SHARP UPPER TAIL ESTIMATES AND LIMIT SHAPES FOR THE KPZ EQUATION VIA THE TANGENT METHOD

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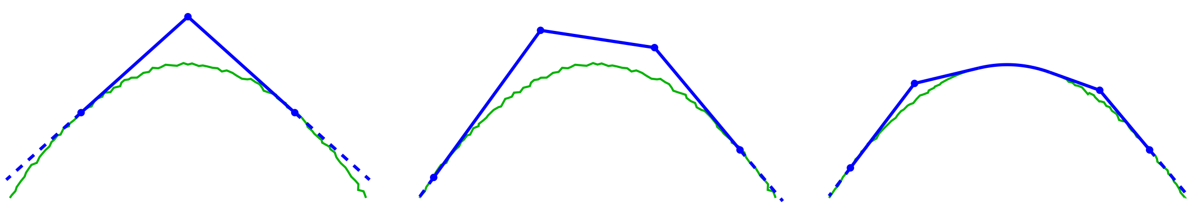
Abstract. We develop a new probabilistic and geometric method to obtain several sharp results pertaining to the upper tail behavior for the Kardar-Parisi-Zhang (KPZ) equation in a unified way. The arguments make crucial use of Brownian resampling invariance properties of Gibbs measures on infinite ensembles of random continuous curves into which the KPZ equation (under narrow-wedge initial data) can be embedded. We obtain sharp one-point upper tail estimates, uniformly in the time parameter, showing that the probability of the value at zero being larger than θ is $\exp(-\frac{4}{3}\theta^{3/2}(1+o(1)))$, for a general class of initial data including the narrow-wedge. Our arguments also yield the first known density estimates for the narrow-wedge case, which are optimal. A key intermediate step is developing a precise understanding of the profile when conditioned on the value at zero equaling θ. Our method further allows one to obtain multi-point asymptotics which were out of reach of previous approaches. As an example, we prove sharp explicit two-point upper tail estimates. All of our arguments also apply to the zero-temperature case of the Airy$_2$ process; even here, while some of the one-point estimates were already available due to its connections to random matrix theory, the two-point asymptotics are new. Finally, to showcase the reach of the method, we obtain the same results in a purely non-integrable setting under only assumptions of stationarity and extremality in the class of Gibbs measures, thus providing evidence in favour of a conjecture suggested by Sheffield and formulated by Corwin-Hammond on a characterization of such line ensembles. Our method bears resemblance to the tangent method introduced by Colomo-Sportiello and mathematically realized by Aggarwal in the context of the six-vertex model.



Contents

1. Introduction, results, and proof ideas

The Kardar-Parisi-Zhang (KPZ) equation, introduced in [KPZ86], is a paradigmatic non-linear stochastic PDE which plays a fundamental role in the class of stochastic growth models known as the KPZ universality class. A particularly important case of the KPZ equation is known as the narrow-wedge solution, corresponding to an appropriate version of the Dirac delta mass initial condition.

A topic of major interest with several recent advances is the tail behavior of the KPZ equation which is also the main subject of investigation of this paper. Before reviewing the literature and the relevant background as well introducing the objects of study formally, we start with a quick glimpse of our main results and techniques.

Most previous work obtaining quantitative probability bounds for the narrow-wedge-solution to the KPZ equation has relied on exactly solvable structure, manifested in the form of explicit formulas for quantities like finite dimensional distributions or Laplace transforms, which are amenable to analytic techniques to obtain asymptotics. Still, it has proven difficult to obtain the sharp behaviour in most cases. In this work we develop a new probabilistic and geometric method that allows a unified treatment of a large class of initial data including, in particular, the narrow-wedge case, yielding sharp results for several quantities related to the upper tail.

We start with matching (up to first order in the exponent) upper and lower bounds on the one point upper tail at depth θ, obtaining the decay of $\exp(-\frac{4}{3}\theta^{3/2}(1+o(1)))$, with explicit lower order error terms. Our arguments also yield sharp density estimates for the narrow-wedge case, which were previously inaccessible. Further, the methods allow access to multi-point upper tails for the narrow-wedge, which had also been out of reach of previous approaches. Towards this, we obtain sharp estimates on two-point upper tails.

The proofs extend to the "zero-temperature" or infinite time analogue of the narrow-wedge solution, the parabolic Airy$_2$ process. While the latter admits determinantal expressions owing to connections to random matrices which had been analyzed previously to obtain tail estimates, many of our results are new even in this case.

Our framework relies on a resampling property known as the Brownian Gibbs property enjoyed by the narrow-wedge solution (after embedding it in the family of random continuous curves known as the KPZ line ensemble introduced in [CH16]) as well as associated monotonicity properties. The techniques bear resemblance to the tangent method proposed by Colomo-Sportiello [CS16] and mathematically realized by Aggarwal [Agg20] to determine limit shapes in the context of the six-vertex model at the ice point.

It is worth remarking that while we do make use of earlier (non-sharp) one-point upper tail estimates for the KPZ equation from [CG20a], obtained there by exactly solvable tools, our arguments work in much greater generality. Indeed, to showcase the power and reach of our method, we also obtain the same sharp asymptotics in a purely non-integrable zero-temperature setting where we work with Brownian Gibbs ensembles whose laws are extremal in the space of such Gibbs measures and which enjoy a certain stationarity property. In particular, in this setting no integrable formulas are available and the arguments rely on purely qualitative assumptions. This also lends evidence to a conjecture suggested by Scott Sheffield (and formulated in [CH14]) which characterizes all ensembles with the mentioned properties.

1.1. Background on the KPZ universality class and KPZ equation.

The KPZ universality class refers to a broad class of models of one-dimensional stochastic growth. These models include those of last passage percolation, exclusion processes, and polymer models, among others. Models in the class feature an observable called a *height* function whose behaviour is governed by three features: local smoothening, non-linear slope-dependent growth, and space-time roughening. The evolution of the height function subject to these forces is paradigmatically captured in the KPZ equation, the stochastic PDE given by

$$\partial_t \mathcal{H} = \frac{1}{4}\partial_x^2 \mathcal{H} + \frac{1}{4}(\partial_x \mathcal{H})^2 + \xi, \tag{1}$$

where $\mathcal{H} : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\xi : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a space-time white noise. The three terms on the right-hand side each represent one of the features mentioned just above: the first, a Laplacian, is a smoothening force; the second is a non-linear slope-dependent term; and the third is a roughening white noise. (The coefficients we have chosen in the KPZ equation are slightly different from the usual ones; any choice can be transformed to any other by a suitable scaling, and we have picked the one above for certain convenient features it offers which we will point out shortly.)

The solution theory for (1) is highly non-trivial because of the combination of the non-linear term $(\partial_x \mathcal{H})^2$ and the space-time white noise ξ: the latter suggests that $\mathcal{H}$ will be rough, i.e., differentiable only in the sense of distributions, which renders $(\partial_x \mathcal{H})^2$ not well-defined. There has been much recent work on handling these issues, including regularity structures [Hai13], paracontrolled distributions [GIP15], and energy methods [GJ14, GP17].

All these notions of solution agree with a physically relevant notion known as the Cole-Hopf solution, which is the one that has been most used in studies of probabilistic properties of the KPZ equation. To describe it we need the multiplicative stochastic heat equation (SHE), given by

$$\partial_t \mathcal{Z} = \frac{1}{4}\partial_x^2 \mathcal{Z} + \xi \mathcal{Z},$$

where $\mathcal{Z} : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\xi : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a space-time white noise. The solution theory of the SHE is more straightforward as it is a linear SPDE. The Cole-Hopf solution is *defined* by

$$\mathcal{H}(t,x) = \log \mathcal{Z}(t,x)$$

for all $t > 0$ and $x \in \mathbb{R}$. It can be checked via a purely formal change of variables computation that the SHE becomes the KPZ equation under this substitution; more importantly, the Cole-Hopf solution agrees with the more sophisticated solution theories mentioned above.

That $\mathcal{H}$ is well-defined by the above relation relies on the fact that, almost surely, $\mathcal{Z}(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ simultaenously if the initial data of the latter is bounded with compact support, non-negative, and not identically zero [Mue91]; the same result was extended in [Flo14] to include the Dirac delta initial condition (earlier work [BC95] had established almost sure positivity of $\mathcal{Z}(t,x)$ for any given (t,x) for a class of initial conditions including positive Borel measures with some regularity conditions).

In particular, we can take the initial condition for the SHE to be the Dirac mass at the origin and still have $\mathcal{H} = \log \mathcal{Z}$ be well-defined. This solution, known as the *narrow-wedge* solution, is of central importance for the KPZ equation, as mentioned earlier. We discuss it more next.

1.2. **A scaled version of the narrow-wedge solution.** We will be interested in $\mathcal{H}(t,\cdot)$ for all large t, and we will make statements uniformly in t. For this reason, we will consider a *scaled* version of $\mathcal{H}(t,\cdot)$, which we will denote $\mathfrak{h}^t$, given by

$$\mathfrak{h}^t(x) = \frac{\mathcal{H}(t, t^{2/3}x) + \frac{t}{12}}{t^{1/3}}.$$

Note that we have scaled space by $t^{2/3}$ and fluctuations around $-t/12$ by $t^{1/3}$; this reflects the well-known KPZ 1:2:3 scaling relation. Now, it is known that the family $\{\mathfrak{h}^t\}_{t>t_0}$ is tight for any $t_0 > 0$; indeed, [ACQ11] showed that $\mathfrak{h}^t(0)$ converges in distribution to the GUE Tracy-Widom distribution as $t \to \infty$, and, more recently, it has been shown independently in [QS22, Vir20] (as a special case) that $\mathfrak{h}^t$ converges in distribution as a process, in the topology of uniform convergence on compact sets, to an object known as the parabolic Airy$_2$ process. As a point of terminology, the

KPZ equation at time $0 < t < \infty$ is said to be of *positive temperature*, while the parabolic Airy$_2$ process (which corresponds to $t = \infty$) is a *zero-temperature* model; this language comes from the fact that the solution to the KPZ equation is the free energy of the continuum directed random polymer [AKQ14], where the time parameter t exactly plays the role of inverse temperature.

1.2.1. *Parabolic curvature of* $\mathfrak{h}^t$. A stationarity property of $\mathfrak{h}^t$ will feature heavily in our arguments: namely, that $\mathfrak{h}^t(x) + x^2$ is a stationary process in x for every $t > 0$ (proved in [ACQ11], see also [Nic21]). One reason for our choice of coefficients of the KPZ equation (1) is to have the coefficient of x^2 here be 1. The same stationarity property holds true for the $t \to \infty$ distributional limit of $\mathfrak{h}^t$ as well, i.e., for the parabolic Airy$_2$ process, introduced in [PS02], which we denote $\mathcal{P}$; the other reason for our choice of coefficients in (1) is so that this convergence holds with no additional constant scaling factors.

The narrow-wedge solution to the KPZ equation as well as the parabolic Airy$_2$ process have been extensively studied in the literature. This is because both objects enjoy a great amount of exactly solvable or integrable structure in the sense that there are explicit exact formulas for quantities such as finite dimensional distributions and Laplace transforms which, further, are amenable to analysis. We discuss some of the work on upper tail behaviour next.

1.3. **Upper tail behaviour predictions.** Recall that $\mathcal{P}(x) + x^2$ has the same distribution for each $x \in \mathbb{R}$ by stationarity. It is well-known that this common distribution is the GUE Tracy-Widom distribution, first discovered in random matrix theory as the scaling limit of the largest eigenvalue of the Gaussian Unitary Ensemble [TW94]. The upper tail asymptotics of the GUE Tracy-Widom are also well-known to be

$$\mathbb{P}\left(\mathcal{P}(0) > \theta\right) = \exp\left(-\frac{4}{3}\theta^{3/2}(1 + o(1))\right) \text{ as } \theta \to \infty; \tag{2}$$

see [RRV11] for an explicit statement, but this was widely known much earlier using the determinantal structure of the Airy point process, of which $\mathcal{P}(0)$ is the largest point, whose kernel can be written in terms of the special function known as the Airy function Ai. Indeed, the above asymptotic essentially follows from the classical fact that $\mathrm{Ai}(\theta) = \exp(-\frac{2}{3}\theta^{3/2}(1 + o(1)))$ as $\theta \to \infty$.

It has been known since 2011 that $\mathfrak{h}^t(0)$ converges in distribution to $\mathcal{P}(0)$ as $t \to \infty$ [ACQ11] (some non-rigorous derivations from the mathematical physics [SS10a, SS10b, SS10c] and physics [Dot10, CLDR10] communities were also given at around the same time; see [Cor12] for a discussion of the history). As mentioned, this has recently been strengthened to process-level convergence independently in [QS22, Vir20]. This suggests that, for large t, the asymptotic (2) may hold for $\mathfrak{h}^t$ as well. Indeed, it has been predicted in the physics literature (see for example [KMS16] and references therein) that the same asymptotic should hold for all finite t for large θ for a broad class of deterministic initial data and not just narrow-wedge. Our results will establish these asymptotics for all large θ uniformly in t, and thus as a byproduct confirm the physics predictions.

We emphasize that while the asymptotic (2) is not difficult to derive using the determinantal structure of $\mathcal{P}$, such structure does not hold for the KPZ equation. This has been the major source of technical difficulty in previous works on the latter, which we briefly overview next.

1.4. **Previous work on upper tail asymptotics.** There has been a substantial amount of work on one-point upper tail asymptotics for $\mathfrak{h}^t$ which we briefly summarize here. A number of works [CJK13, CD15, KKX17] have studied upper tails of the SHE directly with various types of non-linearities; [KKX17] obtains upper tail bounds of the form $\mathbb{P}(\mathfrak{h}^t(0) > \theta) \leq \exp(-c\theta^{3/2})$ for large $\theta > \theta_0$ for some $c > 0$; however, the constants c and θ_0 are not uniform in t, i.e., they may depend on t, which is common to all the mentioned results. [CQ13] obtains a uniform-in-t upper bound, but it does not show that the expected $\exp(-c\theta^{3/2})$ behaviour holds in the large deviation regime of θ.

The first uniform-in-t upper tail estimate capturing the correct behaviour on the level of the $\frac{3}{2}$ tail exponent in the entire upper tail is [CG20a], who obtain estimates of the form

$$\exp(-c_1\theta^{3/2}) \le \mathbb{P}(\mathfrak{h}^t(0) > \theta) \le \exp(-c_2\theta^{3/2}) \tag{3}$$

for $t > t_0$, $\theta > \theta_0$, and some constants $c_1, c_2 > 0$, where θ_0, c_1, c_2 depend only on $t_0 > 0$. They also obtain bounds on c_1 and c_2 depending on the regime of θ in question: for instance, for any $\varepsilon > 0$, $c_1 = \frac{4}{3}(1+\varepsilon)$ and $c_2 = \frac{4}{3}(1-\varepsilon)$ can be taken when $\theta < O(\varepsilon^2 t^{2/3})$, i.e., θ up till the start of the large deviation regime. Actually, they obtain that c_2 (though not c_1) can be taken similarly near-optimal far in the large deviation regime as well, but the values of both c_1 and c_2 deteriorate in all other ranges of θ.

[CG20a] also obtain upper tail estimate for general initial data which capture that $\frac{3}{2}$ tail exponent uniformly in t; however, they do not obtain the expected coefficient of $\frac{4}{3}$ in the exponent.

Apart from this, sharp behavior has been obtained in the large deviation, $t \to \infty$ limit, with results of the form $\lim_{t\to\infty} t^{-1} \log \mathbb{P}(\mathfrak{h}^t(0) > yt^{2/3}) = -\frac{4}{3}y^{3/2}$ $(y > 0)$ [DT21] and similar results for general initial data [GL20]. Note however that these types of results do not yield finite-t tail bounds like (3).

The first objective of this work is to obtain estimates of the form of (3) *with the optimal coefficient of* $\frac{4}{3}$ *for the entire upper tail regime and for a wide class of initial data using non-integrable techniques.*

Such an estimate will also of course imply the aforementioned large deviation results. Perhaps more surprisingly, we will see that the techniques allow us to tackle more difficult questions such as that of multi-point asymptotics, in particular the two-point case.

As is often the case in the study of tail behavior, in our analysis a related question is of central importance: what does the profile of $\mathfrak{h}^t$ look like at finite t when conditioned on the upper tail events? The one-point upper tail case has been investigated in the $t \to 0$ limit in [LLT21], verifying predictions from the physics literature [KMS16]; however, the KPZ equation in this regime of t lies in the Gaussian, more precisely Edwards-Wilkinson, universality class, and does not exhibit the non-linear behaviour characteristic of the KPZ universality class. The physics literature does not seem to have predictions for this question for finite but large t, in which case the KPZ equation does lie in the KPZ universality class. As far as we could tell, the probability literature does not seem to have studied the zero temperature case (i.e., of $\mathcal{P}$) of the question of the profile's shape conditioned on the one-point upper tail. Similarly there does not seem to be work on the same question for multi-point upper tail conditionings.

However, there have been some recent studies in zero-temperature prelimiting models (such as last passage percolation and TASEP) investigating such questions about limit shapes and related geometric observables in the large deviation regime [OT19, QT21, BG19, BGS19].

1.5. **Main results on tail asymptotics.**

All our results will apply to both the positive and zero temperature cases. Though in the above discussion we have denoted the zero-temperature objects by $\mathcal{P}$, in order to write the results more concisely we will use $\mathfrak{h}^t$ for both positive $(t < \infty)$ and zero temperature $(t = \infty)$, and state that the results hold for $t \in [t_0, \infty]$. In all of the results t_0 may be chosen to be any positive value, and all constants in all the results will be allowed to depend on t_0; we will omit this point in the statements of the results for conciseness.

1.5.1. *One-point tail and density asymptotics.*

Our first result concerns the asymptotics of the density of $\mathfrak{h}^t(0)$ as the argument goes to $+\infty$. To the best of our knowledge this is the first result for the one-point density of the KPZ equation.

We denote the density of $\mathfrak{h}^t(0)$ at θ by $\frac{1}{\mathrm{d}\theta}\mathbb{P}(\mathfrak{h}^t(0) \in [\theta, \theta + \mathrm{d}\theta])$. That the density exists is not immediately obvious, but follows from results of [CH16], and also from earlier work on the SHE [MN08]; see also Section 2 for a brief discussion of the former.

Theorem 1 (One-point density asymptotics)**.** *There exist constants C and θ_0 such that, for all $t \in [t_0, \infty]$ and $\theta > \theta_0$,*

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \le \frac{1}{\mathrm{d}\theta}\mathbb{P}\Big(\mathfrak{h}^t(0) \in [\theta, \theta + \mathrm{d}\theta]\Big) \le \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

Next we move to one-point tail asymptotics. While this might seem to be a straightforward corollary of the density bound just stated, the proof of the latter in fact relies on the bound on the upper tail which we derive first. In addition, note that for the lower bound on the tail we have a better error term than in the density bound.

Theorem 2 (One-point upper tail bounds)**.** *There exist $\theta_0 > 0$ and $C < \infty$ such that, for all $t \in [t_0, \infty]$ and $\theta > \theta_0$,*

$$\exp\left(-\frac{4}{3}\theta^{3/2} - \theta^{1/2}\log\theta\right) \le \mathbb{P}\Big(\mathfrak{h}^t(0) \ge \theta\Big) \le \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

1.5.2. *Two-point tail asymptotics.* We will consider the probability that $\mathfrak{h}^t$ is greater than $a\theta$ at $-\theta^{1/2}$ and greater than $b\theta$ at $\theta^{1/2}$ for $a \ge b > -1$ (so that $a\theta, b\theta > -\theta$, the value of the parabola $-x^2$ at $x = \pm\theta^{1/2}$).

We note that by stationarity of $x \mapsto \mathfrak{h}^t(x) + x^2$ and equality in distribution of $x \mapsto \mathfrak{h}^t(x)$ and $x \mapsto \mathfrak{h}^t(-x)$ (which holds by the same invariance of the white noise in (1)), the probability we analyze is equivalent to the probability of any two-point event of the form $\{\mathfrak{h}^t(x_1) > -x_1^2 + t_1, \mathfrak{h}^t(x_2) > -x_2^2 + t_2\}$ with $x_1, x_2 \in \mathbb{R}$ and t_1, t_2 large by an appropriate horizontal translation and choice of a, b. We consider $a\theta$ and $b\theta$ as this simplifies some expressions which, nonetheless, are still somewhat technical to look at. However, the main point that should be taken is that they are rather explicit. Let $\mathsf{ConHull}_{a,b} : \mathbb{R} \to \mathbb{R}$ be the convex hull of $x \mapsto -x^2$ and the points $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. The two-point asymptotics depends on the number of extreme points $\mathsf{ConHull}_{a,b}$ has inside $[-\theta^{1/2}, \theta^{1/2}]$; see Figure 1. Moreover, we will assert later in Theorem 10 that $\mathsf{ConHull}_{a,b}$ is the approximate shape that $\mathfrak{h}^t$ adopts under the conditioning of the two-point tail event we are considering here.

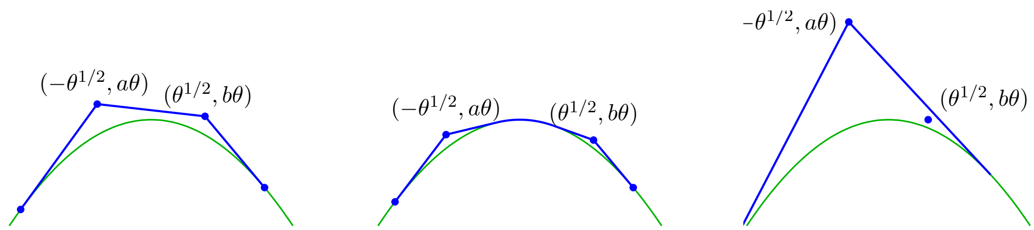


FIGURE 1. The three cases of Theorem 3: from left to right, $\mathsf{ConHull}_{a,b}$ (in blue) has two, infinitely many, and one extreme point inside $[-\theta^{1/2}, \theta^{1/2}]$. (The distance of $\pm\theta^{1/2}$ from the center of the parabola at 0 have been made to differ in the three figures just to visually better emphasize the geometric features of the three cases.)

Theorem 3 (Two-point upper tail bounds)**.** *There exist constants θ_0 and $a_0 = b_0$ such that the following hold for all $t \in [t_0, \infty]$. If (i) $\theta > \theta_0$ and $a \ge b > -1$ or (ii) $\theta > 0$ and $a \ge a_0$, $b \ge b_0$, $a \ge b$, then, if $\mathsf{ConHull}_{a,b}$ has two extreme points inside $[-\theta^{1/2}, \theta^{1/2}]$ (see Figure 1),*

$$\begin{aligned}&\mathbb{P}\Big(\mathfrak{h}^t(-\theta^{1/2}) \ge a\theta, \mathfrak{h}^t(\theta^{1/2}) \ge b\theta\Big)\\ &\qquad= \exp\left(-\frac{\theta^{3/2}}{24}\left[3(a-b)^2 + 24(a+b) + 16\left((1+a)^{3/2} + (1+b)^{3/2}\right) + 32\right] + \mathrm{error}\right);\end{aligned}$$

while if $\mathsf{ConHull}_{a,b}$ *has infinitely many extreme points inside* $[-\theta^{1/2}, \theta^{1/2}]$,

$$\mathbb{P}\Big(\mathfrak{h}^t(-\theta^{1/2}) \geq a\theta, \mathfrak{h}^t(\theta^{1/2}) \geq b\theta\Big) = \exp\left(-\frac{4}{3}\theta^{3/2}\left[(1+a)^{3/2} + (1+b)^{3/2}\right] + \text{error}\right);$$

and finally if $\mathsf{ConHull}_{a,b}$ *has one extreme point inside* $[-\theta^{1/2}, \theta^{1/2}]$,

$$\mathbb{P}\Big(\mathfrak{h}^t(-\theta^{1/2}) \geq a\theta, \mathfrak{h}^t(\theta^{1/2}) \geq b\theta\Big) = \exp\left(-\frac{4}{3}\theta^{3/2}(1+a)^{3/2} + \text{error}\right).$$

Further, the error terms have explicit bounds.

The mentioned explicit error bound is written out in the more technical version of the theorem stated in Section 7 as Theorem 7.1.

It will be clear from the geometric picture developed in the course of the proof of Theorem 3 that a procedure for obtaining k-point asymptotics is also available. See Remark 7.3.

We next point out that the last two cases of the theorem have nice geometric interpretations, which we explain now.

In the final case where there is a single extreme point in the interval, as is apparent by looking at Figure 1 and our earlier remark that $\mathsf{ConHull}_{a,b}$ is the shape of $\mathfrak{h}^t$ on the two-point tail event (as precisely stated ahead in Theorem 10), the event is essentially caused by $\mathfrak{h}^t(-\theta^{1/2}) \geq a\theta$ alone; thus the probability bound is the same as that of the latter event from Theorem 2.

The second case's geometric interpretation is more interesting. Essentially, the interaction or dependence of the events $\{\mathfrak{h}^t(-\theta^{1/2}) \geq a\theta\}$ and $\{\mathfrak{h}^t(\theta^{1/2}) \geq b\theta\}$ is through the line connecting the points $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. In the second case where $\mathsf{ConHull}_{a,b}$ has infinitely many extreme points in $[-\theta^{1/2}, \theta^{1/2}]$, the parabola can be thought of as being a *barrier* to this line, thus preventing the interaction of the two events, and so the probability is the product of the individual probabilities (at least up to first order in the exponent).

The expert reader at this point might be reminded of the *positive association* property of the KPZ equation (which it inherits, for example, from interacting particle system models like the asymmetric simple exclusion process which converge to it, though we provide a proof via a different model). This property says that the probability of an intersection of two *increasing* events (an event A such that if $f \in A$ and $g \geq f$ pointwise, then $g \in A$; eg. A can be the one- or two-point upper tail) is upper bounded by the product of the individual probabilities. This important inequality holds in a large class of lattice statistical mechanics models and is also known as the Fortuin-Kastelyn-Ginibre (FKG) inequality [FKG71], and we will refer to it by this name. The FKG inequality is an indispensable tool in these models, and so it is of interest to understand when it is approximately sharp.

As already alluded to above, the second case of Theorem 3 provides one condition and interpretation for approximate sharpness. As can be seen from Figure 1, this condition is the same as when the line joining $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ intersects the parabola at two points inside $[-\theta^{1/2}, \theta^{1/2}]$ (as then $\mathsf{ConHull}_{a,b}$ will not be piecewise-linear inside $[-\theta^{1/2}, \theta^{1/2}]$, and hence have infinitely many extreme points there). Our next theorem extends this observation by also addressing the case where this line is *tangent* to the parabola. Thus we obtain a criterion for the sharpness of the FKG inequality.

Theorem 4 (Sharpness of the FKG inequality)**.** *Let* $t \in [t_0, \infty]$ *and* $a, b > -1$. *Suppose the line connecting* $(-\theta^{1/2}, a\theta)$ *and* $(\theta^{1/2}, b\theta)$ *intersects the parabola* $-x^2$ *inside* $[-\theta^{1/2}, \theta^{1/2}]$. *Then, as* $\theta \to \infty$ *or as* $a, b \to \infty$,

$$\mathbb{P}\left(\mathfrak{h}^t(-\theta^{1/2}) \geq a\theta, \mathfrak{h}^t(\theta^{1/2}) \geq b\theta\right) = \exp\left(-\frac{4}{3}\theta^{3/2}\left[(1+a)^{3/2} + (1+b)^{3/2}\right](1+o(1))\right),$$

i.e., the probability on the LHS is equal to $\mathbb{P}(\mathfrak{h}^t(-\theta^{1/2}) > a\theta) \cdot \mathbb{P}(\mathfrak{h}^t(\theta^{1/2}) > b\theta)$ *up to first order in the exponent. The* $o(1)$ *term depends on* t_0.

Theorem 4 essentially follows by formulating the tangency condition in an algebraic form and applying the first case of Theorem 3, where $\mathsf{ConHull}_{a,b}$ has two extreme points inside $[-\theta^{1/2}, \theta^{1/2}]$. Simplifying the expression from that case of Theorem 3 will yield Theorem 4.

1.5.3. *Extremal ensembles.* As will be elaborated in some detail in the upcoming Section 1.7 on proof ideas, our approach to proving our results relies on first embedding $\mathfrak{h}^t$ in an infinite family of random continuous curves—called a *line ensemble*—introduced in [CH16]. For this section we will focus on the $t = \infty$ (zero temperature) case, introduced in the earlier work [CH14]; here, the ensemble's top line is the parabolic Airy$_2$ process, which we denoted above by $\mathcal{P}$. While on first glance these ensembles are more complicated objects, they enjoy an explicit Gibbs resampling property in terms of Brownian bridges, which we discuss in more detail in Section 1.7. But in brief, for the zero-temperature case, the resampling is in terms of *non-intersecting* Brownian bridges and is known as the *Brownian Gibbs* property, and the corresponding ensemble in which $\mathcal{P}$ is embedded is known as the *parabolic Airy line ensemble.* In these terms, the laws of these infinite ensembles are examples from a class of Gibbs measures on the space of infinite collections of continuous curves.

In such settings it is an important and natural objective to gain an understanding of the structure of the set of Gibbs measures, which is a convex set. Indeed, this has been an important field of research in areas such as tiling and dimer models [She05, KOS06, Agg19]. For such classifications, the *extremal* Gibbs measures play a special role; a Gibbs measure is extremal if it cannot be written as a non-trivial convex combination of two other Gibbs measures.

The following conjecture regarding the structure of the extremal Gibbs measures appears in [CH14], following a suggestion of Scott Sheffield:

Conjecture 5 (Conjecture 3.2 of [CH14])**.** *Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be a line ensemble such that (i) $\mathcal{L}(x) + x^2$ is stationary under deterministic horizontal shifts, (ii) $\mathcal{L}$ has the Brownian Gibbs property, and (iii) the law of $\mathcal{L}$ is extremal in the set of such Gibbs measures. Then $\mathcal{L}$ is the parabolic Airy line ensemble, up to a trivial deterministic vertical shift of the entire ensemble.*

We will call an ensemble which satisfies the hypotheses of the conjecture an *extremal stationary ensemble.* Proving this conjectured classification result of such Gibbs measures may serve as a useful result to prove universality within KPZ.

Note that despite the hypotheses on the ensemble in the conjecture being purely qualitative—in particular integrable inputs are not available—the conclusion is essentially that the ensemble must be the parabolic Airy line ensemble, which has very precise tail decay and integrable structure. As a first step towards this, our methods are able to yield quantitative evidence in favor of the conjecture. Indeed, our methods establish the same tail decay asymptotics for the top curve of any extremal stationary ensemble as for the parabolic Airy$_2$ process.

Theorem 6 (Tail asymptotics for extremal stationary ensembles)**.** *Theorems 1, 2, 3, and 4 all hold with $t = \infty$ for the top curve of extremal stationary ensembles. In particular, if $\mathcal{L}$ is such an ensemble,*

$$\frac{1}{\mathrm{d}\theta}\mathbb{P}\Big(\mathcal{L}_1(0) \in [\theta, \theta + \mathrm{d}\theta]\Big) = \exp\left(-\frac{4}{3}\theta^{3/2}(1 + o(1))\right).$$

(Note that explicit and implicit constants such as θ_0 or C in Theorems 1–4 and $o(1)$ above will depend on the extremal stationary ensemble measure in question.)

It can be shown using existing results on determinantal point processes that the parabolic Airy line ensemble is extremal in the space of Gibbs measures (see Section 8.1), and so satisfies the hypotheses of the theorem. Thus Theorem 6 represents a purely non-integrable derivation of the upper tails of the parabolic Airy$_2$ process $\mathcal{P}$ (as we will see, the arguments for the tail bounds already stated take an a priori tail bound as input, which so far is only available for $\mathcal{P}$ via integrable arguments).

We also note that there is no a priori control on the fluctuations of such extremal stationary ensembles; for instance, it is not obvious that $\mathcal{L}_1(0)$ even has finite expectation. Even Theorem 6 only implies $\mathbb{E}[\mathcal{L}_1(0) \vee 0] < \infty$ and the finiteness of $\mathbb{E}[|\mathcal{L}_1(0)|]$ is not clear as we currently do not have any bound on the lower tail. This appears difficult and it would be of interest to develop geometric and probabilistic methods to obtain bounds counterpart to the above for the lower tail of extremal ensembles.

It is not hard to deduce further that the asymptotics in Theorem 6 also hold for mixtures of extremal stationary ensembles, so long as the mixture distribution's upper tail is not too heavy.

Finally, we reiterate that the $o(1)$ term depends on the extremal measure in question; indeed, if the conjecture is true, the error term would have to absorb the arbitrary deterministic vertical shift.

1.5.4. *One-point tail asymptotics for general initial data.* Our last result on asymptotics concerns general initial data.

Here we exploit the well-known fact that the one-point distribution of the KPZ equation at a given time with general initial data can be expressed via a convolution formula involving the entire spatial process of the narrow-wedge solution at the same time. Using this it is possible to use asymptotics for the narrow-wedge solution to obtain one-point asymptotics for general initial data.

To formulate this result, we need to define the scaled solution to the KPZ equation under general data, as well as a class of initial conditions under which the solution is well-defined and non-trivial.

Let $\mathcal{H}(t,x)$ be the solution to the KPZ equation (1) started from general initial data $\mathcal{H}(0,\cdot)$. To respect the KPZ scaling, we will allow the initial data to vary with the value of t being considered, but, for brevity, we will still denote it by $\mathcal{H}(0,\cdot)$, omitting the t-dependence in the notation.

More precisely, for a family of functions $f^{(t)} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, we set $\mathcal{H}(0,\cdot)$ by

$$\mathcal{H}(0,y) = t^{1/3} f^{(t)}(t^{-2/3}y) \iff t^{-1/3}\mathcal{H}(0,t^{2/3}y) = f^{(t)}(y). \tag{4}$$

Let $\mathcal{H}(t,x)$ be the solution to the KPZ equation at time t with this t-dependent initial data, and define the scaled solution $\mathfrak{h}^{t,f}$ by

$$\mathfrak{h}^{t,f}(x) = \frac{\mathcal{H}(t,t^{2/3}x) + \frac{t}{12} - \frac{2}{3}\log t}{t^{1/3}}.$$

This scaling of the solution, as well as the initial condition in (4), is convenient since, for example, if $f^{(t)} = f$ for a given function f for all t, then $\mathfrak{h}^{t,f}$ converges in distribution as $t \to \infty$ to the KPZ fixed point at time 1 started from initial condition f by [QS22, Vir20] (see the following page for a brief discussion of this object).

We next list the conditions we impose on the initial data. Essentially the conditions ensure that the solution does not grow too quickly and there is at least some amount of the domain where it is not too negative; otherwise, one runs into pathological settings such as the solution exploding immediately. The form of the assumptions are somewhat standard in the literature by now, and, for example, are similar to but slightly weaker than those adopted in [CH16].

The first thing we require is that f essentially grows at most like x^2. This comes from matching the decay of $\mathfrak{h}^t(x)$, i.e., $-x^2$, since, heuristically, if $\mathcal{H}(0,x)$ grows quadratically in x as $x \to \infty$, then $\mathcal{H}(t,\cdot)$ will blow up at some finite $t > 0$, while a growth faster than quadratic will lead to immediate blow-up. However, because we vary $\mathcal{H}(0,\cdot)$ with t, we in fact need to also limit the coefficient of the quadratic growth to ensure blow up does not occur immediately. More precisely, the time of blow-up T_0 can be seen to be the smallest t such that $\sup_{y\in\mathbb{R}} \mathcal{H}(0,y) - y^2/t = \infty$; using the relation (4), the latter condition is equivalent to $\sup_{y\in\mathbb{R}} f^{(t)}(y) - y^2 = \infty$.

The following is the precise form of the initial data conditions.

Definition 1.1. For K, L, M $\delta > 0$ we say that a function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ satisfies hypothesis $\mathrm{Hyp}(K, L, M, \delta)$ if

- $f(x) \leq x^2 - L|x| + K$ for all $x \in \mathbb{R}$;
- $\mathrm{Leb}\{x \in [-M, M] : f(x) \geq -K\} \geq \delta$ where Leb denotes Lebesgue measure.

Since the form of our initial condition is (4), $f^{(t)} \in \mathrm{Hyp}(K, L, M, \delta)$ for all $t < T_0$ for some $T_0 > 0$ will imply that $\mathfrak{h}^{t,f}$ is defined for all $t < T_0$, since the extra linear term in the first bullet point is enough to handle the random fluctuations beyond the parabola.

Theorem 7 (One-point upper tail bounds for general initial data)**.** *Let $T_0 \in (t_0, \infty]$ and $f^{(t)} \in \mathrm{Hyp}(K, L, M, \delta)$ for some fixed K, L, M, $\delta > 0$ for all $t \in [t_0, T_0)$. There exist $\theta_0 > 0$ and $C < \infty$ such that, for $t \in [t_0, T_0)$ and $\theta > \theta_0$,*

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \leq \mathbb{P}\Big(\mathfrak{h}^{t,f}(0) \geq \theta\Big) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

In fact, as the proofs will show, the upper and lower bounds in Theorem 7 require different parts of the hypotheses in Definition 1.1. For this reason, Theorems 9.1 and 9.2, appearing later in Section 9, separate the two bounds and are more precise about which hypotheses are needed for each.

Unlike in the narrow-wedge case, we have not stated a bound on the one-point densities for general initial data. Such a result would require a more complicated argument, though it is possible that a suitable extension of the narrow-wedge argument would yield it; we discuss this more in Remark 9.4.

As we mentioned, Theorem 7 is proved using a convolution formula which relates $\mathfrak{h}^{t,f}$ with $\mathfrak{h}^t$. This formula was also used in [CG20a] to show uniform-in-t upper tail bounds of the form $\exp(-c\theta^{3/2})$ for general initial data—i.e., the correct tail exponent, but not the sharp coefficient. The lack of sharpness is mainly because the procedure they adopt to go from narrow-wedge solution to the general solution was lossy. We take a different and somewhat simpler approach which allows us to obtain sharp asymptotics.

Our arguments extend easily to the zero temperature analogue for general initial data mentioned above, the KPZ fixed point $\mathfrak{h}^{\mathrm{FP},f}(t, \cdot)$ (where f is the initial condition), which we spend a little time introducing now. This is a Markov process in t, in a certain function space, constructed in [MQR21] and given a variational description in [NQR20] in terms of the directed landscape constructed in [DOV18]. It has recently been proven to be the $t \to \infty$ limit of the KPZ equation under general initial data [QS22, Vir20]; for example, the parabolic Airy_2 process is the KPZ fixed point started from narrow-wedge initial data, and so is the limit of the KPZ equation from the same data.

For our purposes, it will be easier to define the KPZ fixed point via the variational description than the mentioned limit; this is because the limits have not been proven for initial conditions that grow quadratically that we also want to include. Though the variational formula uses the directed landscape, we will only be interested in $\mathfrak{h}^{\mathrm{FP},f}$ for a fixed time and space location, for which the formula can be given in terms of the parabolic Airy_2 process: for each fixed $x \in \mathbb{R}$, in distribution,

$$\mathfrak{h}^{\mathrm{FP},f}(1, x) = \sup_{y \in \mathbb{R}}\Big(\mathcal{P}(y) + f(y + x)\Big). \tag{5}$$

One can get a formula for $\mathfrak{h}^{\mathrm{FP},f}(t, x)$ from the above by a rescaling invariance (see [MQR21, Theorem 4.5]): $\mathfrak{h}^{\mathrm{FP},f}(t, x) \stackrel{d}{=} t^{1/3}\mathfrak{h}^{\mathrm{FP},f^{\{t\}}}(1, t^{-2/3}x)$, where $f^{\{t\}}(x) = t^{-1/3}f(t^{2/3}x)$.

The following result records the same upper tail bounds for $\mathfrak{h}^{\mathrm{FP},f}$ as for the KPZ equation under general initial data; a similar upper bound without quantitative error terms was also given in [MQR21, Proposition 4.7] using integrable methods.

Theorem 8 (One-point upper tail bounds for KPZ fixed point)**.** *Let K, L, M, $\delta > 0$ be fixed, and suppose that $f^{\{t\}} \in \mathrm{Hyp}(K, L, M, \delta)$ for some $t > 0$. There exist $\theta_0 > 0$ and $C < \infty$ (depending only on the fixed constants and not t) such that, for $\theta > \theta_0$,*

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \le \mathbb{P}\Big(\mathfrak{h}^{\mathrm{FP},f}(t,0) \ge \theta t^{1/3}\Big) \le \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

Remark 1.2. We do not explicitly prove Theorem 8, but the proof can be done quickly: one just mimics the proof of Theorem 7 presented in Section 9 but with the zero-temperature version of the convolution formula mentioned above (i.e., (5)) with the parabolic Airy$_2$ process in place of the narrow-wedge KPZ equation solution. The result for $\mathfrak{h}^{\mathrm{FP},f}(0,t)$ for $t \neq 1$ is obtained by the mentioned scaling properties of the KPZ fixed point.

1.6. **Main results on limit shapes.** We now move on to more geometric statements, i.e., about the form of $\mathfrak{h}^t$ on the respective upper tail events, and which, as it turns out, serve as crucial ingredients in the proofs of the results on asymptotics of upper tail event probabilities stated in the previous subsection.

1.6.1. *One-point limit shape.* Define $\mathsf{Tri}_\theta : [-\theta^{1/2}, \theta^{1/2}] \to \mathbb{R}$ (Tri_θ is short for triangle) by

$$\mathsf{Tri}_\theta(x) = -2\theta^{1/2}|x| + \theta.$$

This is the function obtained by considering the two tangents to the parabola $-x^2$ which pass through $(0, \theta)$; see Figure 2. It is an easy computation that the tangents touch the parabola at $(\pm\theta^{1/2}, -\theta)$, which gives Tri_θ.

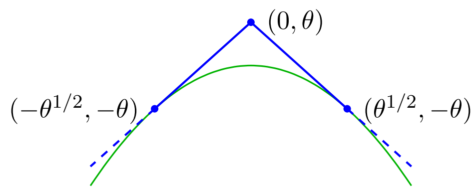


FIGURE 2. The blue solid curve is $\mathsf{Tri}_\theta : [-\theta^{1/2}, \theta^{1/2}] \to \mathbb{R}$.

Theorem 9 (One-point limit shape)**.** *There exist $C < \infty$, $c > 0$, and θ_0 such that, for all $t \in [t_0, \infty]$, $\theta > \theta_0$, and $0 < M < C^{-1}\theta^{3/4}$,*

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},\theta^{1/2}]} |\mathfrak{h}^t(x) - \mathsf{Tri}_\theta(x)| \ge M\theta^{1/4} \;\middle|\; \mathfrak{h}^t(0) = \theta\right) \le \exp(-cM^2).$$

While it may seem slightly odd that we impose an upper bound on M, this is sufficient for our purposes. Moreover, for all M large enough, the probability should behave as $\exp(-c(M\theta^{1/4})^{3/2})$, and it is easy to check that the transition from the Gaussian tail to the former happens when M is of order $O(\theta^{3/4})$. We will in fact impose similar upper bounds on M in many places in the paper so as to ensure that we work only with Gaussian bounds.

The scale of fluctuations, $\theta^{1/4}$, is sharp, at least in the bulk of $[-\theta^{1/2}, 0]$ and $[0, \theta^{1/2}]$ owing to the diffusive nature of Brownian fluctuations. However, the fluctuations are expected to decrease significantly at $\pm\theta^{1/2}$ and beyond, where they should be $O(1)$. While more work using bootstrapping ideas relying on the asymptotics from Theorem 2 should yield some improvement, we don't pursue them to keep things somewhat less technical.

A natural question is what the profile looks like beyond $[-\theta^{1/2}, \theta^{1/2}]$. From the heuristic described later in Section 1.7 one expects it to be close to the parabola $-x^2$, and this is indeed the case; see Proposition 4.2 for details.

Our last result concerns the limit shape under two-point upper tail events.

1.6.2. *Two-point limit shape.* For the two-point limit shape, recall the definition and illustration of $\mathsf{ConHull}_{a,b}$ from Section 1.5.2 and Figure 1. Also define I_{lin} to be the biggest closed set on which $\mathsf{ConHull}_{a,b}$ is piecewise linear (recall that the domain of $\mathsf{ConHull}_{a,b}$ is such that $\mathsf{ConHull}_{a,b}$ is tangent to $-x^2$ at the boundaries); looking at Figure 1 we see that I_{lin} is either one or the union of two closed intervals. Our shape theorem will be restricted to I_{lin}. While it is possible to replace I_{lin} by the entire domain of $\mathsf{ConHull}_{a,b}$, it is technically slightly more cumbersome and not needed to obtain the two-point estimates, and so we do not do it. See Remark 6.4 for a more detailed discussion.

Theorem 10 (Two-point limit shape)**.** *Let $\theta > 0$ and $a \geq b > -1$. For $M > 0$, let $M_{a,b} = M[(1+a)^{1/4} + (1+b)^{1/4}]$. There exist $c > 0$, $C < \infty$, θ_0, and $a_0 = b_0$ such that, if $t \in [t_0, \infty]$, $\theta > \theta_0$ or $a, b \geq a_0, b_0$, and for $0 < M \leq C^{-1}[(1+a)^{3/4} + (1+b)^{3/4}]\theta^{3/4}$,*

$$\mathbb{P}\left(\sup_{x \in I_{\mathrm{lin}}} \mathfrak{h}^t(x) - \mathsf{ConHull}_{a,b}(x) \geq M_{a,b}\theta^{1/4} \;\middle|\; \mathfrak{h}^t(-\theta^{1/2}) = a\theta, \mathfrak{h}^t(\theta^{1/2}) = b\theta\right) \leq \exp(-cM^2).$$

This completes the statements of our main results. Before turning to the proof ideas we remark that while our results are focused on sharp upper tail asymptotics and limit shapes conditioned on upper tail events, there has been a considerable amount of fruitful research done in a number of other directions regarding the KPZ equation. These include integrable formulas [BG16, Gho18], lower tail estimates and large deviations [CG20b, Tsa18, CCR21, CG20a, CC22], studies of correlation and fractal structure of $t \mapsto \mathfrak{h}^t(0)$ [CGH21, DG21], connections to the Kadomtsev–Petviashvili equation [QR19, LD20], and investigations of Brownian regularity [Wu21b, Wu21a], to give a taste.

1.7. **Proof outline.**

The proofs involve several new ideas. Since the formal implementation of them takes a while, to facilitate readability, we include a reasonably detailed description of the key ideas in this section. The subsections can more or less be read independently.

As indicated earlier, the central tool driving our technique is the Brownian resampling properties enjoyed by line ensembles (infinite families of random continuous curves) into which $\mathfrak{h}^t$ for any $t \in (0, \infty]$ can be embedded. We start by discussing these properties in more detail.

1.7.1. *The Brownian Gibbs property.* [CH16] constructed the KPZ line ensemble into which $\mathfrak{h}^t$ embeds (one ensemble for each fixed $t > 0$). The zero temperature case of $t = \infty$ is known as the parabolic Airy line ensemble and was constructed earlier in [CH14].

Thus to avoid notational confusion, in the remainder of the paper we will denote the narrow-wedge solution to the KPZ equation by $\mathfrak{h}_1^t$ and the parabolic Airy_2 process by $\mathcal{P}_1$, and let $\mathfrak{h}^t = (\mathfrak{h}_1^t, \mathfrak{h}_2^t, \ldots)$ and $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots)$ denote the respective line ensembles as a whole.

We next explain the key advantage in considering the line ensembles as a whole instead of their top lines: the resampling properties, called *Brownian Gibbs properties* or simply, Gibbs properties, they enjoy. Let us first describe this in the zero-temperature case of $\mathcal{P}$. We fix $k \in \mathbb{N}$ and an interval $[a, b]$, and condition upon everything in the line ensemble outside of the top k curves on $[a, b]$; in other words, on $\{\mathcal{P}_i(x) : 1 \leq i \leq k, x \notin [a, b] \text{ or } i \geq k+1, x \in \mathbb{R}\}$. The Brownian Gibbs property says that the conditional distribution of $\mathcal{P}_1, \ldots, \mathcal{P}_k$ on $[a, b]$ is given by k independent Brownian bridges of rate *two*, the i^{th} one from $(a, \mathcal{P}_i(a))$ to $(b, \mathcal{P}_i(b))$, conditioned on non-intersection (with each other, and the lower boundary curve $\mathcal{P}_{k+1}$). That the Brownian bridges are rate two is just due to a convention established when the parabolic Airy_2 process was defined in [PS02], but must be kept track of to obtain the correct coefficients in tail asymptotics, as we do.

In the positive temperature case, the conditional distribution after conditioning on the same data for $\mathfrak{h}^t$ can be described similarly in terms of independent rate two Brownian bridges; however here, instead of the hard non-intersection constraint, intersections are energetically penalized in terms of an explicit Radon-Nikodym derivative. The precise expression is somewhat technical and will not be particularly illuminating at this stage, so we defer it to Section 2.

To convey the main ideas, in the remainder of this section we will describe our arguments for $\mathcal{P}$, as its non-intersection resampling is easier to reason about; we will address the differences encountered when working with the positive temperature Gibbs property after outlining the upper tail argument for $\mathcal{P}$. However, to minimize the notation, as in the statements of the main results, we will be using $\mathfrak{h}^t$ instead of $\mathcal{P}$, since $\mathcal{P} = \mathfrak{h}^{t=\infty}$ (combining results of [Wu21b, DM21] and [QS22, Vir20]).

We will primarily focus on the arguments for the one-point case, i.e., tail asymptotics and the shape of the corresponding conditioned profile of the top curve, and say a few words about obtaining the density estimates, as these already contain most of the new ideas in our work; the two-point analogue of these results serve to illustrate the power of the method but do not introduce any substantially new ingredients, while the asymptotics for general initial data rely on refining arguments used in previous works such as [CH16] and [CG20a].

1.7.2. *Three more basic tools: monotonicity, positive association, and control on a conditioned second curve.* We start be specifying the exact properties of the ensembles that we will make use of. In fact, all of our proofs are carried out under certain assumptions we formulate in Section 2.2. This allows our arguments to apply both to the cases of the KPZ and parabolic Airy line ensembles as well the case of extremal stationary ensembles, by simply verifying that all of them satisfy the assumptions.

The first property we use is the Brownian Gibbs property discussed in the previous subsection. The second is that of *positive association* and the FKG inequality, discussed around Theorem 4. That these ensembles possess the FKG inequality is a consequence of the fact that the same holds for Brownian bridges under the hard non-intersection constraint or subject to the Radon-Nikodym derivative in the positive temperature case (which, for brevity, will be termed henceforth as the soft constraint). A crucial ingredient in the proof of this is a certain *monotonicity* property described next, which we make extensive use of in our arguments.

Recall that the Brownian bridge ensembles (under hard or soft constraints) are defined given some boundary data: an lower and/or upper boundary curve, and the values of the Brownian bridges at the boundaries of the interval. The monotonicity property is simply that the law of these Brownian ensembles stochastically increases if the boundary data increases; for example, if the boundary values are kept the same but the lower curve f_1 is increased to f_2 (i.e., $f_1(\cdot) \leq f_2(\cdot)$), there is a coupling of the bridge ensembles such that the bridges associated to the lower curve f_1 are lower than those associated to f_2. And similarly for ordered boundary values. These monotonicity properties were first proven in [CH14] and [CH16] (see also [DM21, Dim21] for more detailed proofs) and have proven indispensable in studies of line ensembles (eg. in [Ham22, Ham19a, Ham20, Ham19b, Wu21a, CHH19, DV21, DOV18]); similar monotonicity properties in other models with nice resampling properties have likewise proved important (eg. [Agg20]).

Let us say a few words on how to go from the positive association property for the Brownian bridge ensembles to the same for the extremal stationary ensembles. Essentially we apply the Gibbs property to a sequence of increasing domains, eg. $[-k, k] \times \{1, \ldots, k\}$ (resampling the top k curves on $[-k, k]$) and use that we know the required positive association for the resulting conditional laws. Extremality is equivalent to the corresponding tail σ-algebras being trivial (see Section 8), and so we obtain the unconditioned statement we desire by taking $k \to \infty$. The extremality assumption is invoked essentially only in carrying out the above and related correlation and monotonicity-in-conditioning arguments described next, which are the final properties needed for our arguments.

When applying the Gibbs property to the top curve we will often need to control the second curve and essentially demand that it still behaves typically, even under the conditioning that $\mathfrak{h}^t_1(0) \geq \theta$ (or other increasing events). To ensure this, we use that the law of such a conditioned $\mathfrak{h}^t_2$ is dominated by the *unconditioned* law of $\mathfrak{h}^t_1$. We will call this the *van den Berg-Kesten (BK) inequality.*[1]

A heuristic to understand why this should be the case is to see that, under the Brownian Gibbs property, $\mathfrak{h}^t_1$ is essentially an upper boundary for $\mathfrak{h}^t_2$. By monotonicity properties of the Gibbs property, as the upper boundary increases (eg. if θ gets larger in the one-point upper tail conditioning event), $\mathfrak{h}^t_2$ gets stochastically larger; so the law of $\mathfrak{h}^t_2$ conditional on $\{\mathfrak{h}^t_1(0) \geq \theta\}$ is stochastically increasing in θ. But in the $\theta \to \infty$ limit, the upper boundary goes to ∞ and disappears, and in that case $\mathfrak{h}^t_2$ can be thought of as becoming an unconditioned $\mathfrak{h}^t_1$ as the latter also has no upper boundary. It is this basic reasoning that allows us to control the lower curve $\mathfrak{h}^t_2$ in all our arguments.

The final tool we need is also a monotonicity statement, but this time in the conditioning variable. It states that the law of $\mathfrak{h}^t_1$, conditional on its values at given points, is stochastically monotone in those values. This will be needed, for example, to compare the conditional law given $\mathfrak{h}^t_1(0) = \theta$ to the same given $\mathfrak{h}^t_1(0) \geq \theta$; the usefulness of such comparisons is that the latter event is increasing, and so is amenable to applying the positive association (FKG) inequality or the above monotonicity properties. The proofs again go via proving the same for Brownian bridge ensembles and, for the case of the extremal ensembles, making use of the triviality of the tail σ-algebra.

It is worth emphasizing that these conditional monotonicity statements are *not* implied by the monotonicity statements mentioned above such as the FKG inequality. One way to see this is by noting that conditionings like $\mathfrak{h}^t_1(0) = \theta$ are not monotone events. We expect that, similar to the monotonicity in boundary data statements, these monotonicity in conditioning statements may prove to be useful in many contexts.

We now move on to the key ideas in the paper.

1.7.3. *Obtaining sharp one-point asymptotics.* We first sketch how to obtain the upper-tail one-point asymptotics of $\exp(-\frac{4}{3}\theta^{3/2}(1 + o(1)))$. The key ideas appear mostly in the proof of Theorem 9 on the shape of the top curve under the conditioning that $\mathfrak{h}^t_1(0) = \theta$, which we assume for the moment. We will also rely on a priori one-point upper tail bounds from [CG20a] as in (3).

First we observe why the exponent of θ is $\frac{3}{2}$. Indeed, by Theorem 9, conditionally on $\{\mathfrak{h}^t_1(0) = \theta\}$, $\mathfrak{h}^t_1(\pm\theta^{1/2})$ is with high probability close to $-\theta$ and hence, on a horizontal interval of scale $\theta^{1/2}$, $\mathfrak{h}^t_1$ oscillates upwards by order θ. If we believe, based on the Gibbs property, that $\mathfrak{h}^t_1$ behaves roughly like a Brownian bridge on the interval $[-\theta^{1/2}, \theta^{1/2}]$ (i.e., assuming the upward push it receives from the lower boundary of the second curve is not too substantial), then the tail is predicted to have a Gaussian form: $\exp(-O(\frac{\theta^2}{\theta^{1/2}})) = \exp(-O(\theta^{3/2}))$.

Following this logic more precisely, and taking into account the effect of the second curve, gives the correct coefficient of $\frac{4}{3}$ for $\theta^{3/2}$. We explain this next.

The upper bound for the one-point tail: Let us pretend that we know that the shape from Theorem 9 holds even if we condition on $\{\mathfrak{h}^t_1(0) \geq \theta\}$ instead of $\{\mathfrak{h}^t_1(0) = \theta\}$ (this is indeed the case, but we do not explicitly prove it); this implies that $\mathfrak{h}^t_1(\pm\theta^{1/2}) \leq -\theta + M\theta^{1/4}$ with probability at least $\frac{1}{2}$, conditionally on $\{\mathfrak{h}^t_1(0) \geq \theta\}$, for large enough M (independent of θ). To simplify the quantities, in the rest of the discussion we will assume that $\mathfrak{h}^t_1(\pm\theta^{1/2}) \leq -\theta$.

As already indicated, ultimately, we wish to apply the Brownian Gibbs property on $[-\theta^{1/2}, \theta^{1/2}]$. This says that, conditional on $\mathfrak{h}^t_1$ on $[-\theta^{1/2}, \theta^{1/2}]^c$ and on $\mathfrak{h}^t_2, \mathfrak{h}^t_3, \ldots$ on $\mathbb{R}$, the distribution of $\mathfrak{h}^t_1$ on $[-\theta^{1/2}, \theta^{1/2}]$ is a Brownian bridge from $(-\theta^{1/2}, \mathfrak{h}^t_1(-\theta^{1/2}))$ to $(\theta^{1/2}, \mathfrak{h}^t_1(\theta^{1/2}))$, conditioned on it lying

[1]The original BK inequality comes from percolation theory, see for example [Gri99], to bound the probability of two events occurring "disjointly" by the product of the probabilities of the events. We use the same terminology here because, by the RSK bijection, the top two curves of the zero temperature line ensemble can be related to weights of pairs of disjoint paths in limiting last passage percolation models, and in this context many cases of the inequality we describe are applications of the classical BK inequality.

above $\mathfrak{h}_2^t$. So there are three pieces of boundary data: $\mathfrak{h}_1^t(-\theta^{1/2})$, $\mathfrak{h}_1^t(\theta^{1/2})$, and $\mathfrak{h}_2^t$. The previous paragraph controls the side values $\mathfrak{h}_1^t(\pm\theta^{1/2})$. What remains is the lower boundary curve $\mathfrak{h}_2^t$.

We obtain this control via the BK inequality from the last subsection, which says that that $\mathfrak{h}_2^t$, conditional on $\mathfrak{h}_1^t \geq \theta$, is stochastically smaller than an unconditioned copy of $\mathfrak{h}_1^t$. So we can use a priori bounds on $\mathfrak{h}_1^t$ to control $\sup_{x\in[-\theta^{1/2},\theta^{1/2}]}\mathfrak{h}_2^t(x)+x^2$ under the conditioned event and conclude that it lies below the parabola $x \mapsto -x^2$ (plus a poly-logarithmic in θ constant which, owing to being sub-polynomial, can be safely ignored) with high probability under the conditioning that $\mathfrak{h}_1^t(0) \geq \theta$.

With this setup, we may apply the Brownian Gibbs property on $[-\theta^{1/2},\theta^{1/2}]$ for the top curve. Now, we are trying to upper bound the probability of the increasing event that $\mathfrak{h}_1^t(0) \geq \theta$, and intuitively this probability increases if we raise the boundary data. Thus we may take the boundary data to be as large as possible, i.e., $(\pm\theta^{-1/2}, -\theta)$ at the sides and the function $x \mapsto -x^2$ below.

This yields, with B a rate two Brownian bridge from $(-\theta^{1/2}, -\theta)$ to $(\theta^{1/2}, -\theta)$, that

$$\begin{aligned}
\mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta\right) &\leq \mathbb{P}\left(B(0) \geq \theta \;\middle|\; B(x) \geq -x^2 \quad \forall x \in [-\theta^{1/2}, \theta^{1/2}]\right)\\
&\leq \frac{\mathbb{P}\left(B(0) \geq \theta\right)}{\mathbb{P}\left(B(x) \geq -x^2 \quad \forall x \in [-\theta^{1/2}, \theta^{1/2}]\right)}.
\end{aligned} \tag{6}$$

Now $B(0)$ is a normal random variable with mean $-\theta$ and variance $2 \times \frac{\theta^{1/2}\times\theta^{1/2}}{2\theta^{1/2}} = \theta^{1/2}$. So, using the form of the Gaussian tail, the numerator is at most

$$\exp(-(\theta+\theta)^2/2\theta^{1/2}) = \exp(-2\theta^{3/2}).$$

The denominator turns out to be lower bounded by $\exp(-\frac{2}{3}\theta^{3/2}(1+o(1)))$ which follows from a straightforward but tedious Brownian computation involving controlling the Brownian bridge at a fine mesh of points in Proposition 3.8. Plugging the above estimates into (6) yields the upper bound of $\exp(-\frac{4}{3}\theta^{3/2})$. Interestingly, a matching upper bound on the denominator can be proved more straightforwardly using tail bounds of a Gaussian observable; see Remark 3.10.

The actual argument has a few more complications than the sketch above, mainly arising from working carefully with the monotonicity properties available to us which we will refrain from explaining further at this point. To obtain the density estimates of Theorem 1 from these tail bounds, it is enough to prove a degree of regularity for the density. This is done in Proposition 5.6 using resampling arguments that we also omit explaining here.

Moving from zero to positive temperature: As mentioned above, in the positive temperature case ($t < \infty$), the Gibbs property involves a softer constraint of Brownian bridges subject to a Radon-Nikodym derivative which energetically penalizes but does not prohibit intersection.

While we postpone giving the precise form of the Radon-Nikodym derivative to Section 2, it can be written as W_{H_t}/Z_{H_t}, where $W_{H_t} \in [0,1]$ and Z_{H_t} is a normalization constant so that the ratio is a probability density. Z_{H_t} is called the *partition function* and it is a deterministic function of the boundary data (i.e., as above, side values and lower curve); H_t is called the Hamiltonian and is needed to specify the precise form of W_{H_t} and Z_{H_t}. The analogue of Z_{H_t} in the zero-temperature, $t = \infty$ case above is the probability of the Brownian bridge staying above the second curve, and W_{H_t} is the corresponding indicator.

We saw above that this probability appeared in the denominator when we were trying to estimate the probability that $\mathfrak{h}_1^t(0) \geq \theta$. Essentially the only difference when working with the positive temperature case is that we need to replace a non-intersection probability by the analogous partition function in all the estimates and this involves lower bounding Z_{H_t} with a parabolic lower boundary. For this, we have a simple lemma (Lemma 3.11) that allows us to transfer lower bounds in the zero temperature case (such as the mentioned Proposition 3.8) to the positive temperature case.

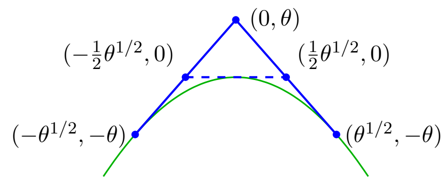


FIGURE 3. The setup for the argument for the lower bound on the upper tail. The interval on which we resample is now $[-\frac{1}{2}\theta^{1/2},\frac{1}{2}\theta^{1/2}]$; note that the boundary points are such that Tri_θ equals zero at them, and so the line connecting $(x,\mathsf{Tri}_\theta(x))$ and $(-x,\mathsf{Tri}_\theta(-x))$ when $x=\frac{1}{2}\theta^{1/2}$ is tangent to $-x^2$. Thus the Brownian bridge defined between these points will avoid $-x^2$ with constant probability, and the FKG inequality will be essentially sharp in lower bounding $\mathbb{P}(\mathfrak{h}_1^t(-\frac{1}{2}\theta^{1/2}),\mathfrak{h}_1^t(\frac{1}{2}\theta^{1/2})\geq 0)$.

The lower bound for the one-point tail: A lower bound could be argued on similar lines to the upper bound, but for one issue: this time, we would want to say that the lower boundary condition (i.e., $\mathfrak{h}_2^t$) does not go too low since, as already indicated, to get the sharp $\exp(-\frac{4}{3}\theta^{3/2})$ bound one has to consider the probability of avoiding the lower boundary. This in turn would require an estimate on the *lower* tail of $\mathfrak{h}_2^t(0)$ and $\inf_{[-\theta^{1/2},\theta^{1/2}]}\mathfrak{h}_2^t(x)$. While lower tail estimates are available for $\mathfrak{h}_2^t(0)$ in both the positive temperature [CG20b, CCR21] and zero temperature [TW94] cases via integrable arguments, and may be upgraded to lower tails for the infimum over an interval by resampling arguments (see [Ham22, Proposition A.2] and [Wu21b, Proposition B.1]), these estimates would not be available for the extremal stationary ensembles that we also wish to make statements about. As mentioned after Theorem 6, it appears quite difficult and interesting to obtain lower tail bounds via the Gibbs property and not relying on integrable inputs.

Observe, however, that the *tightness-in-t* of $\mathfrak{h}_1^t(0)$ on the lower tail side is available for all the ensembles we consider. While on first glance this appears quite weak, our actual argument makes use of this one-point tightness (and so also applies to extremal ensembles) along with a bootstrapping procedure. In fact, the argument initially appears to completely ignore the lower boundary and requires no explicit control on it. For this reason it directly applies to both positive and zero temperature ensembles with no need to transfer non-intersection probability bounds to partition functions. Let us sketch the argument.

We consider the following method of obtaining that $\mathfrak{h}_1^t(0)\geq\theta$: we demand that $\mathfrak{h}_1^t(\pm\frac{1}{2}\theta^{1/2})\geq 0$ and resample $\mathfrak{h}_1^t$ on $[-\frac{1}{2}\theta^{1/2},\frac{1}{2}\theta^{1/2}]$, i.e., we apply the Brownian Gibbs property to $[-\frac{1}{2}\theta^{1/2},\frac{1}{2}\theta^{1/2}]$. Because we are lower bounding the probability of an increasing event, our bound is only worsened by lowering the boundary data: so we may take $\mathfrak{h}_1^t(\pm\frac{1}{2}\theta^{1/2})=0$ and take the second curve to $-\infty$ on $[-\frac{1}{2}\theta^{1/2},\frac{1}{2}\theta^{1/2}]$, i.e., have no lower boundary.

Observe that $\mathbb{P}(\mathfrak{h}_1^t(\frac{1}{2}\theta^{1/2})\geq 0)=\mathbb{P}(\mathfrak{h}_1^t(0)\geq\theta/4)$ by stationarity, and similarly for $\mathfrak{h}_1^t(-\frac{1}{2}\theta^{1/2})$. Letting $p_t(\theta)=\mathbb{P}(\mathfrak{h}_1^t(0)\geq\theta)$—which we will refer to as $p(\theta)$—we see that, by positive association, $\mathbb{P}(\mathfrak{h}_1^t(\pm\frac{1}{2}\theta^{1/2})\geq 0)\geq p(\theta/4)^2$. Letting B be a rate two Brownian bridge from $(-\frac{1}{2}\theta^{1/2},0)$ to $(\frac{1}{2}\theta^{1/2},0)$ and using the form of the Gaussian tail, the above discussion shows that

$$p(\theta)\geq p(\theta/4)^2\cdot\mathbb{P}(B(0)\geq\theta)=p(\theta/4)^2\cdot\exp\left(-\frac{\theta^2}{2\times\frac{1}{2}\theta^{1/2}}\right)=p(\theta/4)^2\cdot\exp\left(-\theta^{3/2}\right),$$

the last factor since $B(0)$ has mean 0 and variance $2\times\frac{\frac{1}{2}\theta^{1/2}\times\frac{1}{2}\theta^{1/2}}{\theta^{1/2}}=\frac{1}{2}\theta^{1/2}$.

Somewhat surprisingly, on iteration, this yields a lower bound of $\exp(-\frac{4}{3}\theta^{3/2}(1+o(1)))$ (it is easy to check $p(\theta)=\exp(-\frac{4}{3}\theta^{3/2})$ satisfies the recurrence) provided that the recurrence is non-trivial, i.e., if one knows that $p(\theta)>0$ for all $\theta>0$. Further to get a result uniform in t, one needs to ensure that the recurrence's initial condition is uniform in t. This can shown by a very similar argument.

The choice of demanding that $\mathfrak{h}_1^t(x) \geq 0$ for $x = \pm\frac{1}{2}\theta^{1/2}$ may appear mysterious, but can be understood using the geometric picture being developed; see Figure 3. Indeed, the choice is made by combining the information on the shape of the profile under the conditioning that $\mathfrak{h}_1^t(0) \geq \theta$ from Theorem 9 (note that $(\pm\frac{1}{2}\theta^{1/2}, 0)$ lie on Tri_θ), the information from Theorem 4 on when the FKG inequality will be sharp (the line joining $(\pm\frac{1}{2}\theta^{1/2}, 0)$ is tangent to $-x^2$), and the intuition that a Brownian bridge starting from height zero will essentially not be affected by the lower boundary $x \mapsto -x^2$ and so will allow the boundary to be safely ignored. Note however that, as we saw above, while we use these mentioned theorems to inform our choice of $x = \pm\frac{1}{2}\theta^{1/2}$, the argument does not actually apply these theorems.

Having explained the arguments for the one-point asymptotics for the KPZ equation, let us end this section by indicating how to go from them to the general initial data tail bounds. Essentially, we combine the convolution formula mentioned above, which expresses $\mathfrak{h}^{t,f}(0)$ in terms of the entire process $\mathfrak{h}_1^t$, with new sharp upper tail asymptotics for statistics like $\sup_{x\in\mathbb{R}} \mathfrak{h}^t(x) + (1-\varepsilon)x^2$ which we obtain from the one-point asymptotics using resampling arguments.

We next turn to sketching the proofs of the shapes of the top curve under the conditionings $\{\mathfrak{h}_1^t(0) = \theta\}$ (which, as already evident, is the main ingredient in the proof of the one point estimate) and $\{\mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta\}$. We will then outline a remaining step in the argument for the extremal stationary ensembles, which relies on an extension of such arguments.

1.7.4. *A heuristic for the shape of the top curve under conditionings.* The basic idea driving the proofs of the shapes of the profile under the two conditionings considered in Theorems 9 and 10 is the following: that Brownian bridges approximately follow the line connecting their endpoints implies that ensembles with a resampling property in terms of Brownian bridges must have shapes that are approximately convex. The argument is fleshed out in Figure 4.

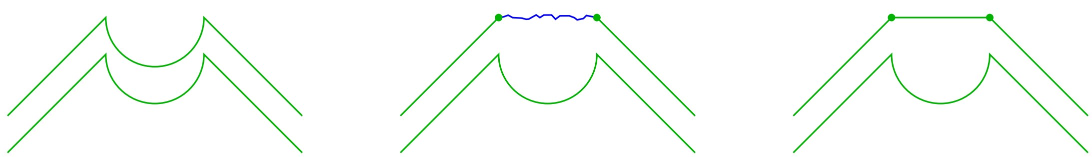


FIGURE 4. On the left panel we consider the situation where the limit shape for the top curve is non-convex on some interval; the non-intersection condition then pushes the second curve down on the same interval. In the second panel we resample the top curve on the interval of non-convexity. The Brownian bridge would typically approximately follow the straight line between its endpoints if it was unconditioned; here it is conditioned to avoid the second curve, but, since the second curve is already lower, the avoidance conditioning is met by the bridge's typical behaviour. Thus as we see in the third panel, the resampling removes the non-convexity from the top curve, thus contradicting the initial non-convexity, since resampling preserves the distribution of the ensemble.

Similar reasoning leads to the statement that the shape of the top curve must be approximately the convex hull (i.e., minimal convex shape above) of the lower curve and constraints imposed by conditioning. This explains the shapes which arise in Theorems 9 and 10, which are indeed respectively the convex hull of $x \mapsto -x^2$ (representing the lower curve) and $(0, \theta)$ (due to the conditioning that $\mathfrak{h}_1^t(0) = \theta$) and the convex hull of $x \mapsto -x^2$, $(-\theta^{1/2}, a\theta)$, and $(\theta^{1/2}, b\theta)$.

Similar observations formed the basis of work by Aggarwal in [Agg20] to obtain the limit shape of the arctic boundary in the six-vertex model at the ice point which admits an encoding in terms of non-intersecting random walks, thus providing mathematical justification of the *tangent method* heuristic introduced by Colomo-Sportiello [CS16].

1.7.5. *Moving from heuristic to the proof of Theorem 9.* Now we sketch the actual arguments we develop to establish Theorem 9 (similar arguments, though with some technical complications, suffice for Theorem 10 on the two-point limit shape as well but we do not discuss this).

Let us focus on proving the shape of the top curve is Tri_θ on the *left* side of 0 only, i.e., in $[-\theta^{1/2}, 0]$; the argument for the right side will clearly be symmetric.

Consider an extension of Tri_θ to $(-\infty, 0]$ given by the same line, i.e., by $x \mapsto \theta + 2\theta^{1/2}x$; see the dotted lines in Figure 2. The crucial point is the following: if we can find two x-coordinates at which $\mathfrak{h}_1^t$ is on the extended version of Tri_θ, then the Brownian Gibbs property gives an approximate linear resampling showing that $\mathfrak{h}_1^t$ will be approximately equal to Tri_θ on the entire interval in between these two points. Then the resampling will be approximately linear in spite of the lower boundary condition (the second curve which must be avoided) because the latter is approximately the inverted parabola $x \mapsto -x^2$. So, as a consequence of the convex hull property of Tri_θ, it is *tangent* to $x \mapsto -x^2$ at the point $x = -\theta^{1/2}$; thus, intuitively, the unconditioned Brownian bridge between two points on Tri_θ should avoid $x \mapsto -x^2$ with uniformly positive probability (as can be checked), and so the conditioned and unconditioned processes can be treated as essentially the same.

So we have to find two points at which $\mathfrak{h}_1^t$ is equal to Tri_θ. We will refer to these as *pinning* points. We already have one pinning point, the one to the right, at 0, since we have conditioned on $\mathfrak{h}_1^t(0) = \theta = \mathsf{Tri}_\theta(0)$. Indeed, this is why we consider this conditioning instead of $\{\mathfrak{h}_1^t(0) \geq \theta\}$, as for the latter we a priori have very little control over the value of $\mathfrak{h}_1^t(0)$.

However, for convenience let us pretend we are actually conditioning on $\{\mathfrak{h}_1^t(0) \geq \theta\}$ (which as an increasing event is easier to work with) while we look for the left pinning point. The pinning point we find will differ based on whether we are trying to lower bound or upper bound the shape of $\mathfrak{h}_1^t$.

Lower bounding the profile: Since we are conditioning on the increasing event $\{\mathfrak{h}_1^t(0) \geq \theta\}$, the FKG inequality says that the top curve under this conditioning is stochastically higher than the unconditioned top curve. The unconditioned top curve is with high probability close to $-\theta$ at $\pm\theta^{1/2}$ (using one-point tightness of $\mathfrak{h}_1^t(x) + x^2$ at $x = \pm\theta^{1/2}$), so $\mathfrak{h}_1^t(-\theta^{1/2})$ is at least $-\theta$ (ignoring any lower order terms for simplicity) under the conditioning as well. This suffices because, again by monotonicity, we can lower $\mathfrak{h}_1^t(-\theta^{1/2})$ from its value to $-\theta$, and this can only lower the profile—not an issue when proving a lower bound. This is a form of pinning, as $(\pm\theta^{1/2}, -\theta)$ lies on Tri_θ.

Next, we apply the Brownian Gibbs property on $[-\theta^{1/2}, 0]$ (it is a symmetric argument for $[0, \theta^{1/2}]$). This tells us that $\mathfrak{h}_1^t$ is a Brownian bridge from $(-\theta^{1/2}, -\theta)$ to $(0, \theta)$, conditioned to avoid $\mathfrak{h}_2^t$. But, again by monotonicity, $\mathfrak{h}_1^t$ must be larger than the *unconditional* (i.e., with $\mathfrak{h}_2^t \equiv -\infty$) Brownian bridge between the two mentioned pinning points. This unconditioned Brownian bridge approximately follows Tri_θ, up to Brownian fluctuations, which occur on scale $(\theta^{1/2})^{1/2} = \theta^{1/4}$. This establishes the lower bound on the profile.

Upper bounding the profile: This side is more delicate, as we cannot ignore the lower boundary $\mathfrak{h}_2^t$ as we did in the lower bound.

We first observe that, again, we do not actually need $\mathfrak{h}_1^t$ at our pinning point to be close to Tri_θ; it is sufficient if the point lies *below* Tri_θ by monotonicity, as we are proving a profile upper bound.

So we need to find a point x_θ such that $\mathbb{P}(\mathfrak{h}_1^t(x_\theta) \geq \mathsf{Tri}_\theta(x_\theta) \mid \mathfrak{h}_1^t(0) \geq \theta)$ is small. It will be convenient to set $x_\theta = -\theta^{1/2}z$. Recalling that $\mathsf{Tri}_\theta(x) = \theta + 2\theta^{1/2}x$, we see using stationarity that the mentioned probability is upper bounded by

$$\frac{\mathbb{P}(\mathfrak{h}_1^t(x_\theta) \geq \mathsf{Tri}_\theta(x_\theta))}{\mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta\right)} = \frac{\mathbb{P}(\mathfrak{h}_1^t(0) \geq \theta(z^2 - 2z + 1))}{\mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta\right)} \leq \frac{\exp\left(-c_1\theta^{3/2}(z-1)^3\right)}{\exp\left(-c_2\theta^{3/2}\right)},$$

the last inequality for some $c_2 > c_1 > 0$, which we obtain by invoking the a priori upper and lower bounds on $\mathbb{P}(\mathfrak{h}_1^t(0) \geq \theta)$ from [CG20a]. Clearly, there exists a large enough z independent of θ such that $c_1(z-1)^3 > c_2$.

Thus we have found a left pinning point at $x_\theta = -\theta^{1/2}z$. To obtain the upper bound on the profile, we wish to apply the Brownian Gibbs property on $[x_\theta, 0]$ (again the argument is symmetric on the other side of zero). However, to implement this we will need some control on the second curve on this interval which forms the lower boundary; if, for example, it has peaks rising much above Tri_θ inside $[x_\theta, 0]$, then there is no way to obtain an upper bound of Tri_θ on $\mathfrak{h}_1^t$.

It is here that it is crucial that x_θ is polynomial in θ. This is because we can break up $[x_\theta, 0]$ into polynomially many unit intervals, and, since one can upgrade one-point upper tail bounds to upper tail bounds on $\sup \mathfrak{h}_2^t(x) + x^2$ (where the supremum is over a unit interval) which decay like $\exp(-c\theta^{3/2})$, a union bound suffices. A subtle point is that this bound has to hold conditional on $\mathfrak{h}_1^t(0) \geq \theta$, which we accomplish by the BK inequality.

At this stage we have two pinning points at which $\mathfrak{h}_1^t$ is equal to Tri_θ and a lower boundary condition which is essentially a parabola on the interval between the pinning points. We know that $\mathfrak{h}_1^t$ is lower than a Brownian bridge conditioned on avoiding the parabola; but, since Tri_θ is tangent to $x \mapsto -x^2$, this conditioning has uniformly positive probability. Thus $\mathfrak{h}_1^t$ lies below essentially an unconditioned Brownian bridge on this interval; this yields the desired upper bound on the profile with a Brownian fluctuation scale of $\theta^{1/4}$, since $[x_\theta, 0]$ is an interval of length $O(\theta^{1/2})$.

Similar arguments also yield bounds on the profile outside of $[-\theta^{1/2}, \theta^{1/2}]$: this is captured in Proposition 4.2, which says that the profile is close to $-x^2$ on an interval of scale $\theta^{1/2}$.

1.7.6. *The arguments for the extremal ensembles.* For the discussion of the proof of Theorem 6 on extremal stationary ensembles, recall that we use the notation $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ for the ensemble.

Now $\mathcal{L}$ has the zero-temperature Gibbs property, which we will make heavy use of. In fact, similar arguments also hold for $t < \infty$, but would not yield tail estimates uniform in t. For this reason we do not present them. We focus on the upper bound on the one-point upper tail.

Let us highlight two parts of the proof of the upper bound on the tail which has been already sketched for $\mathfrak{h}_1^t$. Part 1 was the upper bound on the top curve's profile when conditioned on $\mathfrak{h}_1^t(0) = \theta$, and Part 2 was control on the fluctuations of the second curve on an interval on which we resampled the top curve. (In fact, Part 2 was also needed to prove Part 1, and we will return to this point.)

As we saw in the just concluded sketch, we obtained Part 1 by first finding point $\pm x_\theta$ at which $\mathfrak{h}_1^t$ was with high probability (conditioned on $\mathfrak{h}_1^t(0) = \theta$) below the extended version of Tri_θ (by monotonicity, we could then raise the point to lie on Tri_θ). Then by resampling the top curve on $[-x_\theta, 0]$ and $[0, x_\theta]$, we could show that the top curve remained close to Tri_θ, assuming control on the second curve on the same intervals (it is to obtain this control that Part 2 is needed).

The existence of pinning points was done using a priori bounds on the upper tail of $\mathfrak{h}_1^t(0)$ coming from integrable probability (namely, via analysis in [CG20a] of exact identities). This use of integrable input for pinning is a feature shared between our argument and the argument given by Aggarwal in [Agg20], where, as one part of the larger argument, he obtains a pinning of a curve in the six-vertex model (under a particular choice of parameters) using explicit combinatorial formulas for a certain correlation function, which in turn gives asymptotics for a pinning probability.

For extremal ensembles where we only assume the Brownian Gibbs property and stationarity, we have no such exact formulas or integrable input to use to obtain that pinning. The basic difficulty to be addressed is to make do even without these inputs.

A weak form of pinning: We start by observing that it is possible, using only one-point tightness, stationarity, and parabolic decay of $\mathcal{L}_1$, to obtain a weak form of pinning. By "weak" we mean that we do not attempt to find a point at which $\mathcal{L}_1$ is below Tri_θ, which has slope $-\theta^{1/2}$, but instead where it is below a line of θ-independent slope -1. Indeed, for given $\varepsilon > 0$, we can show that

$$\mathbb{P}\left(\mathcal{L}_1(x) > -|x|\right) \leq \varepsilon \cdot \mathbb{P}(\mathcal{L}_1(0) \geq \theta) \tag{7}$$

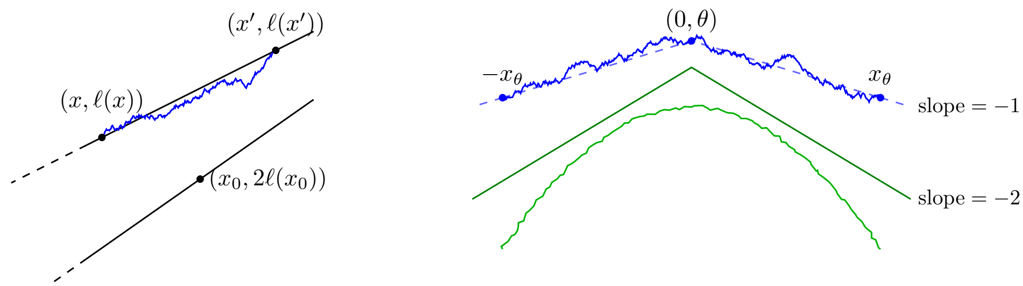


FIGURE 5. *Left panel*: A Brownian bridge whose endpoints lie on a line of given slope is unlikely to intersect another line of different slope if the endpoints of the bridge are well-separated from the second line. *Right panel*: The lower green curve is what $\mathcal{L}_2$ should look like, but which we cannot yet prove; the darker green tent-shaped function above it is the upper bound we are able to prove on $\mathcal{L}_2$, lines of slope ± 2; and the blue curve is the upper bound we are able to prove on $\mathcal{L}_1$ using resampling. In more detail, the blue curve is a Brownian bridge from $(-x_\theta, \theta - x_\theta)$ to $(0, \theta)$ to $(x_\theta, \theta - x_\theta)$ (which are endpoints of lines of slope ± 1, drawn in dashed blue) conditioned to stay above the dark green tent; this latter conditioning has probability lower bounded by a uniformly positive constant by the argument from the left panel. Thus, at $\pm\theta$, $\mathcal{L}_1$ is with high probability below $\theta/2$, as the Brownian bridge that bounds it has mean 0 and standard deviation $O(\theta^{1/2})$ at those locations.

(i.e., the RHS is ε times the probability of the conditioning event) for some x:

$$\mathbb{P}\left(\mathcal{L}_1(x) > -|x|\right) = \mathbb{P}\left(\mathcal{L}_1(0) > x^2 - |x|\right) \to 0$$

as $x \to \infty$ since $x^2 - |x| \to \infty$. But of course, the $x = x_\theta > 0$ which achieves (7) depends on θ in some completely uncontrolled way, another sense in which the pinning is weak. (While the existence of such an x is guaranteed even if the slope of the line was $-\theta^{1/2}$, this would cause a problem later, as we will soon see.)

This is in contrast to the situation when we had a priori inputs, as that yielded that $x_\theta = O(\theta^{1/2})$. This was very important because the exponentially decaying upper tail bound we had on the one-point distribution meant that we could easily control the supremum of $\mathcal{L}_2(x) + x^2$ on an interval whose size was polynomial in θ. (This is where Part 2 was used in Part 1, as mentioned above.) But even with the a priori estimates we cannot control $\mathcal{L}_2$ on arbitrarily large intervals, and here we have mere one-point tightness.

Control on the second curve over all of $\mathbb{R}$*:* To handle this issue, we weaken what we ask from the second curve: we have it remain below $-m|x| + K$ for a constant $m > 0$ (that we choose) and a random K almost surely. (While in reality $\mathcal{P}_2$, the second curve of the parabolic Airy line ensemble, will not remain below $-x^2$ plus any random constant on all of $\mathbb{R}$, it *will* stay below $-\alpha x^2$ plus a random constant for any $0 < \alpha < 1$, and so one might try asking for such a bound on $\mathcal{L}_2$; nevertheless, our proof is not able to furnish this demand on $\mathcal{L}_2$, but can the linear demand. The difference is that a linear bound plays well with Brownian bridge's linear trajectory, as we will see.)

We ask for the above conditionally on $\mathcal{L}_1(0) \geq \theta$, so it may seem natural that K would depend on θ: but recall the BK inequality, which gives that the second curve, conditionally on $\mathcal{L}_1(0) \geq \theta$, is stochastically smaller than an *unconditional* $\mathcal{L}_1$. The latter's law has no θ dependence, so K can be taken to have no θ-dependence. (It is here that we would not have been able to have K be θ-independent if we had taken the slope m to depend on θ.)

So we need to establish that there exists an almost surely finite K such that $\mathcal{L}_1(x) \leq -m|x| + K$ for all $x \in \mathbb{R}$; we will show that for each $m > 0$ there exists such a random K. Establishing this is the main new argument in the extremal ensemble case compared to the arguments sketched above, and we outline it now.

The main tool we have available is the Brownian Gibbs property, which tell us that on any interval the top curve can be viewed as a Brownian bridge between appropriate random endpoints conditioned to avoid $\mathcal{L}_2$. The basic difficulty is that we have no control over $\mathcal{L}_2$: in particular, if it has large peaks at an infinite sequence of random points τ_n, that may force $\mathcal{L}_1(\tau_n)$ to be larger than $-m|\tau_n| + K_n$ for a sequence $K_n \to \infty$.

If the above scenario happens, however, then $\mathcal{L}_1$ is likely to remain high (in particular, higher than $-2m|x|$ say) in between these peaks as well (see the left panel of Figure 5): this is because unconditioned Brownian bridge approximately follows the linear path between its endpoints and so will with positive probability not hit a steeper line, and the lower boundary condition imposed by $\mathcal{L}_2$ only pushes it further up (we need no control on $\mathcal{L}_2$ for this statement!). Importantly, this holds true in a uniform way no matter how large the intervals $[\tau_n, \tau_{n+1}]$ are. But this leads to a contradiction because, by one-point tightness, stationarity, and parabolic decay, we can find a sequence of deterministic points x_n such that $\mathcal{L}_1(x_n) < -2|x|_n$ for all but finitely many n almost surely; this is argued in a similar way to (7) and by invoking the Borel-Cantelli lemma.

Combining the two parts: With this control on the second curve in hand, combined with the weak pinning at $\pm x_\theta$, we can argue with similar ideas as the earlier cases that, conditionally on $\mathcal{L}_1(0) \geq \theta$, the top curve is not too high at a pair of points which are not too far from zero with probability at least $\frac{1}{2}$; we do not take the points to be $\pm\theta^{1/2}$ as earlier since we do not have such strong control yet. Instead, we show that $\mathcal{L}_1(\pm\theta)$ is at most $\theta/2$ with high probability. The argument for this involves resampling on the interval $[-x_\theta, 0]$ and $[0, x_\theta]$, using that the second curve is below $K - 2|x|$, and that a Brownian bridge B is unlikely to hit a line of steeper slope than the slope between its endpoints (which here is -1); see the right panel of Figure 5. Also note the important point that while x_θ can be arbitrarily large, the fluctuations of B at $\pm\theta$ will only be of order $\theta^{1/2}$.

With this control on $\mathcal{L}_1(\pm\theta)$, we can do a resampling on $[-\theta, \theta]$ to say that $\mathbb{P}(\mathcal{L}_1(0) \geq \theta)$ is at most the probability a Brownian bridge B' from $(-\theta, \theta/2)$ to $(\theta, \theta/2)$ which is conditioned to stay above $K - 2|x|$ satisfies $B'(0) \geq \theta$, which is clearly $\exp(-c\theta)$, as again the probability of the conditioning event is uniformly positive.

Now with an exponential tail on the one-point distribution available, the earlier arguments can be applied to yield all the main results for the extremal ensembles as well.

Organization of the paper. In Section 2 we recall the precise definitions of the Brownian Gibbs properties and formulate assumptions on line ensembles under which we will give the rest of the arguments in the paper; these assumptions allow us to argue the parts of the proofs common to both extremal ensembles and the KPZ equation and parabolic Airy$_2$ process (namely those in Sections 4–7) without repetition. In Section 3 we collect various monotonicity and technical tools we will need in the main arguments, though we defer most of their proofs to the appendices as they do not require many new ideas and would obstruct the flow of the main arguments. In Sections 4 and 5 respectively we prove the one-point limit shape and the one-point asymptotics. We do the two-point versions of the same in Sections 6 and 7. In Section 8 we obtain a preliminary one-point upper tail decay for extremal ensembles, which is needed to meet the assumptions stated in Section 2 just mentioned above thus apply the arguments for the sharp estimates to the extremal ensembles. Finally in Section 9 we give the arguments to obtain the one-point asymptotics for general initial data.

To streamline the presentation, there are three appendices. Appendix A proves that the assumptions from Section 2.2 hold in the ensembles of interest and gives proofs for the implicated monotonicity properties. Appendix B.1 collects some calculations involving $\mathfrak{h}^t$ such as of its Hamiltonian for its

Gibbs property. The final Appendix C provides the proofs of various Brownian estimates from Section 3.1.

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2. Setup & line ensemble hypotheses

2.1. **Line ensembles & Brownian Gibbs.** We start by stating formally the definitions of the spaces and objects we will be working with.

Definition 2.1. A line ensemble is a random continuous function defined from $\mathbb{R} \times \mathbb{N}$ to $\mathbb{R}$, where the space of continuous functions $\mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is endowed with the topology of uniform convergence on compact sets and the corresponding Borel σ-algebra.

Definition 2.2 (H-Brownian Gibbs and Brownian Gibbs properties)**.** Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be a line ensemble. For $k \in \mathbb{N}$ and $[\ell, r] \subseteq \mathbb{R}$ a finite interval, define $\mathcal{F}_{\mathrm{ext}}(k, [\ell, r])$ to be the σ-algebra generated by $\{\mathcal{L}_i(x) : (i, x) \notin [\![1, k]\!] \times [\ell, r]\}$, i.e., all the data external to $[\ell, r]$ on the top k curves. Let $H : \mathbb{R} \to [0, \infty)$ be a continuous function (called a *Hamiltonian*).

We say a line ensemble $\mathcal{L}$ has the *H-Brownian Gibbs* property with respect to rate σ^2 Brownian bridge if, for any $k \in \mathbb{N}$ and $[a, b] \subseteq \mathbb{R}$, conditionally on $\mathcal{F}_{\mathrm{ext}}(k, [a, b])$, the Radon-Nikodym derivative of the law $\mathbb{P}_H^{k,a,b,\vec{x},\vec{y},f}$ of $\mathcal{L}$ on $[a, b]$ with respect to the law $\mathbb{P}_{\mathrm{free}}^{k,a,b,\vec{x},\vec{y}}$ of k independent rate σ^2 Brownian bridges on $[a, b]$ with endpoint values $\vec{x}$ and $\vec{y}$ is given by

$$\frac{\mathrm{d}\mathbb{P}_H^{k,a,b,\vec{x},\vec{y},f}}{\mathrm{d}\mathbb{P}_{\mathrm{free}}^{k,a,b,\vec{x},\vec{y}}}(\mathcal{L}_1, \ldots, \mathcal{L}_k) = \frac{W_H^{k,a,b,\vec{x},\vec{y},f}(\mathcal{L}_1, \ldots, \mathcal{L}_k)}{Z_H^{k,a,b,\vec{x},\vec{y},f}},$$

where (with $\mathcal{L}_0 = f$)

$$W_H^{k,a,b,\vec{x},\vec{y},f}(\mathcal{L}_1, \ldots, \mathcal{L}_k) = \exp\left\{-\sum_{i=0}^{k} \int_a^b H\left(\mathcal{L}_{i+1}(u) - \mathcal{L}_i(u)\right) \mathrm{d}u\right\} \tag{8}$$

and

$$Z_H^{k,a,b,\vec{x},\vec{y},f} = \mathbb{E}_{\mathrm{free}}^{k,a,b,\vec{x},\vec{y}}\left[W_H^{k,a,b,\vec{x},\vec{y},f}(\mathcal{L}_1, \ldots, \mathcal{L}_k)\right].$$

The *Brownian Gibbs* property is specified by the above with $H(x) = \infty \cdot \mathbb{1}_{x>0}$; in plain words, the Radon-Nikodym derivative corresponds to conditioning on non-intersection of the curves $\mathcal{L}_1, \ldots, \mathcal{L}_k$ between themselves as well as the lower curve.

In the remainder of the paper we will set $\sigma^2 = 2$.

At a few points in the argument we will need the notion of a stopping domain, which is a random interval analogous to stopping times for Markov processes; the important point is that the Brownian Gibbs property can be applied to stopping domains, as we recall in Lemma 2.4.

Definition 2.3 (Stopping domain and strong Brownian Gibbs)**.** Let $\mathcal{L} : \mathbb{N} \to \mathbb{R}$ be an $\mathbb{N}$-indexed collection of continuous curves and H a Hamiltonian. A pair of random variables $\mathfrak{l}, \mathfrak{r}$ is a *stopping domain* for $\mathcal{L}_1, \ldots, \mathcal{L}_k$ if, for all $\ell < r$,

$$\{\mathfrak{l} \leq \ell, \mathfrak{r} \geq r\} \in \mathcal{F}_{\mathrm{ext}}(k, \ell, r).$$

Define the σ-algebra $\mathcal{F}_{\mathrm{ext}}(k, \mathfrak{l}, \mathfrak{r})$ to be the one generated by events A such that $A \cap \{\mathfrak{l} \leq \ell, \mathfrak{r} \geq r\} \in \mathcal{F}_{\mathrm{ext}}(k, \ell, r)$ for all $\ell < r$. Also define, for $k \in \mathbb{N}$, the set $\mathcal{C}^k = \{(\ell, r, f_1, \ldots, f_k) : \ell < r, f_1, \ldots, f_k \in \mathcal{C}([\ell, r])\}$.

An infinite collection of random continuous curves $\mathcal{L}$ has the *strong H-Brownian Gibbs* property if, for any $k \in \mathbb{N}$, stopping domain $[\mathfrak{l}, \mathfrak{r}]$ with respect to $\mathcal{L}_1, \ldots, \mathcal{L}_k$, and $F : \mathcal{C}^k \to \mathbb{R}$

$$\mathbb{E}[F(\mathfrak{l}, \mathfrak{r}, \mathcal{L}_1|_{[\mathfrak{l},\mathfrak{r}]}, \ldots, \mathcal{L}_k|_{[\mathfrak{l},\mathfrak{r}]}) \mid \mathcal{F}_{\mathrm{ext}}(k, \mathfrak{l}, \mathfrak{r})] = \mathbb{E}_H^{k,a,b,\vec{x},\vec{y},\mathcal{L}_{k+1}}[F(a, b, B_1, \ldots, B_k)],$$

where $\vec{x} = (\mathcal{L}_1(\mathfrak{l}), \ldots, \mathcal{L}_k(\mathfrak{l}))$, $\vec{y} = (\mathcal{L}_1(\mathfrak{r}), \ldots, \mathcal{L}_k(\mathfrak{r}))$, and $B_1, \ldots, B_k$ are distributed according to $\mathbb{E}_H^{k,a,b,\vec{x},\vec{y},\mathcal{L}_{k+1}}$, i.e., Brownian bridges tilted by the Radon-Nikodym factor from Definition 2.2.

In words, the distribution of $\mathcal{L}_1, \ldots, \mathcal{L}_k$ on a stopping domain is still given by Brownian bridges with the appropriate endpoints reweighted as in the usual H-Brownian Gibbs property, except on the random interval $[\mathfrak{l}, \mathfrak{r}]$.

Lemma 2.4 (Lemma 2.5 of [CH16] and Lemma 2.5 of [CH14])**.** *If a line ensemble $\mathcal{L}$ satisfies the H-Brownian Gibbs property, it also satisfies the strong H-Brownian Gibbs property, and similarly for the usual Brownian Gibbs property.*

Definition 2.5 (Parabolic Airy line ensemble)**.** The parabolic Airy line ensemble $\mathcal{P} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is the line ensemble such that the finite dimensional distributions of the ensemble $\mathcal{A}$ given by $\mathcal{A}_i(x) = \mathcal{P}_i(x) + x^2$ can be described as follows: for every $m \in \mathbb{N}$ and real $t_1 < \ldots < t_m$, the point process $\{(\mathcal{A}_i(t_j), t_j) : i \in \mathbb{N}, j \in [\![1, m]\!]\}$ is determinantal with correlation kernel given by the extended Airy kernel $K_{\mathrm{Ai}}^{\mathrm{ext}} : (\mathbb{R} \times (0, \infty))^2 \to \mathbb{R}$, where

$$K_{\mathrm{Ai}}^{\mathrm{ext}}\big((x,t);(y,s)\big) = \begin{cases} \int_0^\infty e^{-\lambda(t-s)} \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda & t \geq s \\ -\int_{-\infty}^0 e^{-\lambda(t-s)} \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda & t < s \end{cases}$$

with Ai the classical Airy function. (The reader is referred to [HKPV09] for background on determinantal point processes.)

Next is the Gibbs property for $t < \infty$; the zero temperature analogue follows in Proposition 2.7.

Proposition 2.6 (Gibbs property of $\mathfrak{h}^t$)**.** *For each $0 < t < \infty$, there exists a line ensemble $\mathfrak{h}^t$ such that $\mathfrak{h}_1^t$ is the lowest indexed curve of $\mathfrak{h}^t$ and the latter has the H_t-Brownian Gibbs property with respect to rate two Brownian bridge, where the Hamiltonian H_t is given by $H_t(x) = 2t^{2/3}\exp(t^{1/3}x)$.*

The existence of the line ensemble $\mathfrak{h}^t$ and its Brownian Gibbs property is proven in [CH16] (see also [Nic21]); in fact, by later results characterizing the law of line ensembles by the distribution of the top curve [Dim21], this line ensemble is also unique in law.

We note that the above specification of the Hamiltonian differs from [CH16, Theorem 2.15 (iii)], which does not include the $2t^{2/3}$ pre-factor in the Hamiltonian. This is due to a neglecting of a Jacobian factor which should be present, and so the correct Hamiltonian for $\mathfrak{h}^t$ is indeed as stated in Proposition 2.6. To clarify this point we write out the proof of Proposition 2.6 in Appendix B.1.

Proposition 2.7 (Theorem 3.1 of [CH14])**.** *$\mathcal{P}$ has the Brownian Gibbs property.*

As mentioned earlier, in the remainder of the paper we will denote $\mathcal{P}$ by $\mathfrak{h}^{t=\infty}$ so that we have a common notation for all $t \in (0, \infty]$. We may also sometimes refer to the Brownian Gibbs property as the H_t-Brownian Gibbs property with $t = \infty$.

Notation. $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of real-valued continuous functions defined on $[a,b]$.

We will often consider conditional probability distributions on conditioning on a σ-algebra $\mathcal{F}$. For these objects we will use the shorthand notation

$$\mathbb{P}_{\mathcal{F}}(\cdot) = \mathbb{P}(\cdot \mid \mathcal{F}).$$

The existence of the regular conditional probability measures we will need is ensured by the fact that we will always take $\mathcal{F}$ to be generated by random variables taking values in a Borel space, and then invoking well-known abstract results such as [Kal21, Theorem 8.5]. Conditional probabilities of the form $\mathbb{P}(\cdot \mid \mathfrak{h}_1^t(0) = \theta)$ and $\mathbb{P}(\cdot \mid \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta)$ are also defined via these regular conditional distributions and their associated probability kernels.

2.2. **Hypotheses on line ensembles.** We adopt the following hypotheses on the ensembles $\mathfrak{h}^t$ where $t \in (0,\infty]$. We will prove our main results under these assumptions; in Appendix A, we will verify that the KPZ, parabolic Airy, and extremal stationary line ensemble satisfy the assumptions. The upper bound in the last assumption, which concerns upper tail bounds, will be proven for extremal stationary ensembles in Section 8; it is to treat the main theorems from Section 1 for this case as well as the KPZ and parabolic Airy cases in a unified way that we have opted to prove our main results under these explicit assumptions.

(i) **Stationarity and Brownian Gibbs:** For each $t > 0$, $\mathfrak{h}^t$ possesses the H_t-Brownian Gibbs property, and for each $t > 0$, the process $x \mapsto \mathfrak{h}_1^t(x) + x^2$ is stationary.

(ii) **Correlation inequalities:** The ensemble $\mathfrak{h}^t$ is positively associated and satisfies the van den Berg-Kesten (BK) inequality, i.e., for increasing events A and B, and any event C (all Borel subsets of $\mathcal{C}([a,b],\mathbb{R})$ for some finite interval $[a,b]$),

$$\begin{aligned}&\mathbb{P}\left(\mathfrak{h}^t \in A, \mathfrak{h}^t \in B\right) \geq \mathbb{P}\left(\mathfrak{h}^t \in A\right)\cdot\mathbb{P}\left(\mathfrak{h}^t \in B\right) \text{ and}\\ &\mathbb{P}\left(\mathfrak{h}_2^t \in A, \mathfrak{h}_1^t \in C\right) \leq \mathbb{P}(\mathfrak{h}_1^t \in A)\cdot\mathbb{P}(\mathfrak{h}_1^t \in C).\end{aligned}$$

(iii) **Monotonicity in conditioning:** For $i = 1$ or $i = 1, 2$, if $y_i^{(1)} \geq y_i^{(2)}$, then the conditional law of $\mathfrak{h}_1^t$ given $\mathfrak{h}_1^t(x_i) = y_i^{(1)}$ stochastically dominates that of the same given $\mathfrak{h}_1^t(x_i) = y_i^{(2)}$.

(iv) **Uniform bounds on the one-point upper tail, and one-point tightness:** For each $t_0 > 0$, there exist $\alpha, \beta > 0$, θ_0, and $c_1, c_2 > 0$ such that, for $\theta > \theta_0$ and $t \in [t_0, \infty)$ or $t = \infty$,

$$\exp(-c_1\theta^\alpha) \leq \mathbb{P}\left(\mathfrak{h}_1^t(0) > \theta\right) \leq \exp(-c_2\theta^\beta).$$

Further, $\{\mathfrak{h}_1^t(x)\}_{t\geq t_0}$ is a tight family for any $x \in \mathbb{R}$.

These assumptions bear a thematic resemblance to those in an earlier paper of the authors [GH20], where upper and lower tail bounds with the correct exponents of $3/2$ and 3 are derived in general last passage percolation models which satisfy the assumptions. Similar to here, the main things assumed or used in [GH20] are parabolic curvature of the profile ((i) here), the FKG and BK inequalities, and a priori tail bounds.

By stochastic domination in Assumption (iii), we mean that for any finite interval $[a,b]$ and increasing function $F : \mathcal{C}([a,b],\mathbb{R}) \to \mathbb{R}$ (where the order on $\mathcal{C}([a,b],\mathbb{R})$ is the usual point-wise order of functions), the expectation of F under the dominating law is lower bounded by the same expectation under the dominated law.

We note that Assumption (i) implies that $\mathfrak{h}^t$ is absolutely continuous to Brownian motion on compact intervals, and so the law of $\mathfrak{h}_1^t(0)$ is absolutely continuous with respect to Lebesgue measure, i.e., has a density (see [CH14, CH16]).

We will verify or prove that the KPZ, parabolic Airy, and extremal stationary line ensembles satisfy these assumptions in Appendices A and A.4, but here we state this fact for easy reference.

Theorem 2.8. *The KPZ, parabolic Airy, and extremal stationary line ensembles satisfy Assumptions (i)–(iv), the latter two with $t = \infty$.*

We will see that Assumption (iv) is in fact available (again from [CG20a]) for the KPZ and parabolic Airy line ensembles with $\alpha = \beta = 3/2$; we have nevertheless formulated the assumption with general tail exponents since, as we saw in the proof outline, the preliminary upper bound on the upper tail we will obtain for extremal stationary ensembles will be only $\exp(-c\theta)$.

With Theorem 2.8 and by invoking theorems that will be stated in later sections we can formally prove Theorems 1–6:

Proofs of Theorems 1–6. Recall that Theorem 6 is simply that Theorems 1–4 apply to extremal stationary line ensembles. Theorems 1–4 follow from Theorem 2.8 applied to meet the hypotheses of Theorem 5.5 (density asymptotics for Theorem 1); Theorems 5.1 and 5.3 (upper and lower bounds on the upper tail for Theorem 2); Theorem 7.1 (two point upper tail estimates for Theorem 3); and Corollary 7.2 (sharpness of FKG for Theorem 4). $\square$

3. Monotonicity & Brownian estimates

In this section we collect many tools that we will be making extensive use of in the main arguments in upcoming sections.

There are mainly two types of tools we will make use of, monotonicity and Brownian estimates. We cover monotonicity tools in Section 3.1 and the Brownian ones in Sections 3.3–3.5. In Section 3.2 we state an estimate relating the tail of the supremum of $\mathfrak{h}_1^t(x) + x^2$ over an interval to the tail of the one-point. In Section 3.6 we apply the Brownian estimates to obtain analogues statements for $\mathfrak{h}_1^t$ which will be useful in later sections.

The proofs of most of the tools are straightforward but tedious, or follow the same lines as arguments already existing in the literature. For this reason such proofs have been deferred to the appendices. However, proofs in Section 3.6 are presented there because they involve interesting applications of the Brownian Gibbs properties.

3.1. **Monotonicity tools.** There are two monotonicity tools. The first is monotonicity of the Brownian bridges under both hard and soft constraints, in the boundary data, Lemma 3.1. The second, Lemma 3.2, is a straightforward consequence of Assumption (iii) on monotonicity in conditioning, which relates conditioning on point values like $\mathfrak{h}_1^t(0) = \theta$ and conditioning on positive probability events like $\{\mathfrak{h}_1^t(0) \geq \theta\}$ (the latter has the benefit of being an increasing event, and so comes within the purview of positive association inequalities from Assumption (ii)).

Lemma 3.1 (Monotonicity in boundary data)**.** *Fix $t > 0$, $k_1 \leq k_2 \in \mathbb{Z}$, $a < b$, two pairs of vectors $w^{(i)}, z^{(i)} \in \mathbb{R}^{k_2-k_1+1}$ and two pairs of measurable functions $(f^{(i)}, g^{(i)})$ for $i \in \{1, 2\}$ such that $w_j^{(1)} \leq w_j^{(2)}$ and $z_j^{(1)} \leq z_j^{(2)}$ for all $j = k_1, \ldots, k_2$ and $f^{(i)} : (a, b) \to \mathbb{R} \cup \{\infty\}$, $g^{(i)} : (a, b) \to \mathbb{R} \cup \{-\infty\}$ and for all $s \in (a, b)$, $f^{(1)}(s) \leq f^{(2)}(s)$ and $g^{(1)}(s) \leq g^{(2)}(s)$.*

For $i \in \{1, 2\}$, let $\mathcal{Q}^{(i)} = \{\mathcal{Q}_j^{(i)}\}_{j=k_1}^{k_2}$ be a $\{k_1, \ldots, k_2\} \times (a, b)$-indexed line ensemble such that $\mathcal{Q}^{(i)}$ has the H_t-Brownian Gibbs property with entrance data $w^{(i)}$, exit data $z^{(i)}$ and boundary data $(f^{(i)}, g^{(i)})$).

There exists a coupling of the laws of $\{\mathcal{Q}_j^{(1)}\}$ and $\{\mathcal{Q}_j^{(2)}\}$ such that almost surely $\mathcal{Q}_j^{(1)}(s) \leq \mathcal{Q}_j^{(2)}(s)$ for all $j \in \{k_1, \ldots, k_2\}$ and all $s \in (a, b)$.

The same is true in the $t = \infty$ (zero-temperature case) if additionally $w_j^{(i)} > w_{j+1}^{(i)}$ and $z_j^{(i)} > z_{j+1}^{(i)}$ for $j = k_1, \ldots, k_2 - 1$ and $i = 1, 2$, and $f^{(i)}(a) > w_{k_1}^{(i)}$, $f^{(i)}(b) > z_{k_1}^{(i)}$, $g^{(i)}(a) < w_{k_2}$, $g^{(i)}(b) < z_{k_2}$ for $i = 1, 2$.

Proof. The positive temperature ($t < \infty$) statements are Lemmas 2.6 and 2.7 of [CH16]. The zero temperature ($t = \infty$) statements are Lemmas 2.6 and 2.7 of [CH14]. See also [DM21] and [Dim21] for more detailed proofs of the respective cases. □

The following lemma is a convenient consequence of Assumption (iii), converting distributional monotonicity to orderings of expectations, that we will be making use of in our applications. We will make use of it many times, usually to convert between conditionings of the form $\mathfrak{h}_1^t(0) = \theta$ (such as in the statement of the limit shape Theorem 4.1) and those of the form $\mathfrak{h}_1^t(0) \leq \theta$ or $\mathfrak{h}_1^t(0) \geq \theta$.

Lemma 3.2 (Monotonicity in conditioning)**.** *Suppose $\mathfrak{h}^t$ satisfies Assumptions (iii). Let F be an increasing function of $(\mathfrak{h}_1^t, \mathfrak{h}_2^t, \ldots)$. Let $y_1, y_2 \in \mathbb{R}$ and $E_1, E_2 \subseteq \mathbb{R}$ be (possibly infinite) intervals such that* $\inf E_i = y_i$ *and $y_i \in E_i$ for $i = 1, 2$. Then, for any $\theta > 0$,*

$$\mathbb{E}\left[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) = y_1, \mathfrak{h}_1^t(\theta^{1/2}) = y_2\right] \leq \mathbb{E}\left[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) \in E_1, \mathfrak{h}_1^t(\theta^{1/2}) \in E_2\right].$$

Similarly, if F is a decreasing function and E_i are intervals such that $\sup E_i = y_i$ *and $y_i \in E_i$ for $i = 1$ and 2,*

$$\mathbb{P}\left[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) = y_1, \mathfrak{h}_1^t(\theta^{1/2}) = y_2\right] \leq \mathbb{E}\left[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) \in E_1, \mathfrak{h}_1^t(\theta^{1/2}) \in E_2\right].$$

The same holds if the conditioning is at a single point, such as 0*, instead of two points $\pm\theta^{1/2}$ as stated above.*

As mentioned, the proof of this is a straightforward consequence of Assumption (iii) and is deferred to Appendix A.3.

3.2. **A bound on the tail of** $\sup_{|x|\leq 1} \mathfrak{h}_1^t(x) + x^2$**.** Next we record a result which bounds the tail of the supremum of $\mathfrak{h}_1^t$ over an interval in terms of the one-point tail. We will make use of this at several points to control the lower curve's fluctuations.

Proposition 3.3. *Under Assumptions (i) and (ii), and assuming that $\{\mathfrak{h}_1^t(x)\}_{t \geq t_0}$ is a tight family for any $x \in \mathbb{R}$, there exist θ_0 and $C < \infty$ such that, for all $t \in [t_0, \infty]$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(\sup_{x \in [-1,1]} \mathfrak{h}_1^t(x) + x^2 \geq \theta\right) \leq 4\theta \cdot \mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta - 2\right).$$

Observe that if we know that the one-point tail decays at least stretched exponentially, as we assume in Assumption (iv), then the tail decay at the level of the exponent is preserved for the supremum. And indeed, if we know that the one-point tail is $\exp(-\frac{4}{3}\theta^{3/2}(1 + o(1)))$, the same gets transferred to the above supremum's tail.

The proof of Proposition 3.3 is a refinement of the "no big max" argument given in [Ham22, Proposition 2.27] and [CH14, Proposition 4.4], and will be given in Appendix B.2.

An analogous result can be proved to bound the lower tail of the infimum over a unit interval; such a result would require an input bound on the one-point lower tail (which in reality, when not too deep in the tail, decays like $\exp(-c\theta^3)$ [CG20b]).

We require such a result only to prove a lower bound on the limit shape of $\mathfrak{h}_1^t$, conditional on $\mathfrak{h}_1^t(0) = \theta$, outside of $[-\theta^{1/2}, \theta^{1/2}]$, namely Proposition 4.2; this result is not stated for extremal stationary ensembles as we do not have access to one-point lower tails there. The following is obtained from [Wu21b, Proposition B.1] (for $t < \infty$) and [Ham22, Proposition A.2] ($t = \infty$); note that the expected cubic tail exponent is not obtained, but this is still sufficient for our purposes.

Proposition 3.4. *Let $\mathfrak{h}^t$ be the KPZ or parabolic Airy line ensemble. Let $k \in \mathbb{N}$. There exist θ_0 and $c > 0$ such that, for $\theta > \theta_0$ and $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\inf_{x\in[-1,1]} \mathfrak{h}_k^t(x) + x^2 \le -\theta\right) \le \exp(-c\theta^{3/2}).$$

3.3. **Gaussian and Brownian bridge estimates.** Here we recall well-known bounds for Gaussian random variables and Brownian bridges. We start with a standard bound on the tail of centered Gaussian random variables.

Lemma 3.5. *For $x \ge (4/3)^{1/2}\sigma$,*

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{\sigma}{4x} \exp\left(-\frac{x^2}{2\sigma^2}\right) \le \mathbb{P}\left(N(0,\sigma^2) \ge x\right) \le \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Proof. We may set $\sigma = 1$ without loss of generality. The upper bound is simply the Chernoff bound. For the lower bound it is well-known via integration by parts that $\mathbb{P}(N(0,1) \ge x) \ge (2\pi)^{-1/2}(x^{-1} - x^{-3})\exp(-x^2/2)$. Using that $x \ge (4/3)^{1/2}$ gives the result. □

Next we recall the well-known distribution of the supremum of a standard Brownian bridge B defined on an interval I. Then we use it to prove Lemma 3.7, which states a bound on the tail of $\sup_J B$ for an interval $J \subseteq I$ such that the bound adapts to the variance of B on J.

Lemma 3.6 (Equation (3.40) in chapter 4 of [RY13])**.** *Let $I \subseteq \mathbb{R}$ be an interval and $B : I \to \mathbb{R}$ a Brownian bridge with $B(\inf I) = B(\sup I) = 0$. Let $\sigma_I^2 = \max_{x\in I} \mathrm{Var}(B(x)) = |I|/4$. For any $M > 0$,*

$$\mathbb{P}\left(\sup_{x\in I} B(x) \ge M\sigma_I\right) = \exp\left(-\frac{1}{2}M^2\right).$$

The following is the mentioned tail $\sup_J B$. It will be useful to use in obtaining estimates on probabilities of events such as that a Brownian bridge stays above a parabola or a line. It is proved in Appendix C.

Lemma 3.7. *Let $I, J \subseteq \mathbb{R}$ be intervals with $J \subseteq I$, and $B : I \to \mathbb{R}$ be a Brownian bridge with $B(\inf I) = B(\sup I) = 0$. Let $\sigma_J^2 = \max_{x\in J} \mathrm{Var}(B(x))$. Then, for all $M > 0$,*

$$\mathbb{P}\left(\sup_{x\in J} B(x) \ge M\sigma_J\right) \le 3\exp\left(-\frac{1}{8}M^2\right).$$

We note that a somewhat analogous statement giving a tail bound in terms of σ_J as defined above is an immediate consequence of the famous Borell-TIS inequality for Gaussian processes (see eg. [AT07, Theorem 2.1.1]); however, the bound coming from the Borell-TIS inequality would only hold for M larger than $\mathbb{E}[\sup_J B]$, which grows with σ_J. This can be a technical nuisance compared to the above statement which only needs $M > 0$.

3.4. **Estimates on parabolic avoidance probabilities.** The Brownian Gibbs property imposes a lower boundary condition of $\mathfrak{h}_2^t$; as is exactly true in the $t = \infty$ case, the lower boundary can be thought of as a curve which the top curve, essentially a Brownian bridge, must avoid. Further we know that $\mathfrak{h}_2^t$ decays like a parabola $-x^2$ and typically has unit order separation from $\mathfrak{h}_1^t$.

Given these facts, it will be crucial to have a good understanding of the probability that a Brownian bridge stays above a parabola, where the starting and ending points of the bridge are at least order one away from the parabola. Such a bound is the next statement.

Proposition 3.8. *Let $z_1 < z_2$ and B be a rate two Brownian bridge on $[z_1, z_2]$ with endpoints at least as high as $(z_1, -z_1^2 + 1)$ and $(z_2, -z_2^2 + 1)$. Then, for all $z_2 - z_1$ large enough,*

$$\mathbb{P}\Big(B(x) > -x^2 \quad \text{for all } x \in [z_1, z_2]\Big) \geq \exp\left(-\frac{1}{12}(z_2 - z_1)^3 - 2(z_2 - z_1)\log(z_2 - z_1)\right)$$

Remark 3.9. The first order term in the exponent looks very similar to the lower tail asymptotics of $\mathfrak{h}_1^t(0)$ (when not too deep in the tail) and the parabolic Airy$_2$ process, namely $\mathbb{P}(\mathfrak{h}_1^t(0) \leq -\theta) \approx \exp(-\frac{1}{12}\theta^3)$ (see for example [CG20b, Theorem 1.1] and [RRV11, Theorem 1.3]). We do not know if this is a coincidence or if some deeper phenomenon is present.

Remark 3.10. Perhaps somewhat surprisingly, an upper bound which matches Proposition 3.8 to first order can be obtained rather quickly. We illustrate this in the simpler symmetric case that $z_1 = -z_2 = -z$ for convenience. In that case the lower bound from Proposition 3.8 is, to first order in the exponent, $\exp(-2z^3/3)$.

Let $\tilde{B}$ be a rate two Brownian bridge from $(-z, 0)$ to $(z, 0)$, i.e., B shifted to have mean zero. Observe that

$$\begin{aligned}\mathbb{P}\left(B(x) > -x^2 \;\; \forall x \in [-z, z]\right) &= \mathbb{P}\left(\tilde{B}(x) - z^2 + 1 > -x^2 \;\; \forall x \in [-z, z]\right) \\ &\leq \mathbb{P}\left(\int_{-z}^{z} (\tilde{B}(x) - z^2 + x^2 + 1)\,\mathrm{d}x > 0\right).\end{aligned}$$

Now $\int_{-z}^{z} \tilde{B}(x)\,\mathrm{d}x$ is a normal random variable with mean zero and variance given by

$$\begin{aligned}\mathbb{E}\left[\left(\int_{-z}^{z} \tilde{B}(s)\,\mathrm{d}s\right)^2\right] &= \int_{-z}^{z}\int_{-z}^{z} \mathrm{Cov}(\tilde{B}(s), \tilde{B}(t))\,\mathrm{d}s\,\mathrm{d}t \\ &= 2 \iint\limits_{-z<s<t<z} 2 \cdot \frac{(s+z)(z-t)}{2z}\,\mathrm{d}s\,\mathrm{d}t = \frac{4}{3}z^3.\end{aligned}$$

Inputting this into the earlier display and evaluating the integrals of the constant terms there gives that

$$\mathbb{P}\left(B(x) > -x^2 \;\; \forall x \in [-z, z]\right) \leq \mathbb{P}\left(N(0, 4z^3/3) > 4z^3/3 - 2z\right).$$

The last probability is at most $\exp(-2z^3/3)$ by Lemma 3.5, up to first order in the exponent.

The fact that a matching upper bound could be obtained by considering the integral may initially appear surprising, but can be explained by observing that the parabola is the function which minimizes the Brownian energy functional under the constraint that the function's integral be larger than a given level (i.e., the function encloses at least a certain area).

The proof of Proposition 3.8 is straightforward but slightly tedious, and relies on first demanding that the Brownian bridge remain above $-x^2$ on a fine mesh, and then controlling the fluctuation of the bridge on the intervals in between the mesh points. It is deferred to Appendix C.

As was indicated in the proof outline in Section 1.7, we will need to have lower bounds also for partition functions with similar parabolic boundary data. The following is a general tool to convert lower bounds on non-avoidance probabilities to lower bounds on analogous partition functions. Though simple to prove, as we will see, this already yields sharp estimates needed to transfer our arguments from zero temperature to positive temperature.

Lemma 3.11. *Suppose $t > 0$ and $z_1 < z_2$. Let p be a given lower boundary curve. Let $x > -z_1^2$ and $y > -z_2^2$, and let B be a rate two Brownian motion from (z_1, x) to (z_2, y). Then, for any* *non-negative function* $g : [z_1, z_2] \to [0, \infty)$, $Z_{H_t}^{1,z_1,z_2,x,y,p}$ *is lower bounded by*

$$\exp\left(-2t^{2/3}e^{-t^{1/6}} \int_{z_1}^{z_2} \exp(-t^{1/3}g(u))\,\mathrm{d}u\right) \cdot \mathbb{P}\left(B(u) > p(u) + g(u) + t^{-1/6}\ \forall u \in [z_1, z_2]\right).$$

Here, the fact that we give a buffer of $t^{-1/6}$ on the right-hand side, and in particular the constant $\frac{1}{6}$, has no significance; it is merely to ensure that we get a decay factor, in this case $\exp(-t^{1/6})$, which goes to zero faster than $t^{2/3}$ as $t \to \infty$.

Proof. By definition, since B is a Brownian bridge distributed as $\mathbb{P}_{\mathrm{free}}^{1,z_1,z_2,x,y}$,

$$\begin{aligned}
&Z_{H_t}^{1,z_1,z_2,x,y,p}\\
&= \mathbb{E}_{\mathrm{free}}^{1,z_1,z_2,x,y}\left[\exp\left\{-\int_{z_1}^{z_2} 2t^{2/3}\exp\left(t^{1/3}(p(u) - B(u))\right)\,\mathrm{d}u\right\}\right]\\
&\geq \mathbb{E}_{\mathrm{free}}^{1,z_1,z_2,x,y}\left[\exp\left\{-\int_{z_1}^{z_2} 2t^{2/3}\exp\left(t^{1/3}(p(u) - B(u))\right)\,\mathrm{d}u\right\}\mathbb{1}_{B(u)>p(u)+g(u)+t^{-1/6}\ \forall u\in[z_1,z_2]}\right]\\
&\geq \mathbb{E}_{\mathrm{free}}^{1,z_1,z_2,x,y}\left[\exp\left\{-\int_{z_1}^{z_2} 2t^{2/3}\exp(-t^{1/3}g(u) - t^{1/6})\,\mathrm{d}u\right\}\mathbb{1}_{B(u)>p(u)+g(u)+t^{-1/6}\ \forall u\in[z_1,z_2]}\right]\\
&= \exp\left(-2t^{2/3}e^{-t^{1/6}}\int_{z_1}^{z_2}\exp(-t^{1/3}g(u))\,\mathrm{d}u\right)\cdot \mathbb{P}\big(B(u) > p(u) + g(u) + t^{-1/6}\ \forall u \in [z_1, z_2]\big). \qquad \square
\end{aligned}$$

The following proposition lower bounding the normalizing constant in positive temperature is an easy consequence of the zero-temperature calculation we performed in Proposition 3.8 and the just-proved Lemma 3.11.

Corollary 3.12. *Suppose* $t > 0$ *and* $z_1 < z_2$. *Let* p *be given by* $p(u) = -u^2$. *Let* $x \geq -z_1^2 + 1 + t^{-1/6}$, $y \geq -z_2^2 + 1 + t^{-1/6}$. *Then, for all* $z_2 - z_1$ *larger than an absolute constant,*

$$Z_{H_t}^{1,z_1,z_2,x,y,p} \geq \exp\left(-\frac{1}{12}(z_2 - z_1)^3 - 3(z_2 - z_1)\log(z_2 - z_1)\right).$$

Proof. We apply Proposition 3.8 and Lemma 3.11 with $p(u) = -u^2$ and $g(u) \equiv 0$, and then observe by direct calculation that $t^{2/3}e^{-t^{1/6}} \leq 5$ for all $t > 0$. Finally we use that $10 \leq \log(z_2 - z_1)$ for sufficiently large $z_2 - z_1$. $\square$

Remark 3.13. As in the zero-temperature case discussed in Remark 3.10, we can prove a matching upper bound on $Z_{H_t}^{1,z_1,z_2,x,y,p}$ when $t \geq t_0$ (with an error term depending on t_0) and $p(u) = -u^2$. We show it in the simpler case that $z_2 = -z_1 = z$ for convenience. Indeed,

$$\begin{aligned}
Z_{H_t}^{1,-z,z,x,y,p} &= \mathbb{E}_{\mathrm{free}}^{1,-z,z,x,y}\left[\exp\left\{-\int_{-z}^{z} 2t^{2/3}\exp\left(t^{1/3}(p(u) - B(u))\right)\,\mathrm{d}u\right\}\right]\\
&\leq \mathbb{E}_{\mathrm{free}}^{1,-z,z,x,y}\left[\exp\left\{-\int_{-z}^{z} 2t^{2/3}\exp\left(t^{1/3}(p(u) - B(u))\right)\mathbb{1}_{B(u)<p(u)}\,\mathrm{d}u\right\}\right]\\
&\leq \mathbb{E}_{\mathrm{free}}^{1,-z,z,x,y}\left[\exp\left\{-\int_{-z}^{z} 2t^{2/3}(-C + (2t_0)^{-1}t^{1/3}(p(u) - B(u)))\mathbb{1}_{B(u)<p(u)}\,\mathrm{d}u\right\}\right],
\end{aligned}$$

using that $\exp(x) \geq -C + (2t_0)^{-1}x$ for all $x \in \mathbb{R}$ for some $C = C(t_0)$; this can be seen to be true immediately using the convexity of $\exp(x)$ for some C, and simple calculus shows that $C = (2t_0)^{-1}(\log((2t_0)^{-1}) - 1)$ works. Using the monotonicity in t of the integrand and that $t \geq t_0$, we see that the previous expression is at most

$$e^{2Ct^{2/3}} \cdot \mathbb{E}_{\mathrm{free}}^{1,-z,z,x,y}\left[\exp\left\{-\int_{-z}^{z} t_0^{-1}t(p(u) - B(u))\mathbb{1}_{B(u)<p(u)}\,\mathrm{d}u\right\}\right]$$

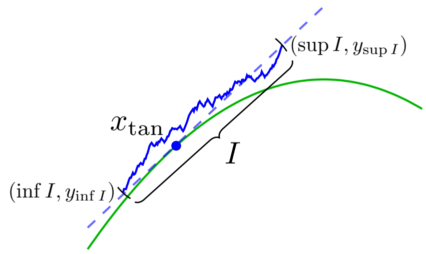

FIGURE 6. A depiction of the setup of Proposition 3.14. There is a Brownian bridge B (in blue) defined on an interval I, and we consider the probability that it avoids the parabola (in green) plus a quantity $\varepsilon M \sigma_{\tan}$, where $\sigma_{\tan}^2$ is the variance of B at $x_{\tan}$; because $x_{\tan}$ is the point at which the line joining B's endpoints is closest to the parabola, the probability of B avoiding $-x^2$ in its neighbourhood is the governing contribution to the overall non-intersection probability, and it is for this reason that we scale the demanded separation between B and $-x^2$ by $\sigma_{\tan}$.

$$\leq e^{2Ct^{2/3}} \cdot \mathbb{E}_{\mathrm{free}}^{1,-z,z,x,y}\left[\exp\left\{\int_{-z}^{z}(B(u)-p(u))\mathbb{1}_{B(u)<p(u)}\,\mathrm{d}u\right\}\right]$$
$$\leq e^{2Ct^{2/3}} \cdot \mathbb{E}_{\mathrm{free}}^{1,-z,z,x,y}\left[\exp\left\{\int_{-z}^{z}(B(u)-p(u))\,\mathrm{d}u\right\}\right].$$

Now, $\int_{-z}^{z} -p(u)\,\mathrm{d}u = 2z^3/3$, while $\int_{-z}^{z} B(u)\,\mathrm{d}u$ is a normal random variable with mean $-z^2+1+t^{-1/6}$ and variance $4z^3/3$ (by the calculation in Remark 3.10). So, integrating the mean of the Gaussian separately, the previous upper bound on the normalizing constant is equal to

$$\exp\left(-2z^3+2z+2zt^{-1/6}+\frac{2}{3}z^3+2Ct^{2/3}\right)\mathbb{E}\left[\exp(N(0,4z^3/3))\right]$$
$$= \exp\left(-\frac{4}{3}z^3+2z+2zt^{-1/6}+\frac{2z^3}{3}+2Ct^{2/3}\right)$$
$$\leq \exp\left(-\frac{2}{3}z^3+6z+2Ct^{2/3}\right).$$

Observe from our expression for $C(t_0)$ that we may take it to be 0 if $t_0 \geq (2e)^{-1}$.

3.5. **Estimates on Brownian fluctuations above parabolas.** While Proposition 3.8 stated a lower bound on the probability of a Brownian bridge staying above a parabola when its endpoints were close to the parabola, we will often be considering cases where the bridge's endpoints are much higher than the parabola, such that the line connecting the endpoints is tangent to the parabola. In such cases we will need to know that the probability of avoidance is basically of constant order.

In the following proposition we make essentially that statement. See Figure 6 for a depiction of the setting of Proposition 3.14.

Proposition 3.14. *Let $I \subseteq \mathbb{R}$ be a finite interval. Suppose $x_{\tan} \in [\inf I + 1, \sup I - 1]$, and let $(\inf I, y_{\inf I})$ and $(\sup I, y_{\sup I})$ be points on the line $\ell^{\tan} : I \to \mathbb{R}$ tangent to the curve $-x^2$ at the point $(x_{\tan}, -x_{\tan}^2)$. Let B be a rate two Brownian bridge from $(\inf I, y_{\inf I})$ to $(\sup I, y_{\sup I})$. Let $\sigma_{\tan}^2 = \mathrm{Var}(B(x_{\tan}))$. Then there exist constants $C > 0$, $\varepsilon_0 > 0$, and M_0 (all universal) such that, for any $M_0 < M < C^{-1}\min(x_{\tan} - \inf I, \sup I - x_{\tan})^{3/2}$,*

$$\mathbb{P}\left(B(x) > -x^2 + \varepsilon_0 M \sigma_{\tan}\ \forall x \in I\right) \geq \exp(-\tfrac{2}{25}M^2). \tag{9}$$

There exists a constant $\varepsilon_0 > 0$ (independent of t) such that the same bound holds with the left-hand side equal to Z_{H_t} for any $t > 0$, where Z_{H_t} is the partition function on I with boundary data $(\inf I, y_{\inf I})$ and $(\sup I, y_{\sup I})$ and lower boundary curve $-x^2 + \varepsilon M \sigma_{\tan}$.

Proposition 3.14 is also proved in Appendix C.

We remind the reader of our remark after Theorem 9 that the upper bound on M in many of our estimates arises for simplicity, as beyond that range a KPZ type tail instead of a Gaussian one will be present. Here, however, the upper bound on M has a different reason: without it, the probability would be zero, as B would not satisfy the event at $x = \inf I$ or $\sup I$ where its value is deterministic. The constant $\frac{2}{25}$ in the exponent is not significant, but it will be useful for some arithmetic that will need to be performed in the upcoming Corollary 3.15.

We prove Proposition 3.14 by controlling point values and fluctuations of B on a sequence of dyadic scales. This multi-scale argument is necessary because we allow the tangency point $x^{\tan}$ to be very close to the boundaries of I, independent of the size of I; allowing this closeness is necessary for our arguments, presented in Section 8, verifying that extremal ensembles satisfy Assumption (iv) (tail probability bounds). One can observe some of the delicacy of the estimate by noting that if the tangency location *was* the boundary, then the left-hand side of (9) would be zero (similar to why we impose the upper bound on M). It is also to ensure that the probability is uniformly bounded below that we consider fluctuations on the scale of $\sigma_{\tan}$.

The situation where Proposition 3.14 will often be needed is to control the probability of a Brownian bridge, reweighted by the Radon-Nikodym derivative associated to the H_t-Brownian Gibbs property, deviating from the line joining its endpoints. This can be accomplished by controlling the ratio of a Brownian bridge deviation probability and the non-intersection probability or partition function, since the numerator W_t of the reweighting factor is at most 1 always, and we record such a bound as an immediate corollary of Lemma 3.7 and Proposition 3.14 next.

Corollary 3.15. *Let I, $x_{\tan}$, and B be as in Proposition 3.14 and satisfy the same assumptions. Let $J \subseteq I$ and $\sigma_J^2 = \sup_{x\in J} \mathrm{Var}(B(x))$. Then there exist constants $C > 0$, $c > 0$, $\varepsilon_0 > 0$, and M_0 (all universal) such that, for any $M_0 < M < C^{-1} \min(x_{\tan} - \inf I, \sup I - x_{\tan})^{3/2}$,*

$$\frac{\mathbb{P}\left(\sup_{x\in J} B(x) - \ell^{\tan}(x) \geq M\sigma_J\right)}{\mathbb{P}\left(B(x) > -x^2 + \varepsilon_0 M\sigma_{\tan}\ \forall x \in I\right)} \leq \exp(-cM^2).$$

The same bound holds with the denominator equal to Z_{H_t} (with ε_0 and c independent of t) for any $t > 0$, where Z_{H_t} is the partition function on I with boundary data $(\inf I, y_{\inf I})$ and $(\sup I, y_{\sup I})$ and lower boundary curve $-x^2 + \varepsilon_0 M\sigma_{\tan}$.

We needed the $\frac{2}{25}$ in Proposition 3.14 to be smaller than the constant $\frac{1}{8}$ in the exponent in the numerator here coming from Lemma 3.7, and this is why we were explicit about the former's value.

The scales σ_J and $\sigma_{\tan}$ respectively of the fluctuations of the bridge and the height above the parabola that we demand the bridge stay above are chosen such that the ratio of the probabilities is small and independent of J or the location of $x_{\tan}$.

The next result concerns the fluctuations of a Brownian bridge which is conditioned to stay above a parabola. It will allow us to control the shape of the profile (conditional on $\mathfrak{h}_1^t(0) = \theta$) beyond $[-\theta^{1/2}, \theta^{1/2}]$ and say that it is close to the parabola $-x^2$, as asserted in Proposition 4.2.

Proposition 3.16. *Let $I = [a, b]$ be an interval. For $0 < H \leq |I|^{1/2} \log |I|$, let $B^{\mathrm{para}} : I \to \mathbb{R}$ be a Brownian bridge from $(a, -a^2 + H)$ to $(b, -b^2 + H)$ conditioned on $B^{\mathrm{para}}(x) > -x^2$ for all $x \in I$. Then there exist absolute constants $C, c > 0$, such that for $|I| > C$ and $M > \log |I|$,*

$$\mathbb{P}\left(\sup_{x\in I} \left(B^{\mathrm{para}}(x) + x^2\right) > 2M|I|^{1/2}\right) \leq \exp(-cM^2).$$

The same statement holds uniformly in $t > 0$ if we replace B^{para} by a Brownian bridge $B^{\mathrm{para},t}$ between the same endpoints as B^{para} but tilted by the Radon-Nikodym derivative given by W_{H_t}/Z_{H_t}, where the latter is associated to the same boundary data and lower boundary curve $-x^2$.

Note that we impose that $M > \log|I|$, unlike previous estimates where M was lower bounded just by an absolute constant. This is to allow us to perform a union bound; essentially, we approximate the parabola by $O(|I|)$ many line segments which are close to the parabola for a unit order length, control fluctuations of Brownian bridges defined on these line segments, and take a union bound over all of them.

Proof of Proposition 3.16. We will adopt the notation that $B^{\mathrm{para}} = B^{\mathrm{para},t}$ with $t = \infty$, and prove the proposition for $B^{\mathrm{para},t}$ for all $t > 0$; to get the $t = \infty$ case, one needs to just replace applications of the H_t-Brownian Gibbs property by the usual Brownian Gibbs property.

We will show that there exists $c > 0$ such that, for any $k \in [a, b-1]$,

$$\mathbb{P}\left(\sup_{x\in[k,k+1]}\left(B^{\mathrm{para},t}(x) + x^2\right) > 2M|I|^{1/2}\right) \leq \exp(-cM^2). \tag{10}$$

This suffices to prove the proposition by a union bound over $O(|I|)$ many values of k, since we have set $M > \log|I|$ and $|I|$ large enough.

Now (10) is easy to prove by considering the line $\ell_k^{\tan}$ which is tangent to $-x^2$ at k. Indeed, let B_k be a Brownian bridge from $(a, \ell_k^{\tan}(a) + |I|^{1/2}\log|I|)$ to $(b, \ell_k^{\tan}(b) + |I|^{1/2}\log|I|)$. By concavity of the function $-x^2$ and the assumed upper bound on H, these points are above $(a, -a^2 + H)$ and $(b, -b^2 + H)$ respectively. So, by the H_t-Brownian Gibbs property and monotonicity Lemma 3.1, B_k, tilted by W_{H_t}/Z_{H_t} where the latter are as in the statement of the proposition, is stochastically larger than $B^{\mathrm{para},t}$. Thus,

$$\begin{aligned}\mathbb{P}&\left(\sup_{x\in[k,k+1]}\left(B^{\mathrm{para},t}(x) + x^2\right) > 2M|I|^{1/2}\right)\\ &\qquad\leq Z_{H_t}^{-1}\mathbb{E}\left[\mathbb{1}_{\sup_{x\in[k,k+1]}(B_k(x)+x^2)>2M|I|^{1/2}}W_{H_t}\right],\end{aligned}$$

where W_{H_t} and Z_{H_t} are associated with the same boundary values as B_k and lower boundary curve $-x^2$, on the interval I.

Next we observe that on $[k, k+1]$, $+x^2$ differs from $-\ell_k^{\tan}(x)$ by a unit order, and that $W_{H_t} \leq 1$, and so the right-hand side is upper bounded by

$$Z_{H_t}^{-1}\mathbb{E}\left[\mathbb{1}_{\sup_{x\in[k,k+1]}\left(B_k(x)-\ell_k^{\tan}(x)\right)>\frac{3}{2}M|I|^{1/2}}\right].$$

By Corollary 3.15, this is upper bounded by $\exp(-cM^2)$, so we are done. ☐

3.6. **Corresponding estimates for $\mathfrak{h}_1^t$.** Here we use some of the Brownian estimates proved in the previous subsection to obtain analogous statements for $\mathfrak{h}_1^t$. For the most part, these are the estimates that will be used in the main arguments, and their proofs also showcase the typical way in which Brownian estimates will be combined with the Gibbs property.

The first statement is the analogue of Corollary 3.15 on the probability of a Brownian bridge (reweighted according to parabolic boundary data) deviating significantly from the line connecting its endpoints, when that line is tangent to the parabola.

We focus on deviations above the line; in order to ensure that we can consider a Brownian bridge whose endpoints give a line tangent to the parabola, the estimate includes the event BT ("below tangent") that the values of $\mathfrak{h}_1^t$ lie below the tangent line. By monotonicity, we can increase the endpoints then to lie on the tangent line.

Similar to previous estimates in this section, there are a number of technical choices made regarding the size of intervals, locations of points, etc. These choices are generally not crucial or significant and are only to make certain arithmetic hold or be more convenient in the argument.

Proposition 3.17. *Suppose Assumption (iv) hold with $\beta = \frac{3}{2}$. Let $I \subseteq \mathbb{R}$ be an interval with $|I| \geq 4$ and $(\inf I, y_{\inf I})$ and $(\sup I, y_{\sup I})$ be points on the line $\ell^{\tan}_{x_{\tan}}$ tangent to the curve $-x^2$ at the point $(x_{\tan}, -x_{\tan}^2)$ with $x_{\tan} \in [\inf I + |I|^{1/2}, \sup I - |I|^{1/2}]$. Let BT be the event $\{\mathfrak{h}^t_1(\inf I) \leq y_{\inf I}, \mathfrak{h}^t_1(\sup I) \leq y_{\sup I}\}$ of being below the tangent and J_k the interval of length $2k$ centred at $x_{\tan}$, where k is such that $J_k \subseteq I$.*

Let $x_1, x_2 \notin I$ and $z_1, z_2 \in \mathbb{R}$. Then there exist absolute constants $c > 0$, M_0, and $C > 0$ such that, for all $\max(M_0, 2k^2) < M < C^{-1}|I|^{3/4}$,

$$\mathbb{P}\left(\sup_{x\in J_k} \mathfrak{h}^t_1(x) + x^2 > M|I|^{1/2}, \mathsf{BT} \;\middle|\; \mathfrak{h}^t_1(x_1) = z_1, \mathfrak{h}^t_1(x_2) = z_2\right) \leq \exp(-cM^2) \quad \text{and}$$

$$\mathbb{P}\left(\sup_{x\in I} \mathfrak{h}^t_1(x) - \ell^{\tan}_{x_{\tan}}(x) > M|I|^{1/2}, \mathsf{BT} \;\middle|\; \mathfrak{h}^t_1(x_1) = z_1, \mathfrak{h}^t_1(x_2) = z_2\right) \leq \exp(-cM^2).$$

Taking $x_1 = x_2 \notin I$ and $z_1 = z_2$ gives an analogous single point conditioning version.

We do not strictly need $\beta = \frac{3}{2}$ for Proposition 3.17 to hold. We have assumed it to simplify some calculations, and because we will be applying it only after an upper tail bound with $\beta = \frac{3}{2}$ has been proven for extremal ensembles (we will verify this holds for the KPZ and parabolic Airy line ensembles using results of [CG20a] and [RRV11] recorded in Theorem A.7).

Proof of Proposition 3.17. We start with proving the first inequality.

Let $\varepsilon > 0$ be the constant from Proposition 3.14 and $\sigma^2_{\tan} = (x_{\tan} - \inf I)(\sup I - x_{\tan})/|I|$. Note that $\sigma^2_{\tan} \geq |I|^{1/2}(1 - |I|^{-1/2}) \geq \frac{1}{2}|I|^{1/2}$ by our assumption that $x_{\tan} \in [\inf I + |I|^{1/2}, \sup I - |I|^{1/2}]$, since the minimum is attained at one of the boundaries, and since $|I| > 4 \implies |I|^{-1/2} \leq 2^{-1}$. To bound the probability appearing in the statement, we first break up the probability based on the occurrence of a favourable event concerning the second curve.

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in J_k} \mathfrak{h}^t_1(x) + x^2 \geq M|I|^{1/2}, \mathsf{BT} \;\middle|\; \mathfrak{h}^t_1(x_1) = z_1, \mathfrak{h}^t_1(x_2) = z_2\right)\\
&\quad\leq \mathbb{P}\left(\sup_{x\in J_k} \mathfrak{h}^t_1(x) + x^2 \geq M|I|^{1/2},\ \mathsf{BT},\ \sup_{x\in I} \mathfrak{h}^t_2(x) + x^2 \leq \varepsilon M\sigma_{\tan} \;\middle|\; \mathfrak{h}^t_1(x_1) = z_1, \mathfrak{h}^t_1(x_2) = z_2\right)\\
&\qquad + \mathbb{P}\left(\sup_{x\in I} \mathfrak{h}^t_2(x) + x^2 > \varepsilon M\sigma_{\tan} \;\middle|\; \mathfrak{h}^t_1(x_1) = z_1, \mathfrak{h}^t_1(x_2) = z_2\right).
\end{aligned} \tag{11}$$

By Lemma 3.18 and since $\sigma_{\tan} \geq 2^{-1/2}|I|^{1/4}$, the second term of (11) is upper bounded by

$$|I| \exp(-c\varepsilon^{3/2}M^{3/2}|I|^{3/8}) \leq \exp(-c\varepsilon^{3/2}M^2),$$

the last inequality for large enough M_0 since we have assumed $M_0 < M \leq C^{-1}|I|^{3/4}$ (for the next part of the argument we will take C as in Proposition 3.14) and reducing the value of c.

Next we analyze the first term of (11). Let $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(1, I)$. Observe that $\{\sup_{x\in I} \mathfrak{h}^t_2(x) + x^2 \leq \varepsilon M\sigma_{\tan}\} \in \mathcal{F}$. Then we see that the first term equals

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\sup_{x\in J_k} \mathfrak{h}^t_1(x) + x^2 \geq M|I|^{1/2} \;\middle|\; \mathfrak{h}^t_1(x_1) = z_1, \mathfrak{h}^t_1(x_2) = z_2\right) \mathbb{1}_{\mathsf{BT}, \sup_{x\in I} \mathfrak{h}^t_2(x) + x^2 \leq \varepsilon M\sigma_{\tan}}\right]$$

Under $\mathbb{P}_{\mathcal{F}}$, $\mathfrak{h}^t_1$ is distributed as a Brownian bridge from $(\inf I, \mathfrak{h}^t_1(\inf I))$ to $(\sup I, \mathfrak{h}^t_1(\sup I))$ tilted by the Radon-Nikodym derivative W_{H_t}/Z_{H_t} associated with lower boundary curve $\mathfrak{h}^t_2$ on I.

On the event $\mathsf{BT} \cap \{\sup_{x\in I} \mathfrak{h}_2^t(x) + x^2 \le \varepsilon M|I|^{1/2}\}$, Lemma 3.1 (monotonicity) says that $\mathfrak{h}_1^t$ under $\mathbb{P}_{\mathcal{F}}$ is stochastically dominated by a Brownian bridge B from $(\inf I, y_{\inf I})$ to $(\sup I, y_{\sup I})$, tilted by $\tilde{W}_{H_t}/\tilde{Z}_{H_t}$ associated to the lower boundary curve $-x^2 + \varepsilon M\sigma_{\tan}$ on I.

Since B does not depend on any data in $\mathcal{F}$ and $\tilde{W}_{H_t} \le 1$, the previous display is bounded above by

$$\tilde{Z}_{H_t}^{-1} \cdot \mathbb{P}\left(\sup_{x\in J_k} B(x) + x^2 \ge M|I|^{1/2}\right).$$

Note that since J_k is an interval of length $2k$ centred at the tangent location, the tangent line $\ell^{\tan}(x)$ differs from $-x^2$ by at most k^2 on J_k. Thus we may upper bound the probability in the above expression with the same probability, but with $+x^2$ replaced by $-\ell^{\tan}(x)$ and M reduced to say $M/2$ since we have assumed $M/2 > k^2$.

By Corollary 3.15 and our choice of ε, the latter probability is bounded by $\exp(-cM^2)$, so we have proved the first inequality of Proposition 3.17; the second inequality is proved in the same way by considering the event $\{\sup_{x\in I} \mathfrak{h}_1^t(x) - \ell_\beta^{\tan}(x) > M\theta^{1/4}\}$ in place of $\{\sup_{x\in J_k} \mathfrak{h}_1^t(x) + x^2 \ge M\theta^{1/4}\}$. □

The next result controls fluctuations of the second curve $\mathfrak{h}_2^t$ under conditionings of the first curve. As was seen in the proof outline in Section 1.7, this kind of control will be frequently needed in our arguments. The argument makes use of Assumption (ii), which allows us to dominate the conditioned $\mathfrak{h}_2^t$ by an unconditioned $\mathfrak{h}_1^t$.

Lemma 3.18. *Let $t_0 > 0$, $I \subseteq \mathbb{R}$ be an interval with $|I| > 10$, and $C < \infty$. Under Assumptions (i)–(iv), for $a, b \in \mathbb{R}$, $\theta \in \mathbb{R}$, $M > (4\log|I|)^{1/\beta}$, and $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\sup_{x\in I} \mathfrak{h}_2^t(x) + x^2 > M \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta\right) \le \exp(-\tfrac{1}{2}M^\beta).$$

The same bound holds if we instead condition on a single point, eg. $\mathfrak{h}_1^t(0) = \theta$.

Proof. We prove the two-point conditioning case as the one-point is exactly the same. By Lemma 3.2 (monotonicity of conditioning),

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in I} \mathfrak{h}_2^t(x) + x^2 > M \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta\right)\\
&\qquad \le \mathbb{P}\left(\sup_{x\in I} \mathfrak{h}_2^t(x) + x^2 > M \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) \ge a\theta, \mathfrak{h}_1^t(\theta^{1/2}) \ge b\theta\right).
\end{aligned}$$

Now by Assumption (ii) (BK inequality), the previous line is upper bounded by

$$\mathbb{P}\left(\sup_{x\in I} \mathfrak{h}_1^t(x) + x^2 > M\right) \le (|I|+1) \cdot \mathbb{P}\left(\sup_{x\in[-1,0]} \mathfrak{h}_1^t(x) + x^2 > M\right),$$

the inequality by a union bound and the stationarity of $\mathfrak{h}_1^t(x) + x^2$, and where we bound by $|I|+1$ to handle the case when $|I|$ is not an integer. By Proposition 3.3 and Assumption (iv),

$$(|I|+1) \cdot \mathbb{P}\left(\sup_{x\in[-1,0]} \mathfrak{h}_1^t(x) + x^2 > M\right) \le 4M \cdot (|I|+1) \cdot \exp(-M^\beta) \le \exp\left(-\tfrac{1}{2}M^\beta\right)$$

since $M > (4\log|I|)^{1/\beta}$ and $|I| > 10$. □

4. One-point limit shape

In this section we prove Theorem 9 on the shape of the profile of $\mathfrak{h}_1^t$ on $[-\theta^{1/2},\theta^{1/2}]$ under the conditioning that $\mathfrak{h}_1^t(0)=\theta$. But first we reformulate it under our general assumptions as Theorem 4.1. Recall the role of β from Assumption (iv), namely that $\mathbb{P}(\mathfrak{h}_1^t(0)>\theta)\le\exp(-c_2\theta^\beta)$.

Theorem 4.1. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). There exist $C<\infty$, $c>0$, and θ_0 such that, for all $\theta>\theta_0$, $0<M<C^{-1}\theta^{3/4}$, and $t\in[t_0,\infty]$,*

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},\theta^{1/2}]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\ge M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\le\exp(-cM^{4\beta/3})$$

and

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},\theta^{1/2}]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\le -M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\le\exp(-cM^2)+4\cdot\mathbb{P}\left(\mathfrak{h}_1^t(0)\le-\tfrac12 M\theta^{1/4}\right).$$

We separate the upper and lower deviation probabilities as the latter involves the one-point lower tail of $\mathfrak{h}_1^t$. While bounds for this are available for the KPZ equation and the parabolic Airy$_2$ process, thus allowing us to obtain a bound of $\exp(-cM^2)$ overall for both deviations as stated in Theorem 9, we have no lower tail bounds for extremal ensembles. Fortunately it is enough for our arguments in Section 8 for the extremal case to know that the lower tail is tight, without any quantitative control.

Proof of Theorem 9. We know from Theorem 2.8 that Assumptions (i)–(iv) holds with $\alpha=\beta=3/2$ for $\mathfrak{h}_1^t$ for $t\in[t_0,\infty]$ uniformly for any $t_0>0$, so the first bound of Theorem 4.1 reads $\exp(-cM^2)$, as claimed in Theorem 9.

We now need to show that the second bound of Theorem 4.1 is also at most $\exp(-cM^2)$, i.e., we have to show the same bound for $\mathbb{P}(\mathfrak{h}_1^t(0)\le-\frac12 M\theta^{1/4})$. For this we quote [CG20b, Theorem 1.1] for $t_0<t<\infty$ and [RRV11, Theorem 1.1] for $t=\infty$. ☐

Proof of Theorem 4.1: $\mathfrak{h}_1^t$ is above $\mathsf{Tri}_\theta-O(\theta^{1/4})$ with high probability. We first show the second inequality above. In fact, by a union bound and symmetry, it is enough to prove the inequality when the supremum inside the probability is over only $[-\theta^{1/2},0]$. Now,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\le -M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\\
&\quad\le\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\le -M\theta^{1/4},\mathfrak{h}_1^t(-\theta^{1/2})>-\theta-\tfrac12 M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\\
&\qquad+\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2})\le-\theta-\tfrac12 M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right).
\end{aligned}\tag{12}$$

Let us bound the second term. We see from Lemma 3.2 that

$$\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2})\le-\theta-\frac12 M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\le\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2})\le-\theta-\frac12 M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)\le\theta\right).$$

We may assume θ is large enough (uniformly in $t\in[t_0,\infty]$ using Assumption (iv)) that $\mathbb{P}(\mathfrak{h}_1^t(0)\le\theta)\ge\frac12$. Then we obtain that, for all large enough θ,

$$\begin{aligned}
\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2})\le-\theta-\frac12 M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)\le\theta\right)&\le 2\cdot\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2})\le-\theta-\frac12 M\theta^{1/4}\right)\\
&=2\cdot\mathbb{P}\left(\mathfrak{h}_1^t(0)\le-\frac12 M\theta^{1/4}\right),
\end{aligned}$$

the last line by stationarity of $\mathfrak{h}_1^t(x)+x^2$.

Now we return to the first term of (12). We want to bound the numerator using the H_t-Brownian Gibbs property. Let $\mathcal{F}=\mathcal{F}_{\mathrm{ext}}(1,[-\theta^{1/2},\theta^{1/2}])$. Recall the notation $\mathbb{P}_{\mathcal{F}}(\cdot)=\mathbb{P}(\cdot\mid\mathcal{F})$ introduced on page 24. Then the first term of (12) is

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\leq -M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\mathbb{1}_{\mathfrak{h}_1^t(-\theta^{1/2})>-\theta-\frac{1}{2}M\theta^{1/4}}\right].$$

By the H_t-Brownian Gibbs property, under $\mathbb{P}_{\mathcal{F}}$, $\mathfrak{h}_1^t$ is a rate two Brownian bridge from $(-\theta^{1/2},\mathfrak{h}_1^t(-\theta^{1/2}))$ to $(0,\mathfrak{h}_1^t(0))$ subject to the Radon-Nikodym derivative given by W_{H_t}/Z_{H_t} associated to these boundary values and lower boundary data $\mathfrak{h}_2^t$ (see Definition 2.2). Now, we are restricted to the situation that $\mathfrak{h}_1^t(0)=\theta$ and $\mathfrak{h}_1^t(-\theta^{1/2})\geq-\theta-\frac{1}{2}M\theta^{1/4}$. By monotonicity (Lemma 3.1), the probability that we are considering, that $\mathfrak{h}_1^t$ is below Tri_θ, is increased if we lower the endpoints of the Brownian bridge as much as possible (i.e., so that they are $-\theta-\frac{1}{2}M\theta^{1/4}$ and θ at the corresponding points) and take the lower boundary curve to $-\infty$, i.e., remove the lower boundary conditioning from the Brownian bridge. Thus, we see

$$\begin{aligned}\mathbb{P}_{\mathcal{F}}&\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\leq -M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\mathbb{1}_{\mathfrak{h}_1^t(-\theta^{1/2})>-\theta-\frac{1}{2}M\theta^{1/4}}\\&\leq\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}B(x)-\mathsf{Tri}_\theta(x)\leq -M\theta^{1/4}\right),\end{aligned}$$

where B is a rate two Brownian bridge from $(-\theta^{1/2},-\theta-\frac{1}{2}M\theta^{1/4})$ to $(0,\theta)$. Observe that $\mathbb{E}[B(x)]=\mathsf{Tri}_\theta(x)-M\theta^{-1/2}$. So,

$$\begin{aligned}\mathbb{P}_{\mathcal{F}}&\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\leq -M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\mathbb{1}_{\mathfrak{h}_1^t(-\theta^{1/2})>-\theta-\frac{1}{2}M\theta^{1/4}}\\&\leq\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}B(x)-\mathbb{E}[B(x)]\leq-\frac{1}{2}M\theta^{1/4}\right)\\&\leq\exp(-cM^2),\end{aligned}$$

using tail bounds on the supremum of Brownian bridge from Lemma 3.6. Tracing the steps back to (12), we obtain

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\leq -M\theta^{1/4},\mathfrak{h}_1^t(-\theta^{1/2})>-\theta-\frac{1}{2}M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\leq\exp(-cM^2).$$

Returning to (12) and using the earlier bound for its second term, we see that

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\leq -M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right)\leq\exp(-cM^2)+2\cdot\mathbb{P}\left(\mathfrak{h}_1^t(0)\leq-\tfrac{1}{2}M\theta^{1/4}\right),$$

which completes the proof of one side of the theorem.

Proof that $\mathfrak{h}_1^t$ is below $\mathsf{Tri}_\theta+O(\theta^{1/4})$ with high probability. As in the previous bound, we will only bound

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},0]}\mathfrak{h}_1^t(x)-\mathsf{Tri}_\theta(x)\geq M\theta^{1/4}\;\Big|\;\mathfrak{h}_1^t(0)=\theta\right),$$

i.e., we look at the curve to the left of 0 only.

Recall from Section 1.7.5 and the figure there that we want to find a point *on* the tangency line such that, even under the large deviation conditioning, $\mathfrak{h}_1^t(x)$ is below that point with high probability. We take $x_0 = 2\theta^\gamma$, where $\gamma = (2\beta)^{-1}(\alpha + \frac{3}{2})$,[2] with α and β as in Assumption (iv). Now, the value of the tangent (i.e., Tri_θ extended in the obvious linear way to $\mathbb{R}$) at $-x_0$ is $\theta - 2\theta^{1/2}x_0 = \theta - 4\theta^{\gamma+1/2}$ and the value of the parabola is $-x_0^2 = -4\theta^{2\gamma}$; clearly the height separating the tangent and the parabola is $\theta - 2\theta^{1/2}x_0 + x_0^2 = (x_0 - \theta^{1/2})^2 = (2\theta^\gamma - \theta^{1/2})^2$. Consider the probability

$$\mathbb{P}\left(\mathfrak{h}_1^t(-x_0) > \theta - 2\theta^{1/2}x_0 \mid \mathfrak{h}_1^t(0) > \theta\right) = \mathbb{P}\big(\mathfrak{h}_1^t(-x_0) + x_0^2 > (x_0 - \theta^{1/2})^2 \mid \mathfrak{h}_1^t(0) > \theta\big).$$

Using the tail bounds from Assumption (iv) and the stationarity from Assumption (i), this probability can be bounded, for $\theta > \theta_0(t_0)$, as

$$\mathbb{P}\Big(\mathfrak{h}_1^t(-x_0) + x_0^2 > (2\theta^\gamma - \theta^{1/2})^2 \mid \mathfrak{h}_1^t(0) > \theta\Big) \le \exp\left(-c_2(2\theta^\gamma - \theta^{1/2})^{2\beta} + c_1\theta^\alpha\right); \tag{13}$$

since $\alpha \ge \beta$, it follows that $\gamma > \frac{1}{2}$, so the previous right-hand side is at most $\exp(-c_2\theta^{2\gamma\beta} + c_1\theta^\alpha) \le \exp(-\theta^{3/2})$ for all large enough θ since our choice of γ satisfies $2\gamma\beta = \alpha + \frac{3}{2}$. Also observe that, since $M < C^{-1}\theta^{3/4}$, $\exp(-\theta^{3/2}) \le \exp(-cM^2)$.

Now, similar to the proof lower bounding, we include some auxiliary events to aid us in the analysis. Apart from control over $\mathfrak{h}_1^t(-x_0\theta^{1/2})$, we will need an upper bound on the second curve's deviation above $-x^2$. We adopt the shorthand

$$\left\{\mathfrak{h}_1^t \in A_{\theta,M}\right\} = \left\{\sup_{x\in[-\theta^{1/2},0]} \mathfrak{h}_1^t(x) - \mathsf{Tri}_\theta(x) \ge M\theta^{1/4}\right\}$$

for notational convenience. Now,

$$\begin{aligned}
&\mathbb{P}\left(\mathfrak{h}_1^t \in A_{\theta,M} \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right)\\
&\quad\le \mathbb{P}\left(\mathfrak{h}_1^t \in A_{\theta,M},\, \mathfrak{h}_1^t(-x_0) \le \theta - 2\theta^{1/2}x_0,\, \sup_{x\in[-x_0,0]} \mathfrak{h}_2^t(x) + x^2 \le \varepsilon M\theta^{1/4} \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right)\\
&\qquad + \mathbb{P}\Big(\mathfrak{h}_1^t(-x_0) > \theta - 2\theta^{1/2}x_0 \;\Big|\; \mathfrak{h}_1^t(0) = \theta\Big)\\
&\qquad + \mathbb{P}\left(\sup_{x\in[-x_0,0]} \mathfrak{h}_2^t(x) + x^2 > \varepsilon M\theta^{1/4} \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right),
\end{aligned} \tag{14}$$

where $\varepsilon > 0$ is the t-independent constant from Proposition 3.14. We already saw that the second term is bounded above by $\exp(-cM^2)$, with c independent of t for $\theta > \theta_0(t_0)$ coming from Assumption (iv). By Lemma 3.18, the third term is bounded by $\exp(-cM^{4\beta/3})$, again with c independent of t as ε is independent of t. So we may focus on the first term. The argument is similar to that of Proposition 3.17 and essentially extends that proposition to the case where $\beta < 3/2$.

As in the lower bound, we apply the Brownian Gibbs property. Similarly to before, let $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(1, [-x_0, 0])$. Then we see that the first term in the last display is equal to

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}(\mathfrak{h}_1^t \in A_{\theta,M} \mid \mathfrak{h}_1^t(0) = \theta)\mathbb{1}_{\mathfrak{h}_1^t(-x_0)\le\theta-2\theta^{1/2}x_0,\, \sup_{x\in[-x_0,0]}\mathfrak{h}_2^t(x)+x^2\le\varepsilon M\theta^{1/4}}\right]$$

By the Brownian Gibbs property, under $\mathbb{P}_{\mathcal{F}}$ and $\{\mathfrak{h}_1^t(0) = \theta\}$, $\mathcal{P}_1$ is a rate two Brownian bridge from $(-x_0, \mathfrak{h}_1^t(-x_0))$ to $(0, \mathfrak{h}_1^t(0)) = (0, \theta)$ subject to the Radon-Nikodym derivative given by $\tilde{W}_{H_t}/\tilde{Z}_{H_t}$ associated with the mentioned boundary values and lower boundary data $\mathfrak{h}_2^t$. By monotonicity, the probability of this Brownian bridge lying in $A_{\theta,M}$ increases by raising its endpoints and raising the

[2]This choice of γ is made purely to ensure that $\mathbb{P}(\mathfrak{h}_1^t(0) > \theta^{2\gamma})/\mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta) \le \exp(-\theta^{3/2})$; the expression for γ then follows from the upper and lower bounds in terms of α and β on the one-point probability from Assumption (iv).

lower boundary condition. Thus the previous display is upper bounded by (using that indicators are bounded by 1)

$$\mathbb{E}\left[\mathbb{1}_{B\in A_{\theta,M}}\frac{W_{H_t}}{Z_{H_t}}\right] \leq \mathbb{P}\left(B\in A_{\theta,M}\right)\cdot Z_{H_t}^{-1},$$

where B is a rate two Brownian bridge from $(-x_0, \theta - 2\theta^{1/2}x_0)$ to $(0,\theta)$ and W_{H_t} and Z_{H_t} are associated with the same boundary values along with the lower boundary curve $-x^2 + \varepsilon M\theta^{1/4}$.

Now, since Tri_θ is the line tangent to $-x^2$ at $(-\theta^{1/2}, -\theta)$ we may apply Corollary 3.15 with $I = [-x_0, 0]$, $J = [-\theta^{1/2}, 0]$, and $x_{\tan} = -\theta^{1/2}$ to see that the previous display is upper bounded by $\exp(-cM^2)$ when $\theta > \theta_0(t_0)$. □

As promised before, while Theorem 9 is focused on the profile inside $[-\theta^{1/2}, \theta^{1/2}]$, we do have a result for the shape outside the interval.

Proposition 4.2. *Let $\mathfrak{h}^t$ be the KPZ line ensemble or parabolic Airy line ensemble. Pick $L > 1$. There exist constants $c > 0$ and $\theta_0 > 0$ such that, for $\theta > \theta_0$ and $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\sup_{x:|x|\in[\theta^{1/2}, L\theta^{1/2}]}\left|\mathfrak{h}_1^t(x) + x^2\right| > \theta^{1/4}\log\theta \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right) \leq \exp(-c(\log\theta)^2).$$

Observe that we do not state this result for extremal ensembles. This is because the probability bound on deviations of $\mathfrak{h}_1^t(x) + x^2$ below $-\theta^{1/4}\log\theta$ relies on quantitative one-point lower tails for the infimum of $\mathfrak{h}_1^t$ over an interval, which are unavailable for extremal ensembles.

Next note that we need the deviation to be of order $\theta^{1/4}\log\theta$ instead of $\theta^{1/4}$; the reason is the same as in Proposition 3.16.

As the reader might already realize, while the true fluctuation scale inside $[-\theta^{1/2}, \theta^{1/2}]$ is $\theta^{1/4}$ (at least in the bulk of the intervals $[-\theta^{1/2}, 0]$ and $[0, \theta^{1/2}]$), the true fluctuation scale on intervals of the form $[-L\theta^{1/2}, \theta^{1/2}]$ and $[\theta^{1/2}, L\theta^{1/2}]$ for $L > 1$ is expected to be $\log\theta$ (rather, $O(1)$ at any given point in such intervals, with the logarithmic factor coming when taking a supremum of fluctuations over the whole interval). While a refinement of our approach is expected to yield improvements over the stated $\theta^{1/4}\log\theta$ bound we do not pursue this.

Proof of Proposition 4.2, the lower bound on the limit shape: By Assumption (iii) (monotonicity in conditioning),

$$\begin{aligned}
&\mathbb{P}\left(\inf_{x:|x|\in[\theta^{1/2}, L\theta^{1/2}]}\mathfrak{h}_1^t(x) + x^2 < -\theta^{1/4}\log\theta \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right)\\
&\qquad\leq \mathbb{P}\left(\inf_{x:|x|\in[\theta^{1/2}, L\theta^{1/2}]}\mathfrak{h}_1^t(x) + x^2 < -\theta^{1/4}\log\theta \;\middle|\; \mathfrak{h}_1^t(0) \leq \theta\right)\\
&\qquad\leq 2\cdot\mathbb{P}\left(\inf_{x:|x|\in[\theta^{1/2}, L\theta^{1/2}]}\mathfrak{h}_1^t(x) + x^2 < -\theta^{1/4}\log\theta\right),
\end{aligned}$$

the last inequality by picking θ_0 large enough that $\mathbb{P}(\mathfrak{h}_1^t(0) \leq \theta) \geq \frac{1}{2}$ for all $\theta > \theta_0$. Now by taking a union bound over $O(\theta^{1/2})$ many unit intervals and applying Proposition 3.4, the previously displayed probability can be upper bounded by $\exp(-c\theta^{3/8}(\log\theta)^{3/2})$.

Proof of the upper bound on the limit shape: By symmetry, it is enough to prove the bound when the supremum inside the probability is over $x \in [-L\theta^{1/2}, \theta^{1/2}]$, i.e., only on the left side of zero.

Similar to the argument in Theorem 4.1, the idea is to show that, conditionally on $\{\mathfrak{h}_1^t(0) = \theta\}$, $\mathfrak{h}_1^t$ is "pinned" close to the parabola $-x^2$ at $-L\theta^{1/2}$ and at $-\theta^{1/2}$ and then apply Proposition 3.16. From Theorem 4.1 we have this pinning at $-\theta^{1/2}$. We next prove that we also have it at $-L\theta^{1/2}$.

Consider the line $\ell_L^{\tan}$ tangent to $-x^2$ at $(-L\theta^{1/2}, -L^2\theta)$. Let $K > L$ be such that, for some $c > 0$,

$$\mathbb{P}\left(\mathfrak{h}_1^t(-K\theta^{1/2}) > \ell_L^{\tan}(-K\theta^{1/2}) \mid \mathfrak{h}_1^t(0) = \theta\right) \le \exp(-c\theta^{3/2}). \tag{15}$$

Such a K exists because, by stationarity of $\mathfrak{h}_1^t(x)+x^2$ and monotonicity in conditioning (Lemma 3.2), we can upper bound the previous probability by

$$\begin{aligned}\mathbb{P}\left(\mathfrak{h}_1^t(-K\theta^{1/2}) > \ell_L^{\tan}(-K\theta^{1/2}) \mid \mathfrak{h}_1^t(0) \ge \theta\right) &\le \frac{\mathbb{P}(\mathfrak{h}_1^t(-K\theta^{1/2}) > \ell_L^{\tan}(-K\theta^{1/2}))}{\mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta)} \\ &= \frac{\mathbb{P}(\mathfrak{h}_1^t(0) > \ell_L^{\tan}(-K\theta^{1/2}) + K^2\theta)}{\mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta)},\end{aligned}$$

and $\ell_L^{\tan}(-K\theta^{1/2})+K^2\theta$ is such that, for any $C < \infty$, there is a K such that $\ell_L^{\tan}(-K\theta^{1/2})+K^2\theta > C\theta$. By choosing a C large enough and using the upper and lower bounds on $\mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta)$ from Theorem 2, we obtain (15).

We will now show that we have the pinning at $-L\theta^{1/2}$ with high probability, i.e., letting $\mathsf{Pin}_{L,M}^\theta = \{\mathfrak{h}_1^t(-L\theta^{1/2}) \le -L^2\theta + M\theta^{1/4}\}$, where M will ultimately be taken to be $\log\theta$. We want to show that

$$\mathbb{P}\left((\mathsf{Pin}_{L,M}^\theta)^c \mid \mathfrak{h}_1^t(0) = \theta\right) \le \exp(cM^2). \tag{16}$$

To prove (16), we may assume from (15) and Theorem 4.1 with high probability that, at $-K\theta^{1/2}$ and $-\theta^{1/2}$, $\mathfrak{h}_1^t$ is lower than the tangent line $\ell_L^{\tan}$. More precisely, we see from (15) and Theorem 4.1 that

$$\begin{aligned}&\mathbb{P}\left((\mathsf{Pin}_{L,M}^\theta)^c \mid \mathfrak{h}_1^t(0) = \theta\right) \\ &\quad\le \mathbb{P}\left((\mathsf{Pin}_{L,M}^\theta)^c, \mathfrak{h}_1^t(-K\theta^{1/2}) \le \ell_L^{\tan}(-K\theta^{1/2}), \mathfrak{h}_1^t(-\theta^{1/2}) \le -\theta + M\theta^{1/4} \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right) \\ &\quad\quad + \exp(-c\theta^{3/2}) + \exp(-cM^2).\end{aligned}$$

The first term is bounded by $\exp(-cM^2)$ by Proposition 3.17 by taking $I = [-K\theta^{1/2}, -\theta^{1/2}]$ and $x_{\tan} = -L\theta^{1/2}$. We need $M \le \theta^{3/4}$ to apply this bound, but this is satisfied as we have taken $M = \log\theta$. So, overall, we obtain (16) for such M.

With the pinning of $\mathfrak{h}_1^t$ at $-L\theta^{1/2}$ and $-\theta^{1/2}$ established, we may move on to the fluctuations of $\mathfrak{h}_1^t$ on $J = [-L\theta^{1/2}, -\theta^{1/2}]$. Let $A_{L,\theta}(\mathfrak{h}_1^t) = \{\sup_{x\in J} \mathfrak{h}_1^t(x) + x^2 > \theta^{1/4}\log\theta\}$, where we have now made $M = \log\theta$ explicit. Then we see that

$$\begin{aligned}&\mathbb{P}(A_{L,\theta}(\mathfrak{h}_1^t) \mid \mathfrak{h}_1^t(0) = \theta) \\ &\quad\le \mathbb{P}\left(\mathfrak{h}_1^t \in A_{L,\theta}, \mathsf{Pin}_{L,\log\theta}^\theta, \mathsf{Pin}_{1,\log\theta}^\theta, \sup_{x\in J}\mathfrak{h}_2^t(x) + x^2 \le (\log\theta)^{4/3} \mid \mathfrak{h}_1^t(0) = \theta\right) \\ &\quad\quad + \mathbb{P}\left((\mathsf{Pin}_{L,\log\theta}^\theta)^c \cup \mathsf{Pin}_{1,\log\theta}^\theta)^c \mid \mathfrak{h}_1^t(0) = \theta\right) \\ &\quad\quad + \mathbb{P}\left(\sup_{x\in J}\mathfrak{h}_2^t(x) + x^2 > (\log\theta)^{4/3} \mid \mathfrak{h}_1^t(0) = \theta\right).\end{aligned} \tag{17}$$

By Theorem 9 (with $M = \log\theta$) and (16), we know that

$$\mathbb{P}\left((\mathsf{Pin}_{L,\log\theta}^\theta)^c \cup (\mathsf{Pin}_{1,\log\theta}^\theta)^c \;\middle|\; \mathfrak{h}_1^t(0) = \theta\right) \le \exp(-c(\log\theta)^2),$$

and, from Lemma 3.18,

$$\mathbb{P}\left(\sup_{x\in J}\mathfrak{h}_2^t(x) + x^2 > \log\theta \mid \mathfrak{h}_1^t(0) = \theta\right) \le \exp(-c(\log\theta)^2).$$

Thus it only remains to bound the first term of the (17).

We again make use of the H_t-Brownian Gibbs property. Let $\mathcal{F}$ be the σ-algebra generated by the lower curves and the top curve outside of J (which recall is $[-L\theta^{1/2}, -\theta^{1/2}]$). Then

$$\begin{aligned}\mathbb{P}\left(A_{L,\theta}(\mathfrak{h}_1^t), \mathsf{Pin}^\theta_{L,\log\theta}, \mathsf{Pin}^\theta_{1,\log\theta}, \sup_J \mathfrak{h}_2^t(x) + x^2 \le (\log\theta)^{4/3} \;\Big|\; \mathfrak{h}_1^t(0) = \theta\right) \\ = \mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(A_{L,\theta}(\mathfrak{h}_1^t) \;\Big|\; \mathfrak{h}_1^t(0) = \theta\right) \mathbb{1}_{\mathsf{Pin}^\theta_{L,\log\theta}, \mathsf{Pin}^\theta_{1,\log\theta}, \sup_{x\in J}\mathfrak{h}_2^t(x)+x^2\le(\log\theta)^{4/3}}\right].\end{aligned}$$

Under $\mathbb{P}_{\mathcal{F}}$, $\mathfrak{h}_1^t$ on J is a Brownian bridge from $(-L\theta^{1/2}, \mathfrak{h}_1^t(-L\theta^{1/2}))$ to $(-\theta^{1/2}, \mathfrak{h}_1^t(-\theta^{1/2}))$, tilted by the Radon-Nikodym derivative given by $\tilde{W}_{H_t}/\tilde{Z}_{H_t}$ with the same boundary values and lower boundary curve $\mathfrak{h}_2^t$. On the events mentioned, by monotonicity Lemma 3.1, this Brownian bridge is stochastically dominated by a Brownian bridge B from $(-L\theta^{1/2}, -L^2\theta + \theta^{1/4}\log\theta)$ to $(-\theta^{1/2}, -\theta + \theta^{1/4}\log\theta)$ which is reweighted by W_{H_t}/Z_{H_t} associated with the same boundary data and lower curve $-x^2 + (\log\theta)^{4/3}$. Since $A_{L,\theta}$ is an increasing event, we see that the expectation in the immediately previous display is bounded above by

$$Z_{H_t}^{-1}\mathbb{E}\left[\mathbb{1}_{B\in A_{L,\theta}} W_{H_t}\right].$$

We want to bound this by applying Proposition 3.16 after using that $W_{H_t} \le 1$. To have the expression fit into the framework of that proposition, we need to shift B and the event $A_{L,\theta}$ down by $(\log\theta)^{4/3}$. Doing so an applying Proposition 3.16 with $H = \theta^{1/4}\log\theta - (\log\theta^{4/3})$ yields that the previous display is upper bounded by $\exp(-c(\log\theta)^2)$. Tracing this bound back completes the proof of the upper bound on the limit shape in Proposition 4.2 $\square$

5. One-point estimates

In this section we prove the one-point estimates under Assumptions (i)–(iv), thereby proving Theorems 1 and 2. We prove the upper bound on the tail in Section 5.1, the lower bound in Section 5.2, and both the density bounds in Section 5.3.

5.1. Upper bound on the tail.

Here is the reformulation of the upper bound half Theorem 2 under our assumptions. The lower bound is stated as Theorem 5.3.

Theorem 5.1. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). There exist $\theta_0 > 0$ and $C < \infty$ such that, for $\theta > \theta_0$ and all $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\mathfrak{h}_1^t(0) \ge \theta\right) \le \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

Remark 5.2. The above bound trivially also applies to $\mathfrak{h}_k^t(0)$ for any k by stochastic ordering of the curves. However observe that, in the zero-temperature case, i.e., when $t = \infty$, Assumption (ii) and Theorem 5.1 imply an improved better tail bound for $\mathfrak{h}_2^t(0)$:

$$\begin{aligned}\mathbb{P}(\mathfrak{h}_2^t(0) \ge \theta) = \mathbb{P}(\mathfrak{h}_2^t(0) \ge \theta, \mathfrak{h}_1^t(0) \ge \theta) = \mathbb{P}(\mathfrak{h}_2^t(0) \ge \theta \mid \mathfrak{h}_1^t(0) \ge \theta) \cdot \mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta) \\ \le \mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta)^2 \le \exp\left(-\frac{8}{3}\theta^{3/2} + 2C\theta^{3/4}\right).\end{aligned} \tag{18}$$

In the first equality we used that, since $t = \infty$, $\mathfrak{h}^t$ is a non-intersecting ensemble and so $\mathfrak{h}_2^t(0) \ge \theta \implies \mathfrak{h}_1^t(0) \ge \theta$.

Now, the BK inequality in Assumption (ii) can be extended in a natural fashion to the k^{th} curve (bounding an upper tail probability of the k^{th} curve by the first curve's probability to the k^{th} power), which in conjunction with a similar argument as above would yield that, for $t = \infty$,

$$\mathbb{P}(\mathfrak{h}_k^t(0) \ge \theta) \le \exp\left(-\frac{4}{3}k\theta^{3/2} + Ck\theta^{3/4}\right). \tag{19}$$

We expect the first order term of $\frac{4}{3}k\theta^{3/2}$ would be sharp; however, have not included an argument for the lower bound here, and we have not found an estimate of this explicit form in the literature.

An interesting question is whether a counterpart holds in the positive temperature case. For instance, it no longer holds that $\mathbb{P}(\mathfrak{h}_2^t(0) \geq \theta) = \mathbb{P}(\mathfrak{h}_2^t(0) \geq \theta, \mathfrak{h}_1^t(0) \geq \theta)$; instead we would have an additional term of $\mathbb{P}(\mathfrak{h}_2^t(0) \geq \theta, \mathfrak{h}_1^t(0) < \theta)$. By the FKG inequality and Theorem 5.1, this additional term is upper bounded by $\mathbb{P}(\mathfrak{h}_2^t(0) \geq \theta)\mathbb{P}(\mathfrak{h}_1^t(0) < \theta) \leq (1 - \exp(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}))\mathbb{P}(\mathfrak{h}_2^t(0) \geq \theta)$ for all large enough θ. So we obtain essentially the same bound as in Theorem 5.1, i.e., without an extra factor of 2 in the exponent.

We next prove Theorem 5.1; recall the proof outline given in Section 1.7.3.

Proof of Theorem 5.1. We will in fact show that $\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta - 1, \theta]) \leq \exp(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4})$, which will clearly suffice since $\mathbb{P}(\mathfrak{h}_1^t(0) \geq \theta) = \sum_{s=\theta}^{\infty} \mathbb{P}(\mathfrak{h}_1^t(0) \in [s, s+1])$. We consider the event $\mathfrak{h}_1^t(0) \in [\theta - 1, \theta]$ instead of the entire upper tail as under the former event we have an upper bound on $\mathfrak{h}_1^t(0)$ which converts to a high probability upper bound on $\mathfrak{h}_1^t(\pm\theta^{1/2})$ from Theorem 4.1.

Indeed, recall that Theorem 4.1 implies that the probability that $\mathfrak{h}_1^t(\pm\theta^{1/2}) + \theta \geq M\theta^{1/4} \mid \mathfrak{h}_1^t(0) = \theta)$ is small for large enough M independent of $\theta > \theta_0$. By Lemma 3.2 (monotonicity in conditioning), we can convert Theorem 4.1's statement to the same statement conditional on $\mathfrak{h}_1^t(0) \in [\theta, \theta - 1]$). Next, Lemma 3.18 gives that $\mathbb{P}(\sup_{[-\theta^{1/2}, \theta^{1/2}]} \mathfrak{h}_2^t(x) + x^2 \geq (4 \log \theta)^{1/\beta})$ is small. Combining all this, there exists a large enough constant M (independent of θ) such that

$$\begin{aligned}
&\frac{1}{2}\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta - 1, \theta]\right) \\
&\qquad \leq \mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta - 1, \theta]\right) \\
&\qquad\qquad \times \mathbb{P}\left(\mathfrak{h}_1^t(\pm\theta^{1/2}) + \theta \leq M\theta^{1/4}, \sup_{[-\theta^{1/2}, \theta^{1/2}]} \mathfrak{h}_2^t(x) + x^2 \leq (4\log\theta)^{1/\beta} \;\middle|\; \mathfrak{h}_1^t(0) \in [\theta - 1, \theta]\right) \\
&\qquad = \mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta - 1, \theta], \mathfrak{h}_1^t(\pm\theta^{1/2}) + \theta \leq M\theta^{1/4}, \sup_{[-\theta^{1/2}, \theta^{1/2}]} \mathfrak{h}_2^t(x) + x^2 \leq (4\log\theta)^{1/\beta}\right).
\end{aligned}$$

Let $\mathcal{F}$ be the σ-algebra generated by everything outside the top curve on $[-\theta^{1/2}, \theta^{1/2}]$. The probability in the last display is

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\big(\mathfrak{h}_1^t(0) \in [\theta - 1, \theta]\big)\mathbb{1}_{\mathfrak{h}_1^t(\pm\theta^{1/2}) + \theta \leq M\theta^{1/4}, \sup_{[-\theta^{1/2}, \theta^{1/2}]} \mathfrak{h}_2^t(x) + x^2 \leq \log\theta}\right] \\
&\qquad\qquad \leq \mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\big(\mathfrak{h}_1^t(0) \geq \theta - 1\big)\mathbb{1}_{\mathfrak{h}_1^t(\pm\theta^{1/2}) + \theta \leq M\theta^{1/4}, \sup_{[-\theta^{1/2}, \theta^{1/2}]} \mathfrak{h}_2^t(x) + x^2 \leq \log\theta}\right];
\end{aligned}$$

observe that $\{\mathfrak{h}_1^t(0) \geq \theta - 1\}$ is an increasing event.

Now conditionally on $\mathcal{F}$, $\mathfrak{h}_1^t$ is a Brownian bridge from $(-\theta^{1/2}, \mathfrak{h}_1^t(-\theta^{1/2}))$ to $(\theta^{1/2}, \mathfrak{h}_1^t(\theta^{1/2}))$ which is reweighted by the Radon-Nikodym factor $\tilde{W}_{H_t}/\tilde{Z}_{H_t}$ associated to the boundary data $\mathfrak{h}_2^t$. By monotonicity Lemma 3.1 and on the $\mathcal{F}$ measurable event in the indicator, this bridge is stochastically dominated by the Brownian bridge B from $(-\theta^{1/2}, -\theta + M\theta^{1/4})$ to $(\theta^{1/2}, -\theta + M\theta^{1/4})$ reweighted by the factor W_{H_t}/Z_{H_t} associated to the boundary data $-x^2 + (4 \log \theta)^{1/\beta}$. Further $W_{H_t} \leq 1$. Thus we see that the previous display is upper bounded by

$$Z_{H_t}^{-1}\mathbb{P}\left(B(0) \geq \theta - 1\right) \tag{20}$$

Since $B(0)$ is a normal random variable of mean $-\theta + M\theta^{1/4}$ and variance $2 \times \frac{\theta^{1/2}\times\theta^{1/2}}{2\theta^{1/2}} = \theta^{1/2}$, using the normal tail bounds from Lemma 3.5, the numerator is upper bounded by

$$\exp\left(-\frac{(2\theta - M\theta^{1/4} - 1)^2}{2\theta^{1/2}}\right) \leq \exp\left(-2\theta^{3/2} + C\theta^{3/4}\right).$$

Next we turn to the denominator of (20). By Corollary 3.12, for large enough θ such that $M\theta^{1/4} > t_0^{-1/6} + 1 > t^{-1/6} + 1$ so as to satisfy the corollary's hypotheses,

$$Z_{H_t} \geq \exp\left(-\frac{1}{12}(2\theta^{1/2})^3 - 6\theta^{1/2}\log(2\theta^{1/2})\right) \geq \exp\left(-\frac{2}{3}\theta^{3/2} - C\theta^{3/4}\right).$$

Together, this implies that

$$\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta - 1, \theta]\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right)$$

for all large θ, completing the proof of Theorem 5.1. □

5.2. **Lower bound on the tail.**

Theorem 5.3. *Let $\mathfrak{h}^t$ satisfy Assumptions (i) and (ii). There exist $C > 0$ and θ_0 such that, for $\theta > \theta_0$ and all $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta\right) \geq \exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{1/2}\log\theta\right).$$

Observe that the above statement is made without using Assumption (iv) which gives an a priori lower bound on the upper tail. Thus the above actually verifies the lower bound part of the upper tail in Theorem 6.

We start by proving a lower bound with the right exponent of 3/2 but a suboptimal constant; again this is by Assumption (iv), but we use only Assumptions (i) and (ii).

Lemma 5.4. *Let $\mathfrak{h}^t$ satisfy Assumptions (i) and (ii). There exists θ_0 such that, for $\theta > \theta_0$ and $t \in [t_0, \infty]$,*

$$\mathbb{P}\Big(\mathfrak{h}_1^t(0) \geq \theta\Big) \geq \exp\big(-5\theta^{3/2}\big).$$

Proof. For an M to be chosen, consider the favourable event

$$\mathsf{Fav} = \left\{\mathfrak{h}_1^t(-\theta^{1/2}) \geq -\theta - M\right\} \cap \left\{\mathfrak{h}_1^t(\theta^{1/2}) \geq -\theta - M\right\}.$$

By stationarity and positive association of $\mathfrak{h}_1^t(x) + x^2$, and tightness of $\{\mathfrak{h}_1^t(0)\}_{t\geq t_0}$, it follows that $\mathbb{P}(\mathsf{Fav}) \geq 1/2$ for $M > M_0(t_0)$ sufficiently large uniformly for all $t \in [t_0, \infty]$.

Consider the σ-algebra $\mathcal{F} = \mathcal{F}_{\text{ext}}(1, [-\theta^{1/2}, \theta^{1/2}])$. The Brownian Gibbs property says that the distribution of $\mathfrak{h}^t$ on $[-\theta^{1/2}, \theta^{1/2}]$, conditionally on $\mathcal{F}$, is that of a rate two Brownian bridge tilted by the Radon-Nikodym derivative W_{H_t}/Z_{H_t} associated to the conditioned boundary data. By monotonicity Lemma 3.1, on Fav, this Brownian bridge stochastically dominates the rate two Brownian bridge B from $(-\theta^{1/2}, -\theta - M)$ to $(\theta^{1/2}, -\theta - M)$ with no lower boundary condition.

Thus we see

$$\mathbb{P}\Big(\mathfrak{h}_1^t(0) \geq \theta\Big) \geq \mathbb{E}\Big[\mathbb{P}_{\mathcal{F}}\Big(\mathfrak{h}_1^t(0) \geq \theta\Big) \cdot \mathbb{1}_{\mathsf{Fav}}\Big] \geq \frac{1}{2} \cdot \mathbb{P}\left(B(0) \geq \theta\right).$$

Now $B(0)$ is a normal random variable with mean $-\theta - M$ and variance $\theta^{1/2}$. Thus using the standard lower bound on the normal probability from Lemma 3.5, we see that, on Fav,

$$\mathbb{P}\Big(\mathfrak{h}_1^t(0) \geq \theta\Big) \geq c\theta^{-3/4} \cdot \exp\left(-\frac{(\theta + \theta + M)^2}{2\theta^{1/2}}\right) \geq \exp(-5\theta^{3/2}),$$

the last inequality for $\theta > M$. This completes the proof. □

Next we prove Theorem 5.3, i.e., obtain the optimal constant in the exponent. We remind the reader of the proof sketch given in Section 1.7.3.

Proof of Theorem 5.3. Suppose we know that there exist $C_n > 0$, γ_n, and θ_n such that for all $\theta > \theta_n$,

$$\mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta\right) \geq \exp(-C_n\theta^{3/2} - \gamma_n \log\theta). \tag{21}$$

We will show that then there exist C_{n+1}, γ_{n+1}, and θ_{n+1} such that, for $\theta > \theta_{n+1}$,

$$\mathbb{P}\left(\mathfrak{h}_1^t(0) \geq \theta\right) \geq \exp(-C_{n+1}\theta^{3/2} - \gamma_{n+1} \log\theta),$$

with the surprising property that $\lim_{n\to\infty} C_n = \frac{4}{3}$; γ_n and θ_n will go to infinity, but at a rate which contributes to the error term of $-\theta^{1/2}\log\theta$. To start the iterations, recall that we have (21) for $n = 0$ for some constant θ_0, $C_0 = 5$, and $\gamma_0 = 0$ by Lemma 5.4.

We define the σ-algebra $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(1, [-\frac{1}{2}\theta^{1/2}, \frac{1}{2}\theta^{1/2}])$. Consider the $\mathcal{F}$-measurable favourable event Fav defined by

$$\mathsf{Fav} := \left\{\mathfrak{h}_1^t(-\tfrac{1}{2}\theta^{1/2}) \geq 0\right\} \cap \left\{\mathfrak{h}_1^t(\tfrac{1}{2}\theta^{1/2}) \geq 0\right\}.$$

From (21), the stationarity of $\mathfrak{h}_1^t(x) + x^2$, and the positive association of $\mathfrak{h}_1^t(-\frac{1}{2}\theta^{1/2})$ and $\mathfrak{h}_1^t(\frac{1}{2}\theta^{1/2})$ from Assumption (ii), we see that

$$\begin{aligned}\mathbb{P}(\mathsf{Fav}) \geq \mathbb{P}(\mathfrak{h}_1^t(0) > \tfrac{1}{4}\theta)^2 &\geq \exp\left(-2C_n 4^{-3/2}\theta^{3/2} - 2\gamma_n\log(\theta/4)\right)\\ &\geq \exp\left(-\tfrac{1}{4}C_n\theta^{3/2} - 2\gamma_n\log\theta\right)\end{aligned} \tag{22}$$

for $\theta > 4\theta_n$. Now,

$$\mathbb{P}\big(\mathfrak{h}_1^t(0) \geq \theta\big) \geq \mathbb{E}\Big[\mathbb{P}_{\mathcal{F}}(\mathfrak{h}_1^t(0) \geq \theta)\cdot \mathbb{1}_{\mathsf{Fav}}\Big]$$

The Brownian Gibbs property says that $\mathfrak{h}_1^t$, on $[-\frac{1}{2}\theta^{1/2}, \frac{1}{2}\theta^{1/2}]$ and conditionally on $\mathcal{F}$, is distributed as a Brownian bridge with the appropriate endpoints and tilted by W_{H_t}/Z_{H_t} associated to the boundary conditions. As usual, on Fav, this Brownian bridge stochastically dominates the Brownian bridge B with between $(-\frac{1}{2}\theta^{1/2}, 0)$ and $(\frac{1}{2}\theta^{1/2}, 0)$ and no lower boundary condition. Now, $B(0)$ is a normal random variable with mean zero and variance $\sigma^2 = \frac{1}{2}\theta^{1/2}$. Thus on Fav and using (22) and a lower bound on normal tails from Lemma 3.5, the previous display is lower bounded by

$$\exp\left(-\theta^{3/2}\left[1 + \tfrac{1}{4}C_n\right] - (2\gamma_n + 1)\log\theta\right)$$

(lower bounding the constant factor in the normal bound by $\theta^{-1/4}$ for convenience).

Thus we have shown that, if $\mathbb{P}(\mathfrak{h}_1^t(0) > \theta) \geq \exp(-C_n\theta^{3/2} - \gamma_n\log\theta)$ for $\theta > \theta_n$, then the same holds true with n replaced by $n+1$, with $C_{n+1} = 1 + C_n/4$, $\gamma_{n+1} = 2\gamma_n + 1$, and $\theta_{n+1} = 4\theta_n$. It is easy to solve these recurrences to get $\theta_n = 4^n\theta_0$, $\gamma_n = 2^n - 1$ and (using $C_0 = 5$)

$$C_n = \frac{4}{3}\left(1 + 11\cdot 4^{-n-1}\right).$$

This establishes that, for $n \in \mathbb{N}$ and $4^n\theta_0 < \theta \leq 4^{n+1}\theta_0$,

$$\mathbb{P}\left(\mathfrak{h}_1^t(0) > \theta\right) \geq \exp\left(-\frac{4}{3}(1 + 11\cdot 4^{-n-1})\theta^{3/2} - 2^n\log\theta\right).$$

Given $\theta > \theta_0$, we choose n such that the condition for the previous display holds. Now observing that $4^{-n-1} \leq \theta_0\theta^{-1}$ and $2^n \leq (\theta/\theta_0)^{1/2}$ display yields the claim. □

5.3. **Density bounds.** In this subsection we prove Theorem 5.5 on the upper and lower bounds on the density of $\mathfrak{h}_1^t(0)$, which, recall, we denote by $\frac{1}{\mathrm{d}\theta}\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta])$:

Theorem 5.5. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). There exist constants C and θ_0 such that, for all $t \in [t_0, \infty]$ and $\theta > \theta_0$,*

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right) \leq \frac{1}{\mathrm{d}\theta}\mathbb{P}\Big(\mathfrak{h}^t(0) \in [\theta, \theta + \mathrm{d}\theta]\Big) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

Starting with the basic fact that

$$\int_\theta^{\theta+1} \frac{1}{\mathrm{d}\theta}\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta])\,\mathrm{d}\theta = \mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + 1]) \leq \mathbb{P}(\mathfrak{h}_1^t(0) \geq \theta),$$

and given the upper bound we have on the right-hand side via Theorem 2, to obtain a bound on the density at a particular value instead of its integral, we will seek to establish that the density satisfies some regularity property. For instance, it would be sufficient to know that the density is decreasing at least for all large arguments. However, as plausible as it might sound, proving this does not seem straightforward. Instead, we establish that it doesn't decay too fast which also turns out to suffice allowing us to approximate the integral by the density's value at a point.

Proposition 5.6. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). There exist $M > 0$ and θ_0 such that, for all $t \in [t_0, \infty]$, $\theta > \theta_0$, and $s > 0$,*

$$\begin{aligned}&\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta + s, \theta + s + \mathrm{d}\theta]\right)\\ &\qquad\qquad \geq \frac{1}{2}\cdot\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta]\right)\cdot\exp\left(-2s\theta^{1/2} - Ms\theta^{-1/4} - s^2\theta^{-1/2}\right).\end{aligned}$$

In particular there exists (a slightly larger) $M > 0$ such that, if $0 < s \leq \theta^{1/4}$,

$$\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta + s, \theta + s + \mathrm{d}\theta]\right) \geq \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta]\right)\cdot\exp\left(-2s\theta^{1/2} - M\right). \tag{23}$$

While our eventual goal of obtaining an upper bound on the density does allow a lot of room in the comparison estimate, note that the coefficient of -2 in front of $s\theta^{1/2}$ in the exponential is in fact sharp: anticipating that the density at θ is approximately $\exp(-\frac{4}{3}\theta^{3/2})$, we see that

$$\begin{aligned}\exp\left(-\frac{4}{3}(\theta + s)^{3/2}\right) = \exp\left(-\frac{4}{3}\theta^{3/2}(1 + s\theta^{-1})^{3/2}\right) &\approx \exp\left(-\frac{4}{3}\theta^{3/2} - \frac{4}{3}\theta^{3/2}\cdot\frac{3}{2}s\theta^{-1}\right)\\ &= \exp\left(-\frac{4}{3}\theta^{3/2} - 2s\theta^{1/2}\right).\end{aligned}$$

The sharpness of the estimate in Proposition 5.6 turns out to be crucial in obtaining the tight lower bound on the density in Theorem 1.

Before turning to the proof of Proposition 5.6, we give the proof of Theorem 5.5. To prove the lower bound in Theorem 5.5, we will need a preliminary uniform lower bound on the density. This is a consequence of Proposition 5.6 as well as both the upper and lower bounds on the one-point upper tail from Theorems 5.1 and 5.3.

Lemma 5.7. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). There exists $c > 0$ and $\theta_0 > 0$ such that, for all $t \in [t_0, \infty]$,*

$$\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta]\right)\Big|_{\theta = \theta_0} \geq c.$$

Proof. Let $g(\theta)$ be the density of $\mathfrak{h}_1^t(0)$. First we observe that, for any $\theta_0 > 0$

$$\sup_{x \in [\frac{1}{2}\theta_0, \theta_0]} g(x) \geq 2\theta_0^{-1}\mathbb{P}(\mathfrak{h}_1^t(0) \in [\tfrac{1}{2}\theta_0, \theta_0]).$$

It is easy to check that a uniform lower bound on the right-hand side follows from the upper and lower bounds on the one-point upper tail from Theorem 2 if we take θ_0 large enough. Then to see that the previous display implies a lower bound on $g(\theta_0)$, we apply Proposition 5.6's statement that, for $s > 0$,

$$g(\theta_0) \geq \frac{1}{2} g(\theta_0 - s) \cdot \exp\left(-2s(\theta_0 - s)^{1/2} - Ms(\theta_0 - s)^{-1/4} - s^2(\theta_0 - s)^{-1/2}\right),$$

with $s \in [0, \frac{1}{2}\theta_0]$ such that that $g(\theta_0 - s) \geq \frac{1}{2}\sup_{x \in [\frac{1}{2}\theta_0, \theta_0]} g(x)$. □

Proof of Theorem 5.5. Let $f(\theta) = \log \frac{1}{\mathrm{d}\theta}\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta])$. We start with the lower bound.

Let θ_0 be the value from Lemma 5.7, which is uniform in $t > t_0$, and we assume that θ is such that $\theta^{1/4} > \theta_0$. In terms of f we observe that (23) of Proposition 5.6 (substituting $\theta - s$ for θ) implies that, for $0 < s \leq \frac{1}{2}\theta^{1/4}$ (which implies that $0 < s \leq (\theta - s)^{1/4}$ as needed for (23)),

$$f(\theta) \geq f(\theta - s) - 2s(\theta - s)^{1/2} - M.$$

We iterate this inequality $2\theta^{3/4} - 1$ times with $s = \frac{1}{2}\theta^{1/4}$, and one last time with $s \in [0, \frac{1}{2}\theta^{1/4}]$ such that $\theta - \frac{1}{2}\theta^{1/4}(2\theta^{3/4} - 1) - s = \theta_0$. This yields

$$f(\theta) \geq -2 \cdot \tfrac{1}{2}\theta^{1/4} \sum_{i=1}^{2\theta^{3/4}} \left(\theta - i \cdot \tfrac{1}{2}\theta^{1/4}\right)^{1/2} - M\theta^{3/4} + f(\theta_0); \tag{24}$$

in the first term the final summand should be $\theta_0^{1/2}$ instead of 0, but we absorb this discrepancy into the $M\theta^{3/4}$ term. Since we know by our choice of θ_0 that $f(\theta_0)$ is bounded from below uniformly in $t > t_0$, we may absorb $f(\theta_0)$ also into the $M\theta^{3/4}$ term for large enough θ. Now,

$$\theta^{1/4} \sum_{i=1}^{2\theta^{3/4}} \left(\theta - i \cdot \tfrac{1}{2}\theta^{1/4}\right)^{1/2} = \theta^{3/2} \cdot \theta^{-3/4} \sum_{i=1}^{2\theta^{3/4}} \left(1 - i \cdot \tfrac{1}{2}\theta^{-3/4}\right)^{1/2}$$

and so equals $\theta^{3/2}$ times the (right) Riemann sum of $\int_0^2 (1 - \frac{1}{2}x)^{1/2}\,\mathrm{d}x = \frac{4}{3}$. Since $(1 - \frac{1}{2}x)^{1/2}$ is decreasing and we are considering the right Riemann sum, it follows that

$$\theta^{1/4} \sum_{i=1}^{2\theta^{3/4}} \left(\theta - i \cdot \tfrac{1}{2}\theta^{1/4}\right)^{1/2} \leq \frac{4}{3}\theta^{3/2}.$$

Substituting this into (24) yields that $f(\theta) \geq -\frac{4}{3}\theta^{3/2} - M\theta^{3/4}$ for some M.

For the upper bound, it follows from the fact that $\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + 1]) \leq \exp(-\frac{4}{3}\theta^{3/2})$. In more detail,

$$\begin{aligned}
\mathbb{P}\left(\mathfrak{h}_1^t(0) \in [\theta, \theta + 1]\right) &= \int_0^1 \frac{1}{\mathrm{d}\theta}\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta + s, \theta + s + \mathrm{d}\theta]) \\
&\geq \mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta]) \int_0^1 \exp(-2s\theta^{1/2} - M)\,\mathrm{d}s \\
&\geq \mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta]) \cdot \exp(-2\theta^{1/2} - M).
\end{aligned}$$

Since $\mathbb{P}(\mathfrak{h}_1^t(0) \in [\theta, \theta+1]) \leq \mathbb{P}(\mathfrak{h}_1^t(0) \geq \theta) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right)$ by Theorem 5.1, we are done. □

Now we turn to the proof of Proposition 5.6. The basic argument relies on a resampling trick which allows us to take a configuration with $\mathfrak{h}_1^t(0) \in [\theta, \theta + \mathrm{d}\theta]$ and construct one where $\mathfrak{h}_1^t(0) \in [\theta + s, \theta + s + \mathrm{d}\theta]$ and in the process compare their "probabilities." However to carry this out efficiently, we will need more information about the distribution of $\mathfrak{h}_1^t$ than is available from just H_t-Brownian Gibbs and monotonicity statements, which have been the main ingredients of previous arguments. The idea here is to condition on more information, in such a way that the conditional distribution of $\mathfrak{h}_1^t(0)$ is more explicitly Gaussian; in the vanilla conditioning of the H_t-Brownian Gibbs property, the distribution of $\mathfrak{h}_1^t(0)$ is Gaussian, but reweighted by a Radon-Nikodym factor which is affected by $\mathfrak{h}_1^t$'s values at other points as well. This makes things hard to control, and the conditioning here will avoid these extra dependencies in the Radon-Nikodym derivative.

In more detail, here, in addition to conditioning on $\mathcal{F}_{\mathrm{ext}}(1,[a,b])$, where $[a,b]$ will be taken as $[-\theta^{1/2},\theta^{1/2}]$, we further include the *bridges* of $\mathfrak{h}_1^t$ on $[-\theta^{1/2},0]$ and $[0,\theta^{1/2}]$ and call the resulting σ-algebra $\mathcal{F}$. The bridge of a function $f: I\to\mathbb{R}$ on an interval $[a,b]\subseteq I$ is the function obtained by affinely shifting f to equal zero at a and b; more explicitly, it is given by

$$x\mapsto f(x)-\frac{b-x}{b-a}f(a)-\frac{x-a}{b-a}f(b).$$

The effect of including this data in the σ-algebra we condition on is that the only information that remains random is the value of $\mathfrak{h}_1^t(0)$. By the H_t-Brownian Gibbs property, the conditional distribution of $\mathfrak{h}_1^t(0)$ is a suitably reweighted normal random variable (where the reweighting is only a function of $\mathfrak{h}_1^t(0)$), and the proof of Proposition 5.6 relies crucially on this fairly explicit representation. An important ingredient for this representation is that, conditionally on the richer σ-algebra, $\mathfrak{h}_1^t$ is given inside of $[-\theta^{1/2},\theta^{1/2}]$ (recall it has been conditioned on outside of $[-\theta^{1/2},\theta^{1/2}]$) by $\mathfrak{h}_1^{t,X}:[-\theta^{1/2},\theta^{1/2}]\to\mathbb{R}$, where

$$\mathfrak{h}_1^{t,x}(u)=\begin{cases}\mathfrak{h}_1^{t,[-\theta^{1/2},0]}(u)+\frac{u+\theta^{1/2}}{\theta^{1/2}}\cdot x+\frac{-u}{\theta^{1/2}}\cdot\mathfrak{h}_1^t(-\theta^{1/2}) & u\in[-\theta^{1/2},0]\\ \mathfrak{h}_1^{t,[0,\theta^{1/2}]}(u)+\frac{\theta^{1/2}-u}{\theta^{1/2}}\cdot x+\frac{u}{\theta^{1/2}}\cdot\mathfrak{h}_1^t(\theta^{1/2}) & u\in[0,\theta^{1/2}]\end{cases}\tag{25}$$

($\mathfrak{h}_1^{t,[-\theta^{1/2},0]}$ is the bridge of $\mathfrak{h}_1^t$ on $[-\theta^{1/2},0]$, and similarly for $\mathfrak{h}_1^{t,[0,\theta^{1/2}]}$), and X is distributed according to the $\mathcal{F}$-conditional distribution of $\mathfrak{h}_1^t(0)$.

Using (25) we can describe the $\mathcal{F}$-conditional distribution of $\mathfrak{h}_1^t(0)$ more precisely: it is a normal distribution with $\mathcal{F}$-measurable mean $\mu=\frac{1}{2}(\mathfrak{h}_1^t(-\theta^{1/2})+\mathfrak{h}_1^t(\theta^{1/2}))$ and variance $\sigma^2=2\times\frac{\theta^{1/2}\times\theta^{1/2}}{2\theta^{1/2}}=\theta^{1/2}$, which is reweighted by a Radon-Nikodym factor $W_t^{\mathrm{pt}}(\mathfrak{h}_1^t(0))/Z_t^{\mathrm{pt}}$, where W_t^{pt} and Z_t^{pt} are given by

$$\begin{aligned}W_t^{\mathrm{pt}}(x)&=\exp\left(-\int_{-\theta^{1/2}}^{\theta^{1/2}}H_t\big(\mathfrak{h}_2^t(u)-\mathfrak{h}_1^{t,x}(u)\big)\,\mathrm{d}u\right)\\ Z_t^{\mathrm{pt}}&=\mathbb{E}_{\mathcal{F}}\left[W^{\mathrm{pt}}(\mathfrak{h}_1^t(0))\right].\end{aligned}$$

Observe that $W^{\mathrm{pt}}(x)$ is increasing in x.

The idea of including the bridge data in Brownian Gibbs resamplings was introduced in [Ham22] and has been used several times in subsequent studies [CHH19, CHHM21]. The correctness of the above description of the conditional distribution is straightforward to verify using the H_t-Brownian Gibbs property; see for example proofs of similar statements in [CHH19, Section 4.1.3].

With this description in hand we can turn to the proof of Proposition 5.6.

Proof of Proposition 5.6. As above, let $\mathcal{F}$ be the σ-algebra generated by $\mathcal{F}_{\mathrm{ext}}(1,[a,b])$ and the bridges of $\mathfrak{h}_1^t$ on $[-\theta^{1/2},0]$ and $[0,\theta^{1/2}]$. Conditional on $\mathcal{F}$, the distribution of $\mathfrak{h}_1^t(0)$ is that of a normal random variable with an $\mathcal{F}$-measurable mean μ (whose formula is given above) and $\sigma^2=\theta^{1/2}$ reweighted by $W_t^{\mathrm{pt}}(\mathfrak{h}^t(0))/Z_t^{\mathrm{pt}}$.

Condition on $\mathcal{F}$, let X be a normal random variable of mean μ and variance σ^2, and set $\mathsf{Fav}=\{\mu\geq-\theta-M\theta^{1/4}\}$ for an M to be chosen (so that, from Theorem 4.1 on the conditional limit shape, Fav has uniformly positive probability when M is large). Then,

$$\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta+s,\theta+s+\mathrm{d}\theta]\right)=\frac{1}{\mathrm{d}\theta}\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\mathfrak{h}_1^t(0)\in[\theta+s,\theta+s+\mathrm{d}\theta]\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{\mathrm{d}\theta}\mathbb{E}\left[\mathbb{E}_{\mathcal{F}}\left[\mathbb{1}_{X\in[\theta+s,\theta+s+\mathrm{d}\theta]}W^{\mathrm{pt}}(X)\right](Z^{\mathrm{pt}})^{-1}\right]\\
&= \frac{1}{\mathrm{d}\theta}\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(X\in[\theta+s,\theta+s+\mathrm{d}\theta]\right)W^{\mathrm{pt}}(\theta+s)(Z^{\mathrm{pt}})^{-1}\right]\\
&\geq \mathbb{E}\left[(2\pi\theta^{1/2})^{-1/2}\exp\left(-\frac{(\theta+s-\mu)^2}{2\theta^{1/2}}\right)W^{\mathrm{pt}}(\theta+s)(Z^{\mathrm{pt}})^{-1}\mathbb{1}_{\mathsf{Fav}}\right]\\
&\geq \mathbb{E}\Bigg[(2\pi\theta^{1/2})^{-1/2}\exp\left(-\frac{(\theta-\mu)^2+s^2+2s(\theta-\mu)}{2\theta^{1/2}}\right)\\
&\qquad\qquad\times W^{\mathrm{pt}}(\theta)(Z^{\mathrm{pt}})^{-1}\mathbb{1}_{\mathsf{Fav}}\Bigg];
\end{aligned}
$$

the last inequality using that $W^{\mathrm{pt}}(x)$ is increasing in x. Next we reinterpret some of the factors in the normal density as $\mathbb{E}_{\mathcal{F}}[\mathbb{1}_{X\in[\theta,\theta+\mathrm{d}\theta]}W^{\mathrm{pt}}(\theta)(Z^{\mathrm{pt}})^{-1}]$ and use that $\theta-\mu\leq 2\theta+M\theta^{1/4}$ on Fav to see that, for $s>0$,

$$
\begin{aligned}
&\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta+s,\theta+s+\mathrm{d}\theta]\right)\\
&\qquad\geq \frac{1}{\mathrm{d}\theta}\mathbb{E}\left[\mathbb{E}_{\mathcal{F}}\left[\mathbb{1}_{X\in[\theta,\theta+\mathrm{d}\theta]}W^{\mathrm{pt}}(\theta)(Z^{\mathrm{pt}})^{-1}\right]\exp\left(-\frac{s^2+2s(\theta-\mu)}{2\theta^{1/2}}\right)\mathbb{1}_{\mathsf{Fav}}\right]\\
&\qquad= \frac{1}{\mathrm{d}\theta}\mathbb{E}\left[\mathbb{E}_{\mathcal{F}}\left[\mathbb{1}_{X\in[\theta,\theta+\mathrm{d}\theta]}W^{\mathrm{pt}}(X)(Z^{\mathrm{pt}})^{-1}\right]\exp\left(-\frac{s^2+2s(\theta-\mu)}{2\theta^{1/2}}\right)\mathbb{1}_{\mathsf{Fav}}\right]\\
&\qquad\geq \frac{1}{\mathrm{d}\theta}\mathbb{E}\Big[\mathbb{P}_{\mathcal{F}}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta]\right)\\
&\qquad\qquad\times\exp\left(-2s\theta^{1/2}-Ms\theta^{-1/4}-\tfrac{1}{2}s^2\theta^{-1/2}\right)\mathbb{1}_{\mathsf{Fav}}\Big]\\
&\qquad\geq \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta],\mathsf{Fav}\right)\cdot\exp\Big(-2s\theta^{1/2}-Ms\theta^{-1/4}-s^2\theta^{-1/2}\Big),
\end{aligned}
$$

the last line using that Fav is $\mathcal{F}$-measurable and the tower property of conditional expectations. Next, by Theorem 4.1, there exists M large enough independent of t and θ such that $\mathbb{P}(\mathfrak{h}_1^t(-\theta^{1/2}),\mathfrak{h}_1^t(\theta^{1/2})\geq -\theta-M\theta^{1/4}\mid\mathfrak{h}_1^t(0)=\theta)\geq\frac{1}{2}$. So, setting M to be such a value,

$$
\begin{aligned}
\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta],\mathsf{Fav}\right) &= \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta]\right)\cdot\mathbb{P}\left(\mathsf{Fav}\mid\mathfrak{h}_1^t(0)=\theta\right)\\
&= \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta]\right)\cdot\mathbb{P}\left(\mu\geq-\theta-M\theta^{1/4}\mid\mathfrak{h}_1^t(0)=\theta\right)\\
&\geq \frac{1}{2}\cdot\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta]\right).
\end{aligned}
$$

Putting it together, we have shown that, for some M and all $s>0$,

$$
\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta+s,\theta+s+\mathrm{d}\theta]\right)\geq\frac{1}{2}\frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}_1^t(0)\in[\theta,\theta+\mathrm{d}\theta]\right)\cdot\exp(-2s\theta^{1/2}-Ms\theta^{-1/4}-s^2\theta^{-1/2}).
$$

□

6. Two-point limit shape

We now prove Theorems 10 and 3 on two-point upper tail limit shapes and asymptotics, in this and the next sections respectively. As already indicated in Section 1.7, while conceptually all the ideas were present in the arguments for the one-point results, the arguments in these two sections are technically more complex and may be somewhat harder to parse, owing mostly to the more complicated formulas that arise in the two-point case.

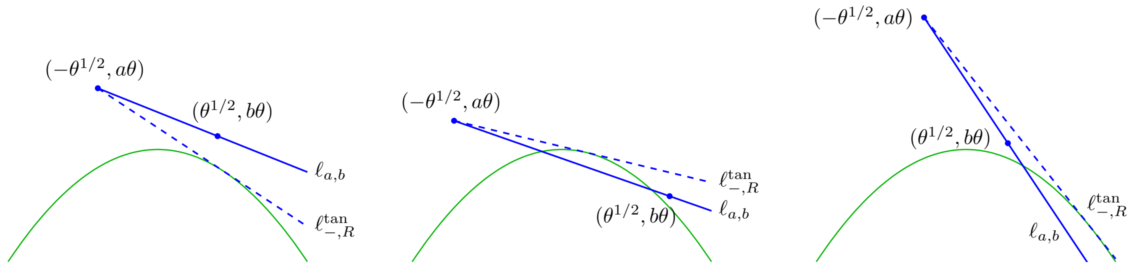

FIGURE 7. A depiction of Lemma 6.1. Assuming $a \geq b > -1$, if $a \leq (\sqrt{1+b}+1)(\sqrt{1+b}+3) \iff (a-b)^2 \leq 8(a+b)$, then $\ell_{a,b}$ stays above $\ell^{\tan}_{-,R}$ to the right of $-\theta^{1/2}$, as depicted in the left panel. Thus $\ell_{a,b}$ intersects the parabola at at most one point. In the middle and right panels two geometrically distinct subcases of $(a-b)^2 > 8(a+b)$ (so that $\ell_{a,b}$ lies below $\ell^{\tan}_{-,R}$ and intersects the parabola at two distinct points) are shown: in the middle panel, the intersection points of $\ell_{a,b}$ with $-x^2$ lie inside $[-\theta^{1/2}, \theta^{1/2}]$, while in the right panel they lie in $(\theta^{1/2}, \infty)$.

As said before, the proofs will make clear that similar arguments would also yield sharp asymptotic expressions for k-point upper tails and limit shapes; see Remark 7.3.

The following technical lemma relates certain geometric conditions that will be relevant in the analysis with algebraic relations, and will be used many times to work with expressions that arise in the proofs. The geometric situations are depicted in Figure 7.

Lemma 6.1. *Let $a \geq b > -1$. Let $\ell_{a,b}$ be the line through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ and $\ell^{\tan}_{-,R}$ be the line through $(-\theta^{1/2}, a\theta)$ which is tangent to $-x^2$ at a point in $[-\theta^{1/2}, \infty)$, i.e., right of $-\theta^{1/2}$.*

Suppose also $(a-b)^2 \leq 8(a+b)$, which, with $a \geq b > -1$, is equivalent to $b \geq (\sqrt{1+a}-1)(\sqrt{1+a}-3)$ and implies $a \leq (\sqrt{1+b}+1)(\sqrt{1+b}+3)$. Then, for $x \in [-\theta^{1/2}, \infty)$, $\ell_{a,b}(x) \geq \ell^{\tan}_{-,R}(x)$; see Figure 7. If $(a-b)^2 > 8(a+b)$, then $\ell_{a,b}(x) < \ell^{\tan}_{-,R}(x)$ for all $x \in [-\theta^{1/2}, \infty)$.

As a consequence, under the condition $(a-b)^2 \leq 8(a+b)$, the convex hull of the function $x \mapsto -x^2$ and the points $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ has both of the points as extreme points, and $\ell_{a,b}$ intersects the parabola at at most one point.

Finally, if $\ell_{a,b}$ and $\ell^{\tan}_{-,R}$ coincide, then $(a-b)^2 = 8(a+b)$.

Proof. The equation of the line $\ell^{\tan}_{-,R}$ that passes through $(-\theta^{1/2}, a\theta)$ and is tangent to $-x^2$ at a point in $[-\theta^{1/2}, \infty)$ is $-2x_0(x+\theta^{1/2}) + a\theta$, where $x_0 = \theta^{1/2}(\sqrt{1+a}-1)$, while the equation of $\ell_{a,b}(x)$ is $(b-a)\theta^{1/2}(x+\theta^{1/2})/2 + a\theta$. Thus the condition that the latter equation lies above the former on $[-\theta^{1/2}, \infty)$ is equivalent to

$$\frac{(b-a)\theta^{1/2}}{2} \geq -2x_0 \iff b \geq a - 4\sqrt{1+a} + 4 = (\sqrt{1+a}-1)(\sqrt{1+a}-3). \tag{26}$$

Viewing the latter inequality as a quadratic in $\sqrt{1+a}$, it is easy to show that it implies that

$$2 - \sqrt{1+b} \leq \sqrt{1+a} \leq 2 + \sqrt{1+b};$$

squaring the second inequality yields $a \leq (\sqrt{1+b}+1)(\sqrt{1+b}+3)$.

It can also be checked that (26) is implied by $(a-b)^2 \leq 8(a+b)$, by solving the latter as a quadratic in b, which yields the following equivalent condition on b when $a \geq -1$:

$$(\sqrt{1+a}-1)(\sqrt{1+a}-3) \leq b \leq (\sqrt{1+a}+1)(\sqrt{1+a}+3).$$

When $b \le a$, the second inequality is automatically satisfied, so (26) along with $-1 < b \le a$ implies $(a-b)^2 \le 8(a+b)$. It is clear that $\ell_{a,b}$ being a tangent line corresponds to equality in all of the preceding inequalities.

A line lying below a tangent line on an infinite ray does not imply the line intersects the parabola at two points. So to see that $(a-b)^2 > 8(a+b)$ is equivalent to $\ell_{a,b}$ having two intersection points (counted with multiplicity) with $-x^2$, we solve the quadratic $-x^2 = \ell_{a,b}(x)$ and note that its discriminant (which needs to be non-negative) is $(b-a)^2/4 - 2(b+a)$. ☐

Recall from the discussions around Theorems 3 and 10 that $\mathsf{ConHull}_{a,b}$ is the convex hull of the parabola $x \mapsto -x^2$ and the points $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$, and I_{lin} is the smallest closed set outside of which $\mathsf{ConHull}_{a,b}(x) = -x^2$. On I_{lin}, $\mathsf{ConHull}_{a,b}$ is piecewise linear. Observe from Lemma 6.1 (see also Figure 1) that I_{lin} is either an interval or a union of two disjoint intervals, depending on whether $\ell_{a,b}$ intersects $-x^2$ inside $[-\theta^{1/2}, \theta^{1/2}]$ or not.

The following is a reformulation of Theorem 10 in terms of the assumptions from Section 2.2.

Theorem 6.2. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). Let $\theta > 0$ and $a \ge b > -1$. For $M > 0$, let $M_{a,b} = M[(1+a)^{1/4} + (1+b)^{1/4}]$. There exist $c > 0$, $C < \infty$, θ_0, and $a_0 = b_0$ such that, if $\theta > \theta_0$ or $a, b \ge a_0, b_0$, and for $0 < M \le C^{-1}[(1+a)^{3/4} + (1+b)^{3/4}]\theta^{3/4}$ and $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\sup_{x \in I_{\mathrm{lin}}} \mathfrak{h}_1^t(x) - \mathsf{ConHull}_{a,b}(x) \ge M_{a,b}\theta^{1/4} \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta\right) \le \exp(-cM^2). \quad (27)$$

Remark 6.3. Note that we do not state a lower bound on the profile, i.e., an estimate for the probability that $\sup_{x \in I_{\mathrm{lin}}} \mathfrak{h}_1^t(x) - \mathsf{ConHull}_{a,b}(x) \le -M[(1+a)^{1/4} + (1+b)^{1/4}]\theta^{1/4}$ under the conditioning that $\mathfrak{h}_1^t(-\theta^{1/2}) = a\theta$ and $\mathfrak{h}_1^t(\theta^{1/2}) = b\theta$. This is merely because we do not need such a bound to prove the two-point upper tail estimate, Theorem 3, while we do require the bound stated in Theorem 6.2. A lower bound on the profile is in fact straightforward to prove following the same strategy used in the proof of the one-point profile lower bound in Theorem 9.

Remark 6.4. Also note that we have restricted the statement of Theorem 6.2 to the set I_{lin} where $\mathsf{ConHull}_{a,b}$ is piecewise linear; in the case that I_{lin} is two disjoint intervals (see the middle panel of Figure 1), this excludes, for example, the portion between the two intervals, where $\mathsf{ConHull}_{a,b}(x) = -x^2$. In fact we could prove a statement similar to Theorem 6.2 including this interval as well by making use of Proposition 3.16; we have not done so simply because it would complicate the statement (eg. we would need $M > \log\theta$ just for this interval in order to apply Proposition 3.16), and we do not need this extra information to prove the two-point asymptotic Theorem 7.1.

Proof of Theorem 10. This is implied by combining Theorem 2.8 that Assumptions (i)–(iv) hold with Theorem 6.2. ☐

Proof of Theorem 6.2. As noted above, there are two cases: (1) I_{lin} is a union of two disjoint intervals (2) I_{lin} is a single interval (and so $\mathsf{ConHull}_{a,b}$ is piecewise linear inside I_{lin}).

Case 1: I_{lin} is a union of two disjoint intervals. Let $I_{\mathrm{lin}} = [x^{\tan}_{-,\ell}, x^{\tan}_{-,r}] \cup [x^{\tan}_{+,\ell}, x^{\tan}_{+,r}]$. Note that (as can be seen by looking at the middle panel of Figure 1) $-\theta^{1/2} \in [x^{\tan}_{-,\ell}, x^{\tan}_{-,r}]$ and $\theta^{1/2} \in [x^{\tan}_{+,\ell}, x^{\tan}_{+,r}]$. We argue (27) with $[x^{\tan}_{-,\ell}, x^{\tan}_{-,r}]$ in place of I_{lin}; the same argument will apply to $[x^{\tan}_{+,\ell}, x^{\tan}_{+,r}]$. Define the event

$$\mathsf{Dev}^{\mathrm{left}}_{-,M} = \left\{ \sup_{x \in [x^{\tan}_{-,\ell}, x^{\tan}_{-,r}]} \mathfrak{h}_1^t(x) - \mathsf{ConHull}_{a,b}(x) \ge M_{a,b}\theta^{1/4} \right\}$$

and similarly $\mathsf{Dev}^{\mathrm{right}}_{-,M}$ by replacing $[x^{\tan}_{-,\ell}, x^{\tan}_{-,r}]$ with its $+$ counterpart. It is enough to show that the conditional probability given the values of $\mathfrak{h}^t_1(\pm\theta^{1/2})$ of each of these events is at most $\exp(-cM^2)$.

Let $\ell^{\tan}_{-,L}$ and $\ell^{\tan}_{-,R}$ be the tangent lines passing through $(-\theta^{1/2}, a\theta)$ on the left and right sides respectively. We now find pinning points on either side of $[x^{\tan}_{-,\ell}, x^{\tan}_{-,r}]$, i.e., points on the left and right of the interval at which $\mathfrak{h}^t_1$ is below $\ell^{\tan}_{-,L}$ and $\ell^{\tan}_{-,R}$ respectively with high probability, conditionally on $\mathfrak{h}^t_1(-\theta^{1/2}) = a\theta$ and $\mathfrak{h}^t_1(\theta^{1/2}) = b\theta$. To obtain a pinning point on the right side, observe (as can be seen from the middle panel of Figure 7) that the assumption that I_{lin} is two disjoint intervals implies that $(\theta^{1/2}, b\theta)$ is below $\ell^{\tan}_{-,R}(\theta^{1/2})$, and so $\theta^{1/2}$ serves as the pinning point. Thus we see, by applying Proposition 3.17 with $I = [-\theta^{1/2}, \theta^{1/2}]$,

$$\mathbb{P}\left(\mathsf{Dev}^{\mathrm{right}}_{-,M} \mid \mathfrak{h}^t_1(-\theta^{1/2}) = a\theta, \mathfrak{h}^t_1(\theta^{1/2}) = b\theta\right) \leq \exp(-cM^2).$$

To find a pinning point on the left we make use of stationarity and parabolic decay, as in the proof of Theorem 4.1. By Lemma 3.2 (monotonicity in conditioning),

$$\begin{aligned}
&\mathbb{P}\left(\mathfrak{h}^t_1(-x_L) > \ell^{\tan}_L(-x_L) \;\middle|\; \mathfrak{h}^t_1(-\theta^{1/2}) = a\theta, \mathfrak{h}^t_1(\theta^{1/2}) = b\theta\right) \\
&\qquad\leq \mathbb{P}\left(\mathfrak{h}^t_1(-x_L) > \ell^{\tan}_L(-x_L) \;\middle|\; \mathfrak{h}^t_1(-\theta^{1/2}) \geq a\theta, \mathfrak{h}^t_1(\theta^{1/2}) \geq b\theta\right) \\
&\qquad\leq \frac{\mathbb{P}\left(\mathfrak{h}^t_1(-x_L) > \ell^{\tan}_L(-x_L)\right)}{\mathbb{P}\left(\mathfrak{h}^t_1(-\theta^{1/2}) \geq a\theta, \mathfrak{h}^t_1(\theta^{1/2}) \geq b\theta\right)}.
\end{aligned}$$

Now we can lower bound the denominator by the FKG inequality (Assumption (ii)) and Theorem 5.3. Upper bounding the numerator by Theorem 5.1, and using stationarity and parabolic decay, we can find $-x_L < -x^{\tan}_{-,\ell}$ which is $O([(1+a)^{1/2} + (1+b)^{1/2}]\theta^{1/2})$ such that, for an absolute constant $c > 0$,

$$\mathbb{P}\left(\mathfrak{h}^t_1(-x_L) > \ell^{\tan}_L(-x_L) \;\middle|\; \mathfrak{h}^t_1(-\theta^{1/2}) = a\theta, \mathfrak{h}^t_1(\theta^{1/2}) = b\theta\right) \leq \exp\left(-c[(1+a)^{3/2} + (1+b)^{3/2}]\theta^3/2\right).$$

Using the previous display we see that

$$\begin{aligned}
&\mathbb{P}\left(\mathsf{Dev}^{\mathrm{left}}_{-,M} \;\middle|\; \mathfrak{h}^t_1(-\theta^{1/2}) = a\theta, \mathfrak{h}^t_1(\theta^{1/2}) = b\theta\right) \\
&\qquad\leq \mathbb{P}\left(\mathsf{Dev}^{\mathrm{left}}_{-,M}, \mathfrak{h}^t_1(-x_L) \leq \ell^{\tan}_L(-x_L) \;\middle|\; \mathfrak{h}^t_1(-\theta^{1/2}) = a\theta, \mathfrak{h}^t_1(\theta^{1/2}) = b\theta\right) \\
&\qquad\quad + \exp\left(-c[(1+a)^{3/2} + (1+b)^{3/2}]\theta^{3/2}\right).
\end{aligned} \tag{28}$$

Now the second term is further bounded by $\exp(-cM^2)$ since $M \leq C^{-1}[(1+a)^{3/4} + (1+b)^{3/4}]\theta^{3/4}$. The first term is bounded by applying Proposition 3.17 to obtain that it is at most $\exp(-cM^2)$.

Case 2: I_{lin} is one interval.

This case can be further divided based on whether $(a-b)^2 \leq 8(a+b)$ or $(a-b)^2 > 8(a+b)$, as the two correspond to different behaviours, as we saw in Lemma 6.1. In the second case we see that $\mathsf{ConHull}_{a,b}$ is actually the convex hull of $(-\theta^{1/2}, a\theta)$ and $x \mapsto -x^2$, i.e., we are in the case of the one-point limit shape. Thus the estimate follows from Theorem 4.1 after doing an appropriate translation and using stationarity of $\mathfrak{h}^t_1$.

In the remaining case of $(a-b)^2 \leq 8(a+b)$, we label I_{lin} as $[x^{\tan}_\ell, x^{\tan}_r]$. These are locations of tangency of the tangents to $-x^2$ which pass through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. For reference in a future proof we record the equations of the tangents as

$$\begin{aligned}
y &= 2x^{\tan}_\ell(x + x^{\tan}_\ell) - (x^{\tan}_\ell)^2, \\
y &= -2x^{\tan}_r(x - x^{\tan}_r) - (x^{\tan}_r)^2,
\end{aligned} \tag{29}$$

where

$$x^{\tan}_\ell = -(1 + \sqrt{1+a})\theta^{1/2} \qquad \text{and} \qquad x^{\tan}_r = (1 + \sqrt{1+b})\theta^{1/2}, \tag{30}$$

as can be calculated by using the information that the lines in (29) pass through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ respectively and solving.

Let $I_{\text{lin}}^{\text{left}} = [-x_\ell^{\tan}, -\theta^{1/2}]$, $I_{\text{lin}}^{\text{mid}} = [-\theta^{1/2}, \theta^{1/2}]$, and $I_{\text{lin}}^{\text{right}} = [\theta^{1/2}, x_r^{\tan}]$. Define the event

$$\mathsf{Dev}_M^{\text{left}} = \left\{ \sup_{x \in I_{\text{lin}}^{\text{left}}} \mathfrak{h}_1^t(x) - \mathsf{ConHull}_{a,b}(x) \geq M_{a,b}\theta^{1/4} \right\},$$

and similarly define $\mathsf{Dev}_M^{\text{cent}}$ and $\mathsf{Dev}_M^{\text{right}}$ with $I_{\text{lin}}^{\text{cent}}$ and $I_{\text{lin}}^{\text{right}}$ in place of $I_{\text{lin}}^{\text{left}}$; thus we are trying to bound

$$\mathbb{P}\left(\mathsf{Dev}_M^{\text{left}} \cup \mathsf{Dev}_M^{\text{cent}} \cup \mathsf{Dev}_M^{\text{right}} \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta \right).$$

The conditional probabilities of $\mathsf{Dev}_M^{\text{left}}$ and $\mathsf{Dev}_M^{\text{right}}$ are bounded by the same argument as was explained for bounding the conditional probability of $\mathsf{Dev}_{-,M}^{\text{left}}$ in Case 1. So we turn to $\mathsf{Dev}_M^{\text{cent}}$.

By applying the Brownian Gibbs property, $\mathbb{P}(\mathsf{Dev}_M^{\text{cent}} \mid \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta)$ is the probability that a Brownian bridge between $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ which is conditioned to stay above $\mathfrak{h}_2^t$ has a deviation greater than $M_{a,b}\theta^{1/4}$. As in Theorem 4.1, we use monotonicity to control $\mathfrak{h}_2^t$ via Lemma 3.18 (with $\beta = 3/2$, as ensured by Theorem 5.1). This yields

$$\begin{aligned}
&\mathbb{P}\left(\mathsf{Dev}_M^{\text{cent}} \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta \right) \\
&\quad \leq \mathbb{P}\left(\mathsf{Dev}_M^{\text{cent}}, \sup_{x \in I_{\text{lin}}^{\text{cent}}} \mathfrak{h}_2^t(x) + x^2 \leq \varepsilon M \theta^{1/4} \;\middle|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta, \mathfrak{h}_1^t(\theta^{1/2}) = b\theta \right) + \exp(-cM^2).
\end{aligned}$$

By applying the Brownian Gibbs property and using monotonicity (Lemma 3.1) we can replace the lower boundary condition by $-x^2 + \varepsilon M\theta^{1/4}$. Since in this subcase the line joining the endpoints of the Brownian bridge is tangent to or lies above $-x^2$ by Lemma 6.1, we get a Gaussian bound on the first term of the previous display by Corollary 3.15. ☐

6.1. **Control on the marginal height when conditioned on the upper tail.** The following lemma will be needed for the upper bound on the two-point upper tail that we prove in the next section. Roughly, as in the proof of Theorem 5.1, we will first upper bound the probability that $\mathfrak{h}_1^t(-\theta^{1/2}) \in [a\theta, a\theta + 1], \mathfrak{h}_1^t(\theta^{1/2}) \in [b\theta, b\theta + 1]$ instead of the probability of the entire upper tail, and to do so we will consider the probability that a Brownian bridge B with appropriate endpoints near the endpoints of $\mathsf{ConHull}_{a,b}$ satisfies $B(-\theta^{1/2}) \geq a\theta, B(\theta^{1/2}) \geq b\theta$, i.e., we come back to the entire upper tail for B (this is due to invoking monotonicity at an earlier point in the argument, as in the one-point case).

To estimate this probability we will break it up into $\mathbb{P}(B(-\theta^{1/2}) \geq a\theta) \cdot \mathbb{P}(B(\theta^{1/2}) \geq b\theta \mid B(-\theta^{1/2}) \geq a\theta)$; to estimate the second probability we will need to know that $B(-\theta^{1/2})$ is not too high above $a\theta$. Observe that such considerations did not arise in the one-point case. The following implies exactly this control on the margin of $B(-\theta^{1/2})$.

Lemma 6.5. *Suppose $a \geq b > -1$ and $(a-b)^2 \leq 8(a+b)$. For $M \in \mathbb{R}$, let $M_{a,b} = M((1+a)^{1/4} + (1+b)^{1/4})$ and let B be a rate two Brownian bridge from $(-x_\ell^{\tan}, -(x_\ell^{\tan})^2 + M_{a,b}\theta^{1/4})$ to $(x_r^{\tan}, -(x_r^{\tan})^2 + M_{a,b}\theta^{1/4})$, with $x_\ell^{\tan}$ and$x_r^{\tan}$ from (30). Then there exists an absolute constant M_0 such that, for $\theta > 0$ and $M > M_0$,*

$$\mathbb{P}\left(B(-\theta^{1/2}) \geq a\theta + 2M_{a,b}\theta^{1/4} \text{ or } B(\theta^{1/2}) \geq b\theta + 2M_{a,b}\theta^{1/4} \;\middle|\; B(-\theta^{1/2}) \geq a\theta, B(\theta^{1/2}) \geq b\theta \right) \leq \frac{1}{2}.$$

Proof. Let y, z be such that $(-x_\ell^{\tan}, y)$ and $(x_r^{\tan}, z)$ lie on the line ℓ connecting $(-\theta^{1/2}, a\theta + M_{a,b}\theta^{1/4})$ and $(\theta^{1/2}, b\theta + M_{a,b}\theta^{1/4})$; see Figure 8. The convexity hypothesis that $(a-b)^2 \leq 8(a+b)$ implies that $y > -(x_\ell^{\tan})^2 + M_{a,b}\theta^{1/4}$ and $z > -(x_r^{\tan})^2 + M_{a,b}\theta^{1/4}$.

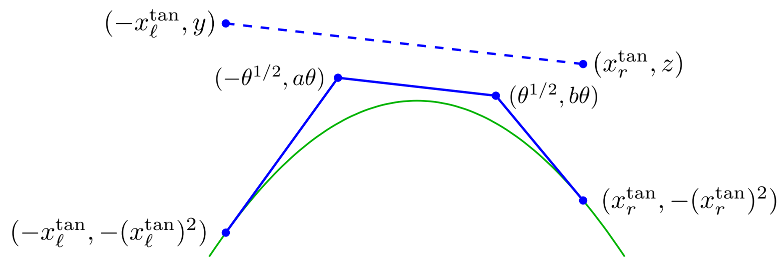

FIGURE 8. The solid blue lines show $\mathsf{ConHull}_{a,b}$ in the main case of Theorem 6.2, when $(a-b)^2 \leq 8(a+b)$. The dotted line above is at a height of $M_{a,b}\theta^{1/4}$ above the line connecting $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ and appears in the proof of Lemma 6.5.

Let $\tilde{B}$ be a rate 2 Brownian bridge from $(-x_\ell^{\tan}, y)$ to $(x_r^{\tan}, z)$. Let $I = [-x_\ell^{\tan}, x_r^{\tan}]$. Note that $\ell(x) - M_{a,b}\theta^{1/4}$ passes through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. By monotonicity (Lemma 3.1),

$$\begin{aligned}
&\mathbb{P}\left(B(-\theta^{1/2}) \geq a\theta + 2M_{a,b}\theta^{1/4} \text{ or } B(\theta^{1/2}) \geq b\theta + 2M_{a,b}\theta^{1/4} \;\middle|\; B(-\theta^{1/2}) \geq a\theta, B(\theta^{1/2}) \geq b\theta\right) \\
&\quad \leq \mathbb{P}\left(\tilde{B}(-\theta^{1/2}) \geq a\theta + 2M_{a,b}\theta^{1/4} \text{ or } \tilde{B}(\theta^{1/2}) \geq b\theta + 2M_{a,b}\theta^{1/4} \;\middle|\; \tilde{B}(x) \geq \ell(x) - M_{a,b}\theta^{1/4}\right) \\
&\quad \leq \mathbb{P}\left(\sup_{x\in I} \tilde{B}(x) - \ell(x) \geq M_{a,b}\theta^{1/4} \;\middle|\; \inf_{x\in I} \tilde{B}(x) - \ell(x) \geq -M_{a,b}\theta^{1/4}\right) \leq 2\exp(-cM_{a,b}^2),
\end{aligned}$$

for $c = 2(x_r^{\tan} - x_\ell^{\tan})^{-1} = 2(2+\sqrt{1+a}+\sqrt{1+b})^{-1}$, using Lemma 3.6 on the tail of the supremum of a Brownian bridge and (30) for the second equality. For large enough M (independent of a, b) this probability is less than $1/2$, as required. □

7. TWO-POINT ESTIMATES

Here we prove Theorem 3, after reformulating under the assumptions from Section 2.2. Recall that Theorem 3 has three cases depending on the number of extreme points of the convex hull $\mathsf{ConHull}_{a,b}$ inside $[-\theta^{1/2}, \theta^{1/2}]$. Using Lemma 6.1, it can be seen that the number of extreme points of the convex hull can be characterized in terms of the intersection of $\ell_{a,b}$ (the line passing through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$) with $-x^2$, along with an algebraic condition on a and b. This turns out to be convenient for the proof.

The reader can take a look at Figure 1 for a depiction of the three cases below (the panels are in the same order as the cases).

Theorem 7.1. *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). There exist constants θ_0 and $a_0 = b_0$ such that the following hold for all $t \in [t_0, \infty]$. If (i) $\theta > \theta_0$ and $a \geq b > -1$ or (ii) $\theta > 0$ and $a \geq a_0$, $b \geq b_0$, $a \geq b$, then, if $(a-b)^2 \leq 8(a+b)$,*

$$\begin{aligned}
&\mathbb{P}\Big(\mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta, \mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta\Big) \\
&\qquad = \exp\left(-\frac{\theta^{3/2}}{24}\left[3(a-b)^2 + 24(a+b) + 16\left((1+a)^{3/2} + (1+b)^{3/2}\right) + 32\right] + \text{error}\right).
\end{aligned}$$

while if $(a-b)^2 > 8(a+b)$ and $\ell_{a,b}$ intersects $-x^2$ inside $[-\theta^{1/2}, \theta^{1/2}]$,

$$\mathbb{P}\Big(\mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta, \mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta\Big) = \exp\left(-\frac{4}{3}\theta^{3/2}\left[(1+a)^{3/2} + (1+b)^{3/2}\right] + \text{error}\right),$$

and if $(a-b)^2 > 8(a+b)$ *but* $\ell_{a,b}$ *intersects* $-x^2$ *outside* $[-\theta^{1/2}, \theta^{1/2}]$,

$$\mathbb{P}\Big(\mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta, \mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta\Big) = \exp\left(-\frac{4}{3}\theta^{3/2}(1+a)^{3/2} + \mathrm{error}\right),$$

The error term may be lower bounded, up to a universal constant factor, by $-((1+a)^{1/2} + (1+b)^{1/2})\theta^{1/2}\log[(1+a)(1+b)\theta]$ *in the first and second case, and* $-(1+a)^{3/4}\theta^{3/4}$ *in the third.*

It may be upper bounded, again up to a universal constant factor, by $((1+a)^{3/4} + (1+b)^{3/4})\theta^{3/4}$ *for the first case,* $((1+a)^{1/2} + (1+b)^{1/2})\theta^{1/2}\log[(1+a)(1+b)\theta]$ *for the second case, and* $(1+a)^{3/4}\theta^{3/4}$ *in the third.*

Proof of lower bound of Theorem 7.1. We prove the bounds case-by-case, in increasing order of complexity of the argument.

Case 1: $(a-b)^2 > 8(a+b)$ *and* $\ell_{a,b}$ *intersects* $-x^2$ *inside* $[-\theta^{1/2}, \theta^{1/2}]$. Here the lower bound follows immediately from the FKG inequality (Assumption (ii)) and the one-point lower bound Theorem 5.3. The error term is $-C[(1+a)^{1/2} + (1+b)^{1/2}]\theta^{1/2}\log[(1+a)(1+b)\theta]$.

Case 2: $(a-b)^2 > 8(a+b)$ *and* $\ell_{a,b}$ *intersects* $-x^2$ *outside* $[-\theta^{1/2}, \theta^{1/2}]$. Observe from Lemma 6.1 that the tangent line from $(-\theta^{1/2}, a\theta)$ to the right lies above $(\theta^{1/2}, b\theta)$. So by the one-point limit shape (Theorem 4.1) and monotonicity in conditioning (Lemma 3.2), for large enough constant M,

$$\mathbb{P}\left(\mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta \;\Big|\; \mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta + M\theta^{1/4}\right) \geq \mathbb{P}\left(\mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta \;\Big|\; \mathfrak{h}_1^t(-\theta^{1/2}) = a\theta + M\theta^{1/4}\right) \geq \frac{1}{2}.$$

So we see that

$$\begin{aligned}
\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta,\, \mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta\right) &\geq \mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta + M\theta^{1/4},\, \mathfrak{h}_1^t(\theta^{1/2}) \geq b\theta\right) \\
&\geq \frac{1}{2}\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) \geq a\theta + M\theta^{1/4}\right) \\
&\geq \exp\left(-\frac{4}{3}\theta^{3/2}(1+a)^{3/2} - C(1+a)^{3/4}\theta^{3/4}\right),
\end{aligned}$$

using Theorem 5.3 for the last line. This completes the proof of the lower bound in this case.

Case 3: $(a-b)^2 \leq 8(a+b)$. This is the main case. The idea is to consider the convex hull of the parabola $-x^2$ and the two points $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. By assuming $a \geq b > -1$ and $(a-b)^2 \leq 8(a+b)$, we have ensured (see Lemma 6.1) that the convex hull is the parabola outside an interval, and a piecewise linear function inside, with both $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ as extreme points; see Figure 8. In other words, $\ell_{a,b}$ lies above the parabola.

We want to get a lower bound on the two-point probability by considering the event where $\mathfrak{h}_1^t$ lies above a well-chosen point on each of the tangency lines, and then compute the probability that a Brownian bridge with these two boundary points passes above $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. Notice that the Brownian bridge will ignore the lower boundary condition, which is allowed when proving lower bounds since, by monotonicity, the lower boundary would only push the bridge up.

We will lower bound the probability of $\mathfrak{h}_1^t$ being above the two boundary points using the FKG inequality. We know that the FKG inequality will be sharp in lower bounding the probability of $\{\mathfrak{h}_1^t(x_1) > 0, \mathfrak{h}_1^t(x_2) > 0\}$ for any $x_1 < 0 < x_2$; recall from Theorem 4 that the sharpness holds if the line connecting $(x_1, 0)$ and $(x_2, 0)$ is tangent to $-x^2$, which is clearly the case here, with the tangency location at zero. (Though we do not actually apply Theorem 4, as that will be proven as a consequence of the theorem we are currently proving.)

This leads us to choose the boundary points such that they lie on the tangency lines passing through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ and have y-coordinates zero. A simple calculation using the tangency line equations from (29) shows that these two x-coordinates are

$$-x_\ell^0 = -\frac{1}{2}x_\ell^{\tan} \quad \text{and} \quad x_r^0 = \frac{1}{2}x_r^{\tan},$$

where $x_\ell^{\tan}$ and $x_r^{\tan}$ are given by (30).

Let $\mathcal{F}$ be the σ-algebra generated by the lower curves on $\mathbb{R}$ and the top curve outside of $[-x_\ell^0, x_r^0]$. Now

$$\mathbb{P}_{\mathcal{F}}\Big(\mathfrak{h}_1^t(-\theta^{1/2}) > a\theta, \mathfrak{h}_1^t(\theta^{1/2}) > b\theta\Big) \geq \mathbb{P}_{\mathcal{F}}\Big(\mathfrak{h}_1^t(-\theta^{1/2}) > a\theta, \mathfrak{h}_1^t(\theta^{1/2}) > b\theta\Big) \cdot \mathbb{1}_{\mathfrak{h}_1^t(-x_\ell^0)\geq 0, \mathfrak{h}_1^t(x_r^0)\geq 0}.$$

By the H_t-Brownian Gibbs property and monotonicity Lemma 3.1, the conditional probability on the RHS on the specified events is lower bounded by the probability that a rate two Brownian bridge B from $(-x_\ell^0, 0)$ to $(x_r^0, 0)$ is greater than $a\theta$ and $b\theta$ at $-\theta^{1/2}$ and $\theta^{1/2}$ respectively. (Note that we have removed the lower boundary condition.) The latter probability is

$$\mathbb{P}\big(B(-\theta^{1/2}) \geq a\theta, B(\theta^{1/2}) \geq b\theta\big) \geq \mathbb{P}\big(B(-\theta^{1/2}) \geq a\theta\big) \cdot \mathbb{P}\big(B(\theta^{1/2}) \geq b\theta \mid B(-\theta^{1/2}) = a\theta\big).$$

Each of these are tail probabilities of normal distributions, and we calculate them now. First, $B(-\theta^{1/2})$ is distributed as a normal random variable with mean 0 and variance

$$\sigma_-^2 = 2\frac{(x_r^0 + \theta^{1/2})(x_\ell^0 - \theta^{1/2})}{x_r^0 + x_\ell^0} = 2\theta^{1/2} \cdot \frac{(\sqrt{1+a}-1)(\sqrt{1+b}+3)}{2+\sqrt{1+a}+\sqrt{1+b}},$$

so, by Lemma 3.5,

$$\begin{aligned}
\mathbb{P}\big(B(-\theta^{1/2}) \geq a\theta\big) &= \mathbb{P}\left(N(0,\sigma_-^2) \geq a\theta\right) \\
&\geq \exp\left(-\frac{1}{2\sigma_-^2}a^2\theta^2 + O(\log\theta)\right) \\
&= \exp\left(-a^2\theta^{3/2}\frac{(2+\sqrt{1+a}+\sqrt{1+b})}{(\sqrt{1+a}-1)(\sqrt{1+b}+3)} + O(\log\theta)\right).
\end{aligned}$$

Next we turn to computing $\mathbb{P}(B(\theta^{1/2}) \geq b\theta \mid B(-\theta^{1/2}) = a\theta)$. Observe that, conditionally on $B(-\theta^{1/2}) = a\theta$, the distribution of $B(\theta^{1/2})$ is a normal random variable with mean and variance given by

$$\begin{aligned}
\mu &= \frac{x_r^0 - \theta^{1/2}}{x_r^0 + \theta^{1/2}} \cdot a\theta + \frac{2\theta^{1/2}}{x_r^0 + \theta^{1/2}} \cdot 0 = \frac{a(\sqrt{1+b}-1)}{\sqrt{1+b}+3}\theta \\
\sigma_+^2 &= \frac{(x_r^0 - \theta^{1/2})2\theta^{1/2}}{x_r^0 + \theta^{1/2}} = \frac{4(\sqrt{1+b}-1)}{\sqrt{1+b}+3}\theta^{1/2}
\end{aligned}$$

Thus, noting that $\mu \leq b\theta$ by convexity (i.e., that $\mu \leq b\theta$ is implied by $(a-b)^2 \leq 8(a+b) \implies a \leq (\sqrt{1+b}+1)(\sqrt{1+b}+3)$ by Lemma 6.1),

$$\begin{aligned}
\mathbb{P}(B(\theta^{1/2}) \geq b\theta \mid B(-\theta^{1/2}) = a\theta) &\geq \mathbb{P}(N(\mu, \sigma_+^2) \geq b\theta) \\
&\geq \exp\left(-\frac{1}{2\sigma_+^2}(b\theta-\mu)^2 + O(\log\theta)\right) \\
&= \exp\left(-\theta^{3/2}\frac{\left(-a(\sqrt{1+b}-1) + b(\sqrt{1+b}+3)\right)^2}{8\left(\sqrt{1+b}-1\right)\left(\sqrt{1+b}+3\right)} + O(\log\theta)\right).
\end{aligned}$$

Finally, $\mathbb{P}(\mathfrak{h}_1^t(-x_\ell^0) > 0, \mathfrak{h}_1^t(-x_r^0) > 0)$ can be lower bounded using the FKG inequality and Theorem 5.3 as (for all large enough θ or large enough a, b)

$$\begin{aligned}
&\mathbb{P}\big(\mathfrak{h}_1^t(-x_\ell^0) > 0, \mathfrak{h}_1^t(-x_r^0) > 0\big)\\
&\qquad\geq \mathbb{P}\big(\mathfrak{h}_1^t(-x_\ell^0) > 0\big)\cdot\mathbb{P}\big(\mathfrak{h}_1^t(-x_r^0) > 0\big)\\
&\qquad\geq \exp\left(-\frac{4}{3}\left[(x_r^0)^3 + (x_\ell^0)^3\right] + O\left((x_r^0)^{1/2}\log x_r^0 + (x_\ell^0)^{1/2}\log x_\ell^0\right)\right)\\
&\qquad= \exp\left(-\frac{\theta^{3/2}}{6}\left(a(\sqrt{1+a}+3) + b(\sqrt{1+b}+3) + 4(2+\sqrt{1+a}+\sqrt{1+b})\right)\right)\\
&\qquad\qquad\times\exp\left\{O\left(\theta^{1/2}[(1+a)^{1/2} + (1+b)^{1/2}]\log\left(\theta(1+a)(1+b)\right)\right)\right\}.
\end{aligned}$$

The lower bound on $\mathbb{P}(\mathfrak{h}_1^t(-\theta^{1/2}) > a\theta, \mathfrak{h}_1^t(\theta^{1/2}) > b\theta)$ is given by

$$\mathbb{P}\big(B(-\theta^{1/2}) \geq a\theta\big)\cdot\mathbb{P}\big(B(\theta^{1/2}) \geq b\theta \mid B(-\theta^{1/2}) = a\theta\big)\cdot\mathbb{P}(\mathfrak{h}_1^t(-x_\ell^0) > 0, \mathfrak{h}_1^t(x_r^0) > 0),$$

which, using the expressions for each of the terms and algebraically simplifying the expressions in the resulting exponent, is the same as the claim.

Proof of upper bound of Theorem 7.1 We again prove the statement case-by-case in order of increasing complexity of the argument. (Cases 1 and 2 are switched compared to the lower bound arguments).

Case 1: $(a-b)^2 > 8(a+b)$ and $\ell_{a,b}$ intersects $-x^2$ outside $[-\theta^{1/2}, \theta^{1/2}]$. In this case the bound follows immediately from Theorem 5.1 (one-point upper tail) and stationarity, since always $\mathbb{P}(A\cap B) \leq \mathbb{P}(A)$.

In the remaining two cases we will in fact prove that the claimed upper bounds are upper bounds for the probability of the event

$$A(\theta, a, b) = \left\{\mathfrak{h}_1^t(-\theta^{1/2}) \in [a\theta, a\theta+1], \mathfrak{h}_1^t(\theta^{1/2}) \in [b\theta, b\theta+1]\right\};$$

clearly this suffices, by summing over a and b (and applying the probability bound to each summand corresponding to the case in the theorem statement it falls under). Indeed, that the sum will have the same form claimed in the statement follows from the fact that the summands decay exponentially.

Case 2: $(a-b)^2 > 8(a+b)$ and $\ell_{a,b}$ intersects $-x^2$ inside $[-\theta^{1/2}, \theta^{1/2}]$. Let Fav be given by

$$\mathsf{Fav} = \left\{\max_{x\in\{-x_\ell^{\tan}, x_\ell^{\text{in}}, x_r^{\text{in}}, x_r^{\tan}\}} \mathfrak{h}_1^t(x) + x^2 \leq M\theta^{1/4}, \sup_{x\in[-x_\ell^{\tan}, x_r^{\tan}]} \mathfrak{h}_2^t(x) + x^2 \leq \log\theta\right\},$$

where $x_\ell^{\text{in}} = (-1+\sqrt{1+a})\theta^{1/2}, x_r^{\text{in}} = (1-\sqrt{1+b})\theta^{1/2}$ are the tangency points corresponding to the tangents to $-x^2$ which pass through $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$. It can be checked that the hypotheses in this case imply that $x_\ell^{\text{in}} < x_r^{\text{in}}$, which also implies that both lie inside $(-\theta^{1/2}, \theta^{1/2})$; we label them "in" as they are the inner ones, i.e., lie inside $(-\theta^{1/2}, \theta^{1/2})$.

Now we use Theorem 6.2 and Lemma 3.18 to see that

$$\mathbb{P}(\mathsf{Fav}) \geq \frac{1}{2},$$

so that

$$\frac{1}{2}\mathbb{P}\left(A(\theta, a, b)\right) \leq \mathbb{P}\left(A(\theta, a, b), \mathsf{Fav}\right).$$

Let $\mathcal{F}$ be the usual sigma algebra associated with $[-x_\ell^{\tan}, x_\ell^{\mathrm{in}}]$. Then we see that the previous probability is equal to

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\mathfrak{h}_1^t(-\theta^{1/2}) \in [a\theta, a\theta+1]\right)\mathbb{1}_{\mathfrak{h}_1^t(\theta^{1/2})\in[b\theta,b\theta+1],\,\mathsf{Fav}}\right].$$

Let B be a Brownian bridge from $(-x_\ell^{\tan}, -(x_\ell^{\tan})^2 + M\theta^{1/4})$ to $(x_\ell^{\mathrm{in}}, -(x_\ell^{\mathrm{in}})^2 + M\theta^{1/4})$ and Z_{H_t} the partition function associated to the same boundary data and lower curve $-x^2+\log\theta$. By monotonicity Lemma 3.1, the previous display is upper bounded by

$$Z_{H_t}^{-1}\mathbb{P}\left(B(-\theta^{1/2}) \geq a\theta\right)\cdot\mathbb{P}\left(\mathfrak{h}_1^t(\theta^{1/2}) \in [b\theta, b\theta+1],\,\mathsf{Fav}\right).$$

The first two terms can be bounded, as in the proof of Theorem 5.1, by using normal tail bounds Lemma 3.5 and the lower bound on the partition function associated to boundary curve $-x^2$ from Lemma 3.11. Doing so yields an overall upper bound on the previous display of

$$\exp\left(-\frac{4}{3}\theta^{3/2}(1+a)^{3/2} + C(1+a)^{3/4}\theta^{3/4}\right)\cdot\mathbb{P}\left(\mathfrak{h}_1^t(\theta^{1/2}) \in [b\theta, b\theta+1],\,\mathsf{Fav}\right).$$

The same argument applied to the second term gives the claim.

Case 3: $(a-b)^2 \leq 8(a+b)$. This is the main case. Let $M_{a,b} = M((1+a)^{1/4} + (1+b)^{1/4})$ and

$$\mathsf{Fav} = \left\{\max\{\mathfrak{h}_1^t(-x_\ell^{\tan}) + (x_\ell^{\tan})^2, \mathfrak{h}_1^t(x_r^{\tan}) + (x_r^{\tan})^2\} \leq M_{a,b}\theta^{1/4}, \sup_{x\in[-x_\ell^{\tan}, x_r^{\tan}]}\mathfrak{h}_2^t(x) + x^2 \leq \log\theta\right\}.$$

Theorem 6.2 (two-point limit shape), Assumption (ii) (conditioned $\mathfrak{h}_2^t$ dominated by unconditioned $\mathfrak{h}_1^t$), Lemma 3.2 (monotonicity in conditioning variable), and Lemma 3.3 (tails of $\sup\mathfrak{h}_1^t$) imply that there is a constant M large enough such that $\mathbb{P}(\mathsf{Fav} \mid A(\theta,a,b)) \geq \frac{1}{2}$ for all large enough θ or large enough a, b. So,

$$\begin{aligned}\frac{1}{2}\mathbb{P}\left(A(\theta,a,b)\right) &\leq \mathbb{P}\left(A(\theta,a,b)\right)\cdot\mathbb{P}\left(\mathsf{Fav} \mid A(\theta,a,b)\right)\\ &= \mathbb{P}\left(A(\theta,a,b),\mathsf{Fav}\right).\end{aligned}$$

Let $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(1, [-x_\ell^{\tan}, x_r^{\tan}])$ be the σ-algebra generated by everything outside the top curve on $[-x_\ell^{\tan}, x_r^{\tan}]$. Conditioning on $\mathcal{F}$, the probability in the last display is $\mathbb{P}_{\mathcal{F}}(A(\theta,a,b))\mathbb{1}_{\mathsf{Fav}}$. By the H_t-Brownian Gibbs property and monotonicity (Lemma 3.1), this conditional probability, on Fav, is upper bounded by

$$Z_{H_t}^{-1}\mathbb{E}\left[\mathbb{1}_{B(-\theta^{1/2})\geq a\theta, B(\theta^{1/2})\geq b\theta}W_{H_t}\right] \leq Z_{H_t}^{-1}\mathbb{P}\left(B(-\theta^{1/2}) \geq a\theta, B(\theta^{1/2}) \geq b\theta\right), \tag{31}$$

where B is a Brownian bridge from $(-x_\ell^{\tan}, -(x_\ell^{\tan})^2 + M\theta^{1/4})$ to $(-x_r^{\tan}, -(x_r^{\tan})^2 + M\theta^{1/4})$ and W_{H_t} and Z_{H_t} are the Boltzmann factor and partition function associated with the same boundary values and lower boundary curve $-x^2+\log\theta$; note that we have replaced the intervals $[a\theta, a\theta+1]$ and $[b\theta, b\theta+1]$ by $[a\theta,\infty)$ and $[b\theta,\infty)$, which is what allows us to apply monotonicity.

We estimate the numerator of (31) first. We write, using Lemma 6.5,

$$\begin{aligned}&\frac{1}{2}\mathbb{P}\left(B(-\theta^{1/2}) \geq a\theta,\, B(\theta^{1/2}) \geq b\theta\right)\\ &\qquad\leq \mathbb{P}\left(B(-\theta^{1/2}) \in [a\theta, a\theta + 2M_{a,b}\theta^{1/4}],\, B(\theta^{1/2}) \in [b\theta, b\theta + 2M_{a,b}\theta^{1/4}]\right)\\ &\qquad\leq \mathbb{P}\big(B(-\theta^{1/2}) \geq a\theta\big)\cdot\mathbb{P}\left(B(\theta^{1/2}) \geq b\theta \;\middle|\; B(-\theta^{1/2}) \in [a\theta, a\theta + 2M_{a,b}\theta^{1/4}]\right)\end{aligned} \tag{32}$$

Now $B(-\theta^{1/2})$ is a normal random variable with mean $\mu_{\ell,0,r}$ and variance $\sigma^2_{\ell,0,r}$ given by

$$\begin{aligned}\mu_{\ell,0,r} &= \frac{x_\ell^{\tan}-\theta^{1/2}}{x_r^{\tan}+x_\ell^{\tan}}(-(x_r^{\tan})^2)+\frac{x_r^{\tan}+\theta^{1/2}}{x_r^{\tan}+x_\ell^{\tan}}(-(x_\ell^{\tan})^2)+M_{a,b}\theta^{1/4}\\ &= -\left(1+2\sqrt{1+a}+\sqrt{(1+a)(1+b)}\right)\theta+M_{a,b}\theta^{1/4};\\ \sigma^2_{\ell,0,r} &= 2\cdot\frac{(x_\ell^{\tan}-\theta^{1/2})(x_r^{\tan}+\theta^{1/2})}{x_r^{\tan}+x_\ell^{\tan}}\\ &= \frac{2\sqrt{1+a}(2+\sqrt{1+b})}{2+\sqrt{1+a}+\sqrt{1+b}}\cdot\theta^{1/2};\end{aligned}$$

Observe that $\mu_{\ell,0,r}-M_{a,b}\theta^{1/4}\le-\theta$ while $a>-1$, so $\mu_{\ell,0,r}-M_{a,b}\theta^{1/4}\le a\theta$. So by standard normal bounds from Lemma 3.5, the first factor in (32) is upper bounded by

$$\exp\left(-\frac{1}{2\sigma^2_{\ell,0,r}}(a\theta-\mu_{\ell,0,r})^2+\frac{1}{2\sigma^2_{\ell,0,r}}M_{a,b}^2\theta^{1/2}\right), \tag{33}$$

the final term included to handle the case that $\mu_{0,\ell,0,r}\ge a\theta$ in which case we cannot apply Lemma 3.5; however then $|a\theta-\mu_{\ell,0,r}|\le M_{a,b}\theta^{1/4}$, and so in that case the above displayed expression is greater than 1, making the bound hold trivially.

The second term of (32) is upper bounded by

$$\mathbb{P}\left(B(\theta^{1/2})\ge b\theta\;\middle|\;B(-\theta^{1/2})=a\theta+2M_{a,b}\theta^{1/4}\right).$$

Now $B(\theta^{1/2})$, when conditioned on $B(-\theta^{1/2})=a\theta+2M_{a,b}\theta^{1/4}$, is distributed as a normal random variable with mean $\mu_{0,0,r}$ and variance $\sigma^2_{0,0,r}$ given by

$$\begin{aligned}\mu_{0,0,r} &= \frac{2\theta^{1/2}}{x_r^{\tan}+\theta^{1/2}}(-(x_r^{\tan})^2+M_{a,b}\theta^{1/4})+\frac{x_r^{\tan}-\theta^{1/2}}{x_r^{\tan}+\theta^{1/2}}(a\theta+2M_{a,b}\theta^{1/4})\\ &\le \frac{-2(\sqrt{1+b}+1)^2+a\sqrt{1+b}}{2+\sqrt{1+b}}\cdot\theta+2M_{a,b}\theta^{1/4};\\ \sigma^2_{0,0,r} &= 2\cdot\frac{2\theta^{1/2}(x_r^{\tan}-\theta^{1/2})}{x_r^{\tan}+\theta^{1/2}}=\frac{4\sqrt{1+b}}{2+\sqrt{1+b}}\cdot\theta^{1/2}.\end{aligned}$$

Above the bound on $\mu_{0,0,r}$ is obtained by replacing the first term of $M_{a,b}\theta^{1/4}$ by $2M_{a,b}\theta^{1/4}$.

Next, we can apply the convexity hypothesis $a\le(\sqrt{1+b})+1)(\sqrt{1+b}+3)$ to see that $\mu_{0,0,r}-2M_{a,b}\le b\theta$. So by the standard normal bounds from Lemma 3.5, the second factor of (32) is at most

$$\exp\left(-\frac{1}{2\sigma^2_{0,0,r}}(b\theta-\mu_{0,0,r})^2+\frac{1}{2\sigma^2_{0,0,r}}4M_{a,b}^2\theta^{1/2}\right), \tag{34}$$

where the second term is again present to handle the situation where $\mu_{0,0,r}\ge b\theta$, in which case $|\mu_{0,0,r}-b\theta|\le 2M_{a,b}\theta^{1/4}$.

By Corollary 3.12, the denominator Z_{H_t} of (31) is lower bounded by

$$\exp\left(-\frac{1}{12}\left(x_\ell^{\tan}+x_r^{\tan}\right)^3-3(x_\ell^{\tan}+x_r^{\tan})\log(x_\ell^{\tan}+x_r^{\tan})\right).$$

Combining the previous bound with (33) and (34), substituting into (31) and (32), and simplifying the exponent, we obtain that (31) is upper bounded by

$$\exp\Bigg(-\frac{\theta^{3/2}}{24}\left[3(a-b)^2+24(a+b)+16\left((1+a)^{3/2}+(1+b)^{3/2}\right)+32\right]$$

$$+ CM_{a,b}\theta^{3/4} + 3(x_\ell^{\tan} + x_r^{\tan})\log(x_\ell^{\tan} + x_r^{\tan}) + CM_{a,b}^2\Big),$$

which completes the proof of Theorem 7.1. □

The following is Theorem 4 under Assumptions (i)–(iv).

Corollary 7.2 (Sharpness of FKG)**.** *Let $\mathfrak{h}^t$ satisfy Assumptions (i)–(iv). Let $t \in [t_0, \infty]$ and $a \geq b > -1$. