Cellular Algebras

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1. Cellular Bases

R - commutative domain with 1

A - associative unital R-algebra, free as an R-module

We want a basis of A with particular properties:

Let $(\Lambda, \geq)$ be a finite poset s.th. for each $\lambda \in \Lambda$, $\mathcal{T}(\lambda)$ is a finite indexing set and $\mathcal{C} = \{c_{st}^{\lambda} \mid \lambda \in \Lambda, s,t \in \mathcal{T}(\lambda)\}$ is a basis of A.

For $\lambda \in \Lambda$, let $\check{A}^{\lambda} = \text{span}\{c_{uv}^{\mu} \mid \mu \in \Lambda, \mu > \lambda, u,v \in \mathcal{T}(\mu)\} \subseteq A$ (R-submodule)

Def'n: $(\mathcal{C}, \Lambda)$ is a cellular basis of A if

① $*: \begin{cases} A \longrightarrow A \\ c_{st}^{\lambda} \longmapsto c_{ts}^{\lambda} \end{cases}$ is an algebra anti-isomorphism of A

② For $\lambda \in \Lambda$, $t \in \mathcal{T}(\lambda)$, $a \in A$ there exist $r_v = r_{vt}^{a}$ such that for all $s \in \mathcal{T}(\lambda)$

$$c_{st}^{\lambda} a \equiv \sum_{v \in \mathcal{T}(\lambda)} r_v c_{sv}^{\lambda} \mod \check{A}^{\lambda}$$

↳ if A has such a basis it is a cellular algebra.

Ex. 1: $A = R[x]$, $\Lambda = \mathbb{N}$ (with the usual ordering)
For $n \in \mathbb{N}$ take $\mathcal{T}(n) = \{n\}$, $c_{st}^{\lambda} = c_{nn}^{n} = x^n$, $\mathcal{C} = \{x^n : n \in \mathbb{N}\}$
$\check{A}^n = x^{n+1} R[x]$ (all terms of degree higher than n)
↳ $* = \text{id}$ is an anti-isomorphism
↳ for $\lambda = n$, $t = n$, $a = \sum_{i=0}^{k} a_i x^i$

$$c_{st}^{\lambda} a = x^n \sum_{i=0}^{k} a_i x^i$$
$$= a_0 x^n + \sum_{i=1}^{k} a_i x^{k+i}$$
$$\equiv a_0 x^n \mod \check{A}^n$$

so for any $s \in \mathcal{T}(\lambda)$ (the only option is $s = n$)

we have $c_{st}^{\lambda} a \equiv \underbrace{a_0}_{r_v} x^n \mod \check{A}^n$

Ex. 2: $A = \text{Mat}_{n \times n}(R)$, $\Lambda = \{n\}$, $\mathcal{T}(n) = \{1, 2, \ldots, n\}$, $\mathcal{C} = \{E_{ij} \mid 1 \leq i,j \leq n\}$
Then $(\mathcal{C}, \Lambda)$ is a cellular basis of A:

↳ consider $n = 2$, $t = 1 \in \mathcal{T}(2)$, $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{M} \in A$

$\sum_{v \in \mathcal{T}[\lambda]} r_{v1}^{M} c_{sv}^{2}$

(s=1) $E_{11} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = aE_{11} + bE_{12}$

(s=2) $E_{21} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = aE_{21} + bE_{22}$

so $\overset{a}{r}{}^{M}_{\underset{v\,t}{11}} = a$, $r^{M}_{21} = b$

Ex. 3: Let $A = \mathcal{H}(S_3) \cong \mathbb{K}\langle b_1, b_2 \rangle / (b_i^2 = (q+q^{-1}) b_i,\ b_1 b_2 b_1 - b_1 = b_2 b_1 b_2 - b_2)$

Let $\Lambda = \{ (3) > (2,1) > (1^3) \}$ (partitions of 3 with lexicographic order)

Given $\lambda \in \Lambda$, take $\mathcal{T}(\lambda)$ to be the set of standard Young tableaux of shape λ, ie

$\mathcal{T}((3)) = \{ s = \boxed{1|2|3} \}$, $\mathcal{T}((2,1)) = \{ t = \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix},\ u = \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \}$, $\mathcal{T}((1^3)) = \{ v = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \}$

Let $c^{(3)}_{ss} = b_1 b_2 b_1 - b_1 = b_2 b_1 b_2 - b_2$

$c^{(2,1)}_{tt} = b_1 b_2$ $\quad c^{(2,1)}_{ut} = b_2$

$c^{(2,1)}_{tu} = b_1$ $\quad c^{(2,1)}_{uu} = b_2 b_1$

$c^{(1^3)}_{vv} = 1$

↳ Claim: this is a cellular basis for $A = \mathcal{H}(S_3)$.

