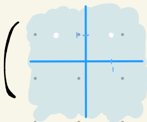
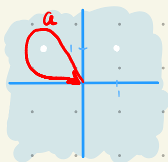
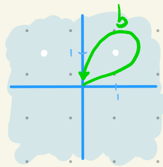
15. A party trick: Brunnian links

1. You have 2 pins, A and B. Find a way to hang a picture subject to the following conditions:
   1. If you remove pin A, the picture falls. If you remove pin B, the picture stays up.
   2. If you remove pin A, the picture stays up. If you remove pin B, the picture stays up.
   3. If you remove pin B, the picture falls. If you remove pin A, the picture stays up.

2. Write your solutions from Exercise 1 in terms of the generators of π_1(), where a and b are:





$[x,y] = xyx^{-1}y^{-1}$

3. Prove that the following identities hold in the free group on the letters a, b, c.
   1. $(ab)^{-1} = b^{-1}a^{-1}$ (Hint: show that the RHS is the unique element x such that $(ab)x = 1$ and $x(ab) = 1$)
   2. $[a,b] = [a, ba]$
   3. $[a,b][b,c] = [aba^{-1}, ca^{-1}]$ (Hint: $(aba^{-1})^{-1} = ab^{-1}a^{-1}$)

4. Recall the solutions for the 2-pin and 3-pin problem:

$$[a,b] = aba^{-1}b^{-1} \quad \xrightarrow{a=1} [1,b] = 1, \qquad \xrightarrow{b=1} [a,1] = 1$$

$$[[a,b],c] \quad \xrightarrow{a=1} [[1,b],c] = [1,c] = 1, \qquad \xrightarrow{b=1} [[a,1],c] = [1,c] = 1, \qquad \xrightarrow{c=1} [[a,b],1] = 1$$

Find a solution to the 4-pin problem, using commutators.

Can you generalize your solution?

5. Solve the "2 out of 3" puzzle: removing any 2 pins makes the picture fall, but removing only 1 pin makes it stay up. You may only use a word (in a, b, c) with 6 letters. Draw your solution.

Generalize your solution to the "(n-1) out of n" puzzle.

6. (Challenge) Interpret the commutator as an OR statement, and use the 3-commutator from Exercise 4 to solve the 2 out of 4 puzzle. You may use the Sage Math code in the second page to check that the picture does not fall when you only remove 1 pin.

7. (Up to you) Solve your own m out of n puzzle.

Sage code: (use https://sagecell.sagemath.org)

```
F.<a,b,c,d> = FreeGroup();
def comm(x,y): return x*y*x^-1*y^-1
def comm3(x,y,z): return x*y*z*x^-1*y^-1*z^-1

rels=[a,b*c]
G=F/rels
f = F.hom(G.gens())
word=comm(d,b*c)*a^2

f(word)==f(1)
```

→ Define the free group

→ Define the commutator

→ Define the 3-commutator

→ Choose relations to impose. Here $a = 1$ and $bc = 1$

→ Define the map f = "apply the relations"

→ choose some word to simplify. Here, $[d, bc]\, a^2$

→ check if the word simplifies to 1. Here, $[d; bc]a^2 = [d; 1] \cdot 1^2 = 1$

Evaluate

```
True
```

→ So correctly, it gives True