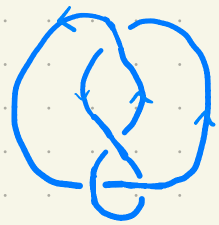
4. A link polynomial.

- $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$
- $\langle \bigcirc \rangle = 1$
- $\langle L \bigcirc \rangle = (-A^{2} - A^{-2})\langle L \rangle$

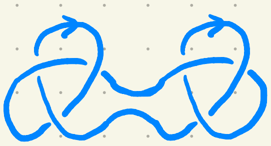
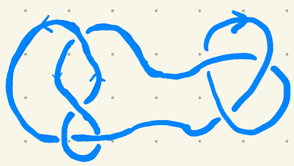
$J(L) := (-A)^{-3\,\mathrm{writhe}(D)} \cdot \langle D \rangle$ $\quad \times_{+1} \quad \times_{-1}$

1. Compute the writhe of the following oriented links:

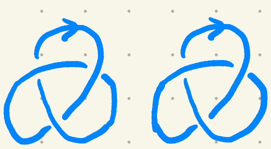
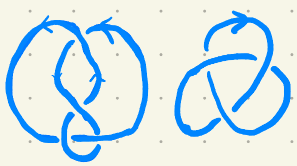
a)  b)  c) 

2. Compute the Kauffman bracket of each of the links in 1. Then compute the Jones polynomial. Argue that none of them are topologically equivalent to one another, or to the unknot.

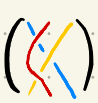
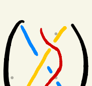
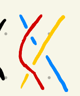
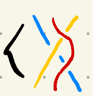
3. Compute the Jones polynomial of the following composite knots $K_1 \# K_2$, by resolving the crossings of K_1 first, and then the crossings of K_2. Do you see why, in general, $\langle K_1 \# K_2 \rangle = \langle K_1 \rangle \cdot \langle K_2 \rangle$? How are writhe (K_1), writhe (K_2) and writhe $(K_1 \# K_2)$ related? Can you prove that $J(K_1 \# K_2) = J(K_1) \cdot J(K_2)$?

a)  b) 

4. Compute the Jones polynomial of the following disjoint unions of knots $K_1 \sqcup K_2$, similarly as in 3. Can you see a relation between $J(K_1 \sqcup K_2)$ and $J(K_1) \cdot J(K_2)$?

a)  b) 

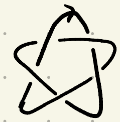
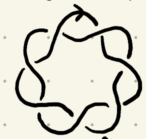
5. Prove that the Jones polynomial satisfies RIII invariance. Hint: prove that

writhe() = writhe () and $\langle$  $\rangle = \langle$  $\rangle$.

6. Explore the following extra topics:

a) Crossing numbers. Recall that $cr(L)$ = minimum number of crossings among all diagrams for L.

The breadth of a polynomial is: $br(P(A))$ = highest power of A in P - lowest.

Consider the $(2n+1)$-foil:  ,  ,  , ...

3-foil 5-foil 7-foil

Let a = number of circles in D for K after resolving all the crossings to $)($

b = number of circles in D for K after resolving all the crossings to $\asymp$

c = number of crossings in D.

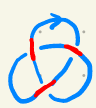
Do some examples to convince yourselves that $a+b = c-2$

Prove that the highest power in $\langle D \rangle$ will be $A^{c} \cdot A^{2(a-1)}$.

Prove that the lowest power in $\langle D \rangle$ will be $A^{-c} \cdot A^{-2(b-1)}$.

Deduce that $br(\langle D \rangle) = 4c$.

Remark: this can be used to show that actually $cr((2n+1)\text{-foil}) = 2n+1$ (next class maybe)

b) Mirror images:  $\mapsto$  (reverse all the crossings)

Prove that the trefoil and its mirror image are not equivalent.

Can you see the relation between $J(L)$ and $J(\text{mirror image}(L))$?

A knot which is equivalent to its mirror image is called ampichiral (the terminology comes from a related notion in chemistry). What should $J(\text{ampichiral link})$ look like?