y$ then $f(x) \neq f(y)$ — Injectivity
- Every $y \in Y$ is equal to $f(x)$ for some $x \in X$. — Surjectivity

Isomorphism of groups $f: X \to Y$:

- f is an isomorphism of sets
- $f(x * y) = f(x) * f(y)$ — "Respects the operation"

1. a) Let $G = \langle a \mid a^7 = 1 \rangle$. Prove that $|G| = 7$. $1, a, a^2, a^3, b, ab, a^2b, \underline{a^3b}$

   b) Let $G = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ ba = a^3 b \rangle$. Prove that $|G| = 8$.

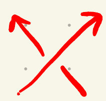
   c) Let $n \geq 2$. Prove that $\langle a \mid a^n = 1 \rangle \cong C_n$ as groups.

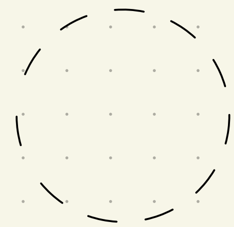
2. Compute the following fundamental groups, using the given diagrams.

   a) $\pi($  $)$

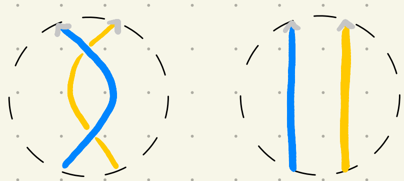
   b) $\pi($  $)$

   c) $\pi($  $)$



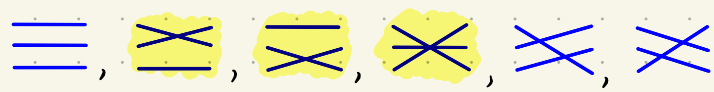


3. Consider link diagrams D_1 and D_2 which only differ in the region inside the dashed circles below. Prove that both D_1 and D_2 will have equivalent relations:



4. Let L be a link and let L' be its mirror image. Prove that $\pi(L) \cong \pi(L')$.

5. (Optional) Prove that $\pi(L)$ is infinite for any link L. (Hint: find a surjective map $\pi(L) \to \mathbb{Z}$)

6. (Challenge). Recall that $S_3 = \{$  $\}$

   Highlighted are the elements known as "transpositions". Prove that if x, y, z are the three transpositions (in any order), then $x^{-1} y x = z$. Use this to prove that a tricoloring of a link L determines a homomorphism $\pi(L) \to S_3$

   ↳ a map such that $f(a*b) = f(a)*f(b)$