The map $*$ is induced by $b_1 \leftrightarrow b_2$.

The multiplication condition:
It suffices to check for $a = b_1, b_2 \in A$

$\underline{\lambda = (3)}$: $s \in \mathcal{T}((3))$, $a = b_1 \in A$

We need $c^{(3)}_{ss} b_1 \equiv r^{b_1}_{ss} c^{(3)}_{ss}$

$$\begin{aligned} c^{(3)}_{ss} b_1 &= (b_1 b_2 b_1 - b_1) b_1 \\ &= (b_1 b_2 - 1) b_1^2 \\ &= (b_1 b_2 - 1)((q+q^{-1}) b_1) \\ &= (q+q^{-1}) c^{(3)}_{ss} \end{aligned}$$

so take $r^{b_1}_{ss} = q + q^{-1}$

↳ Symmetrically $r^{b_2}_{ss} = q + q^{-1}$

$\underline{\lambda = (2,1)}$: $t \in \mathcal{T}((2,1))$, $a = b_1$, $\check{A}^{(2,1)} = \langle c^{(3)}_{ss} \rangle$

$c^{(2,1)}_{tt} b_1 \equiv r_t c^{(2,1)}_{tt} + r_u c^{(2,1)}_{tu} \mod \check{A}^{(2,1)}$

↳ $b_1 b_2 b_1 \equiv r_t\, b_1 b_2 + r_u\, b_1 \mod \check{A}^{(2,1)}$

$c^{(2,1)}_{ut} b_1 \equiv r_t c^{(2,1)}_{ut} + r_u c^{(2,1)}_{uu} \mod \check{A}^{(2,1)}$

↳ $b_2 b_1 \equiv r_t\, b_2 + r_u\, b_2 b_1 \mod \check{A}^{(2,1)}$

This is satisfied by $r_t = 0$, $r_u = 1$

For $a = b_2$: $b_1 b_2 b_2 \equiv r_t b_1 b_2 + r_u b_1 \mod \check{A}^{(2,1)}$

↳ $b_1 (q+q^{-1}) b_2 \equiv r_t b_1 b_2 + r_u b_1 \mod \check{A}^{(2,1)}$

$$b_2^2 \equiv r_t b_2 + r_u b_2 b_1 \mod \check{A}^{(2,1)}$$
$$\hookrightarrow (q+q^{-1}) b_2 \equiv r_t b_2 + r_u b_2 b_1 \mod \check{A}^{(2,1)}$$

This is satisfied by $r_t = q+q^{-1}$, $r_u = 0$

The computations for $u \in \mathcal{T}(\lambda)$ are similar.

$\underline{\lambda = (1^3)}$: $v \in \mathcal{T}((1^3))$, $\check{A}^{(1^3)} = \langle b_1, b_2 \rangle$

$$c_{vv}^{(1^3)} b_1 \equiv r_v c_{vv}^{(1^3)} \mod \check{A}^{(1^3)}$$
$$b_1 \equiv r_v \mod \check{A}^{(1^3)}$$
$\hookrightarrow$ take $r_v = 0$

Similarly: $b_2 \equiv r_v \mod \check{A}^{(1^3)}$
$\hookrightarrow$ take $r_v = 0$

$\square$

2. Properties

Fix $(\mathcal{C}, \Lambda)$ a cellular basis of an algebra A. For $\lambda \in \Lambda$ let A^λ be the R-module with basis $\{c_{uv}^\mu \mid \mu \in \Lambda, \mu \geq \lambda, u,v \in \mathcal{T}(\mu)\}$
Note that $\check{A}^\lambda \subseteq A^\lambda$, and $A^\lambda / \check{A}^\lambda$ has basis $\{c_{st}^\lambda + \check{A}^\lambda \mid s,t \in \mathcal{T}(\lambda)\}$

We note some preliminary consequences of the definition of cellular bases: (Numbering corresponds to the book)

<u>Lemma 2.3</u>: Let $\lambda \in \Lambda$

i) Let $s \in \mathcal{T}(\lambda)$, $a \in A$. For all $t \in \mathcal{T}(\lambda)$
$$a^* c_{st}^\lambda \equiv \sum_{u \in \mathcal{T}(\lambda)} r_u c_{ut}^\lambda \mod \check{A}^\lambda$$
(r_u: the same $r_u \in R$ as in ③)

$\hookrightarrow$ this follows from applying $*$ to part ② of the definition

ii) The R-modules A^λ and $\check{A}^\lambda$ are ideals of A
$\hookrightarrow$ A^λ being an ideal follows from ②, and $\check{A}^\lambda = \sum_{\mu > \lambda} A^\mu$ gives the result for $\check{A}^\lambda$.

iii) Let $s,t \in \mathcal{T}(\lambda)$. There exists $r_{st} \in R$ such that for any $u,v \in \mathcal{T}(\lambda)$, $c_{us}^\lambda c_{tv}^\lambda \equiv r_{st} c_{uv}^\lambda \mod \check{A}^\lambda$
$\hookrightarrow$ use part (i) of this Lemma and part ② of the definition to write $c_{us}^\lambda c_{tv}^\lambda$ as a linear combination of the cellular basis elements.

The cellular basis gives us a filtration of A (via the A^λ), and part (iii) of the Lemma tells us that there is a bilinear form on each quotient $A^\lambda / \check{A}^\lambda$ of the filtration.

Given $\lambda \in \Lambda$, $s \in \mathcal{T}(\lambda)$ we define $C_s^\lambda \subset A^\lambda / \check{A}^\lambda$ as the R-submodule with basis $\{ c_{st}^\lambda + \check{A}^\lambda \mid t \in \mathcal{T}(\lambda) \}$.
This is a right A-module (by ②) and the A-action does not depend on s. This gives $C_s^\lambda \cong C_t^\lambda$ for $s,t \in \mathcal{T}(\lambda)$, so we define the right cell module C^λ as the right A-module with basis $\{ c_t^\lambda \mid t \in \mathcal{T}(\lambda) \}$ where for $a \in A$, $c_t^\lambda a = \sum_{v \in \mathcal{T}(\lambda)} r_v c_v^\lambda$ (the r_v as in ②)

↳ $C^\lambda \cong C_s^\lambda$ for any $s \in \mathcal{T}(\lambda)$ via $c_t^\lambda \mapsto c_{st}^\lambda + \check{A}^\lambda$ for $t \in \mathcal{T}(\lambda)$.

We define the left cell module $C^{*\lambda}$ as the free R-module with basis $\{ c_t^\lambda \mid t \in \mathcal{T}(\lambda) \}$ and A-action $a^* c_t^\lambda = \sum_{v \in \mathcal{T}(\lambda)} r_v c_v^\lambda$ ($a \in A$, r_v as in ②).

This is a left A-module and $C^{*\lambda} \cong \mathrm{Hom}_R(C^\lambda, R)$.

As (A,A)-bimodules, via $c_{st}^\lambda + \check{A}^\lambda \mapsto c_s^\lambda \otimes c_t^\lambda$, for $s,t \in \mathcal{T}(\lambda)$,

$$A^\lambda / \check{A}^\lambda \cong C^{*\lambda} \otimes C^\lambda \cong \bigoplus_{s \in \mathcal{T}(\lambda)} C_s^\lambda$$

so $A^\lambda / \check{A}^\lambda \cong C^{\lambda \oplus |\mathcal{T}(\lambda)|}$ as a right A-module

<u>Lemma 2.7</u>: Let $a \in C^\lambda$, $y \in A^\mu$. Then $ay = 0$ unless $\lambda \geq \mu$.

<u>Pf</u>: Fix $s \in \mathcal{T}(\lambda)$ and identify $C^\lambda \cong C_s^\lambda$. By definition, $ay = 0 \ \forall a \in C_s^\lambda$ iff $c_{st}^\lambda y \in \check{A}^\lambda$ for all $t \in \mathcal{T}(\lambda)$.
$A^\lambda, A^\mu \subset A$ are ideals, so $c_{st}^\lambda y \in A^\lambda \cap A^\mu$, but if $\lambda \ngeq \mu$, $A^\lambda \cap A^\mu \subseteq \check{A}^\lambda$.
□

Lemma 2.3 (iii) tells us that there is a unique bilinear map $\langle \, , \rangle : C^\lambda \times C^\lambda \longrightarrow R$ such that for $s,t \in \mathcal{T}(\lambda)$ $\langle c_s^\lambda, c_t^\lambda \rangle$ is given by $\langle c_s^\lambda, c_t^\lambda \rangle c_{uv}^\lambda \equiv c_{us}^\lambda c_{tv}^\lambda \mod \check{A}^\lambda$. ③

This map is both symmetric and associative:

<u>Prop. 2.9</u>: Let $\lambda \in \Lambda$, $x, y \in C^\lambda$.
(i) $\langle x, y \rangle = \langle y, x \rangle$
(ii) $\langle xa, y \rangle = \langle x, ya^* \rangle$ for all $a \in A$
(iii) $x c_{uv}^\lambda = \langle x, c_u^\lambda \rangle c_v^\lambda$ for all $u, v \in \mathcal{T}(\lambda)$

<u>Def'n</u>: Let $\mathrm{rad}\, C^\lambda = \{ x \in C^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in C^\lambda \}$
This is an A-submodule of C^λ, so we define $D^\lambda := C^\lambda / \mathrm{rad}\, C^\lambda$.

Recall: the Jacobson radical of a module is the intersection of its maximal ideals.

Prop. 2.11: Let R be a field and $\mu \in \Lambda$ be such that $D^\mu \neq 0$
(i) The right A-module D^μ is absolutely irreducible.
(ii) The Jacobson radical of C^μ is $\operatorname{rad} C^\mu$.

Pf: Let $x \neq 0$ be in $C^\mu \setminus \operatorname{rad} C^\mu$, so $\langle x, y\rangle \neq 0$ for some $y \in C^\mu$. We can assume $\langle x, y \rangle = 1$. Since $y \in C^\mu$ we can write $y = \sum_{s \in \mathcal{T}(\mu)} r_s c_s^\mu$ for some $r_s \in R$.

For $t \in \mathcal{T}(\mu)$ let $y_t := \sum_{s \in \mathcal{T}(\mu)} r_s c_{st}^\mu \in A$.

$$
\begin{aligned}
x y_t &= x \sum_{s \in \mathcal{T}(\mu)} r_s c_{st}^\mu \\
&= \sum_{s \in \mathcal{T}(\mu)} r_s \, x c_{st}^\mu \\
\text{(Prop 2.9 (iii))} \quad &= \sum_{s \in \mathcal{T}(\mu)} r_s \langle x, c_s^\mu \rangle c_t^\mu \\
\text{(bilinearity)} \quad &= \langle x, \sum_{s \in \mathcal{T}(\mu)} r_s c_s^\mu \rangle c_t^\mu \\
&= \langle x, y \rangle c_t^\mu \\
&= c_t^\mu
\end{aligned}
$$

So, x generates C^μ as a right A-module, for any $x \in C^\mu \setminus \operatorname{rad} C^\mu$, so D^μ is irreducible and $\operatorname{rad} C^\mu$ is the unique maximal proper submodule of C^μ, so it is equal to the Jacobson radical of C^μ.

The same argument gives us that D^μ is irreducible for any extension field of R, and so is absolutely irreducible.

Prop. 2.12: Let R be a field and let $\lambda, \mu \in \Lambda$, $D^\mu \neq 0$. Let $M \subsetneq C^\lambda$ be a proper submodule and assume $\theta : C^\mu \to C^\lambda / M$ is an A-module homomorphism.

(i) If $\theta \neq 0$ then $\lambda \trianglerighteq \mu$
(ii) If $\mu = \lambda$ then $\exists \, r_\theta \in R$ such that $\theta(z) = M + r_\theta z \;\; \forall z \in C^\mu$, so $\operatorname{Hom}_A(C^\mu, C^\mu / M) \cong R$

Pf: Choose $x, y \in C^\mu$ such that $\langle x, y \rangle = 1$, and for $t \in \mathcal{T}(\mu)$ let $y_t = \sum_{s \in \mathcal{T}(\mu)} r_s c_{st}^\mu$. As seen, $c_t^\mu = x y_t$.

$\theta(x) = M + a_\theta$ for some $a_\theta \in C^\lambda$, so

$$\theta(c_t^\mu) = \theta(x y_t) \overset{\text{(A-module homomorphism)}}{=} \theta(x) y_t = M + a_\theta y_t \quad \text{for any } t \in \mathcal{T}(\lambda).$$

Since $a y_t = 0$ unless $\lambda \trianglerighteq \mu$ (Lemma 2.7), if $\theta \neq 0$ we must have $\lambda \trianglerighteq \mu$ (which proves (i)).

↳ If $\lambda=\mu$ then $a_\theta \in C^\mu$, so

$$\theta(c_t^\mu) = M + a_\theta y_t$$
$$= M + a_\theta \sum_{s\in T(\mu)} r_s c_{st}^\mu$$
$$= M + \sum_{s\in T(\mu)} r_s a_\theta c_{st}^\mu$$

(2.9 (iii)) $$= M + \sum_{s\in T(\mu)} r_s \langle a_\theta, c_s^\mu\rangle c_t^\mu$$

(bilinearity) $$= M + c_t^\mu \langle a_\theta, y\rangle$$

so that θ is the natural projection $C^\mu \to C^\mu/M$ composed with multiplication by $r_\theta = \langle a_\theta, y\rangle$, proving (ii)

Cor 2.13: If R is a field and $\mu, \lambda \in \Lambda$ are such that $D^\mu \neq 0$ and $D^\mu \cong D^\lambda$, then $\mu = \lambda$.

(There exists a nonzero $\theta: C^\mu \to D^\lambda$ so $\lambda \geq \mu$, and by symmetry $\mu \geq \lambda$, so $\mu = \lambda$).

↳ we will soon see that all irreducible A-modules are of this form.

3. Simple Modules in a Cellular Algebra

For this section we will assume $|\lambda| < \infty$ and so $\dim A < \infty$.

Cellular bases give us many filtrations of A.

Def'n: $T \subset \Lambda$ is a poset ideal if $\mu \in T$, $\lambda > \mu$ implies $\lambda \in T$.

For such a subset T let $A(T) \subset A$ be the R-submodule with basis $\{c_{uv}^\mu \mid \mu \in T, u, v \in T(\mu)\}$. Then $A(T) = \sum_{\mu\in T} A^\mu$ is an ideal.

Lemma 2.14: Let $\phi = T_0 \subset T_1 \subset \ldots \subset T_k = \Lambda$ (finite) is a maximal chain of ideals in Λ. Then there is a total ordering $\mu_1, \ldots, \mu_k$ of Λ such that $T_i = \{\mu_1, \ldots, \mu_i\}$ for all i, and $0 = A(T_0) \hookrightarrow A(T_1) \hookrightarrow \ldots \hookrightarrow A(T_k) = A$ is a filtration of A with composition factors $A(T_i)/A(T_{i-1}) \cong C^{*\mu_i} \otimes C^{\mu_i}$.

Pf: Since the chain is maximal, $|T_i \setminus T_{i-1}| = 1$ for $i = 1, \ldots, k$. There is therefore a total ordering $\mu_1, \ldots, \mu_k$ of the elements in Λ such that $j > i$ when $\mu_i > \mu_j$ and $T_i = \{\mu_1, \ldots, \mu_i\}$, $1 \leq i, j \leq k$.

Therefore $\check{A}^{\mu_i} \subseteq A(T_{i-1})$ and $\{ c^{\mu_i}_{uv} + A(T_{i-1}) \mid u,v \in \mathcal{T}(\mu_i) \}$ is a basis of the ideal $A(T_i)/A(T_{i-1})$, so that the R-linear map
$$\begin{cases} A(T_i)/A(T_{i-1}) \longrightarrow C^{*\mu_i} \otimes C^{\mu_i} \\ c^{\mu_i}_{uv} + A(T_{i-1}) \longmapsto c^{\mu_i}_{uv} + \check{A}^{\mu_i} \end{cases}$$
is an (A,A)-bimodule isomorphism for $i = 1, \dots, k$. □

Recall that $C^{*\mu} \otimes_R C^{\mu} \cong (C^{\mu})^{\oplus |\mathcal{T}(\mu)|}$ as a right A-module, so each irreducible composition factor of A is a composition factor of some cell module, which we will investigate.

Lemma 2.15: Suppose $\lambda \in \Lambda$ is minimal, then $C^{\lambda} = D^{\lambda}$ (recall $D^{\lambda} := C^{\lambda}/\text{rad}\, C^{\lambda}$)

Pf: We need to show that $\text{rad}\, C^{\lambda} = 0$.
Suppose $x \in \text{rad}\, C^{\lambda}$, and write $x = \sum_t r_t c^{\lambda}_t$ for some $r_t \in R$.
Fix $s \in \mathcal{T}(\lambda)$ and let $\hat{x} = \sum_t r_t c^{\lambda}_{st}$, so $\hat{x} \in A^{\lambda}$ and $\hat{x} \in \check{A}^{\lambda}$ iff $x = 0$. Since $x \in \text{rad}\, C^{\lambda}$, $\langle x, y \rangle = 0$ for all $y \in C^{\lambda}$, so for $u, v \in \mathcal{T}(\lambda)$
$$\hat{x} c^{\lambda}_{uv} = \sum_{t \in \mathcal{T}(\lambda)} r_t c^{\lambda}_{st} c^{\lambda}_{uv}$$
(3)
$$\equiv \sum_{t \in \mathcal{T}(\lambda)} r_t \langle c^{\lambda}_t, c^{\lambda}_u \rangle c^{\lambda}_{sv}$$
(Bilinearity)
$$= \langle x, c^{\lambda}_u \rangle c^{\lambda}_{sv}$$
$$= 0 \mod \check{A}^{\lambda}$$

So, $\hat{x} a \in \check{A}^{\lambda}$ for all $a \in A$, and so $\hat{x} \cdot 1 = \hat{x} \in \check{A}^{\lambda}$, which gives us $x = 0$ as desired. □

Let $\Lambda_0 = \{ \mu \in \Lambda \mid D^{\mu} \neq 0 \}$, ie μ such that $\langle \, , \, \rangle$ is non-zero on μ ($D^{\mu} = C^{\mu}/\text{rad}\, C^{\mu}$, so $D^{\mu} \neq 0$ if $\text{rad}\, C^{\mu} \neq C^{\mu}$, ie there is some $x \in C^{\mu}$ such that $\langle x, y \rangle \neq 0$ for some $y \in C^{\mu}$)

Thm 2.16 (Graham-Lehrer): Assume R is a field, $|\Lambda| < \infty$.

Then $\{ D^{\mu} \mid \mu \in \Lambda_0 \}$ is a complete set of pairwise inequivalent irreducible A-modules.

Pf: If $D^{\mu} \neq 0$ it is irreducible (Prop. 2.11) and $D^{\mu} \ncong D^{\lambda}$ for $\lambda \neq \mu$ (Corollary 2.13). A has a filtration with composition factors the cell modules of A (Lemma 2.14) so it suffices to prove that every irreducible composition factor of a cell module C^{λ} is isomorphic to D^{μ} for some $\mu \in \Lambda_0$.

By induction (on elements of the poset Λ):

- If $\lambda \in \Lambda$ is minimal, $C^\lambda = D^\lambda \neq 0$ (Lemma 2.15), so $\lambda \in \Lambda_0$.

- If $\lambda \in \Lambda$ is not minimal, let D be an irreducible composition factor of C^λ. Either $D = D^\lambda$ or D is a composition factor of $\operatorname{rad} C^\lambda$.

  Let $T = \{\nu \in \Lambda \mid \lambda \not> \nu\}$. This is a poset ideal in Λ, so $A(T)$ is an ideal of A.

  A^λ annihilates $\operatorname{rad} C^\lambda$ (Prop 2.9 (iii)), but if $\nu \in T \setminus \{\lambda\}$ then $C^\lambda \cdot A^\nu = 0$ (Lemma 2.7), so $\operatorname{rad} C^\lambda \cdot A(T) = 0$, so every composition factor of $\operatorname{rad} C^\lambda$ is a composition factor of $A/A(T)$.

  Extending $\emptyset \subset T \subset \Lambda$ to a maximal chain of poset ideals, Lemma 2.14 gives us a filtration with composition factors isomorphic to cell modules C^ν, $\nu \notin T$ (ie $\lambda > \nu$).

  By induction, since $\nu < \lambda$, every irreducible composition factor of C^ν is isomorphic to some D^μ, $\mu \in \Lambda_0$.

  □

<u>Def'n:</u> For $\mu \in \Lambda_0$, $\lambda \in \Lambda$ define $d_{\lambda\mu} := [C^\lambda : D^\mu]$, the composition multiplicity of the irreducible $D^\mu \subset C^\lambda$. This is well-defined by the Jordan-Hölder Theorem. The decomposition matrix of A is $D = (d_{\lambda\mu})$, $\lambda \in \Lambda$, $\mu \in \Lambda_0$.

<u>Corollary 2.17:</u> Let R be a field. Then D is unitriangular (ie. $d_{\mu\mu} = 1$, $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$)

<u>Pf:</u> $d_{\lambda\mu} \neq 0$ iff there are submodules $M, N \subset C^\lambda$ s. th. $D^\mu \cong N/M$, so there is a nonzero homomorphism $\theta: C^\mu \to C^\lambda/M$ s.th. $\operatorname{im}\theta/M \cong N/M \cong D^\mu$. So, if $d_{\lambda\mu} \neq 0$, $\lambda \geq \mu$ (Prop. 2.12 (i))

If $\lambda = \mu$, $\theta(z) = M + r_\theta z$ $\forall z \in C^\mu$ (Prop. 2.12 (ii)), and $\theta(C^\mu) = C^\mu/M \cong D^\mu$ and D^μ is simple. But by Prop. 2.11 (ii) D^μ is the unique simple quotient of C^μ, so $M = \operatorname{rad} C^\mu$ and $d_{\mu\mu} = 1$.

□

For $\lambda \in \Lambda_0$ we have both a simple D^λ and a principal indecomposable P^λ uniquely determined by $P^\lambda/\operatorname{Rad} P^\lambda \cong D^\lambda$

Lemma 2.18: Assume R is a field and take $\lambda \in \Lambda_0$, $\nu \in \Lambda$.
Then $d_{\nu\lambda} = \dim_R \mathrm{Hom}_A(P^\lambda, C^\nu) = \dim_R(P^\lambda \otimes_A C^{*\nu})$

Def'n: For $\mu \in \Lambda_0$, let $c_{\lambda\mu} := [P^\lambda : D^\mu]$ be the composition multiplicity of $D^\mu \subset P^\lambda$. Then $C := (c_{\lambda\mu})$, $\lambda, \mu \in \Lambda_0$ is the Cartan matrix of A

Lemma 2.19: For P a projective A-module and $k = |\Lambda|$, P has an A-module filtration $\emptyset = P_0 \subseteq P_1 \subseteq \dots \subseteq P_k = P$ such that the nonzero P_i / P_{i-1} are isomorphic to the nonzero modules $P \otimes_A (C^{*\nu} \otimes_R C^\nu)$ with each $\nu \in \Lambda$ occurring exactly once.

↳ $P \otimes_A (C^{*\nu} \otimes_R C^\nu) \cong (P \otimes_A C^{*\nu}) \otimes_R C^\nu \cong (\dim P \otimes_A C^{*\nu}) C^\nu$
So every projective A-module P has a cell module filtration.

Thm 2.20 (Graham-Lehrer): For R a field and $|\Lambda| < \infty$, $C = D^t D$, and so is symmetric.

Pf: Let $\lambda, \mu \in \Lambda_0$ and take $P = P^\lambda$ in Lemma 2.19. Then P^λ has a filtration with composition factors the nonzero $P^\lambda \otimes_A (C^{*\nu} \otimes_R C^\nu)$ where each $\nu \in \Lambda$ occurs at most once.

So, $c_{\lambda\mu} = [P^\lambda : D^\mu] = \sum_{\nu \in \Lambda} [(P^\lambda \otimes_A C^{*\nu}) \otimes_R C^\nu : D^\mu]$
$= \sum_{\nu \in \Lambda} \dim_R (P^\lambda \otimes_A C^{*\nu}) [C^\nu : D^\mu]$
(Lemma 2.18) $= \sum_{\nu \in \Lambda} d_{\nu\lambda} d_{\nu\mu}$

so $C = D^t D$

Theorem (Brauer-Humphreys reciprocity): $[P^\lambda : C^\mu] = [C^\mu : D^\lambda]$

Pf: Let $A = (a_{\lambda\mu})$ with $a_{\lambda\mu} = [P^\lambda : C^\mu]$
Then $c_{\lambda\mu} = [P^\lambda : D^\mu]$
$= \sum_{\alpha \in \Lambda} [P^\lambda : C^\alpha][C^\alpha : D^\mu]$

means $C = AD$
By Thm 2.20, $C = D^T D$, so $D^T D = AD$
If D is square, by Corollary 2.17 it is invertible, and so $D^T = A$, and
$[C^\mu : D^\lambda] = [P^\lambda : C^\mu]$

↳ we will see more details about this later.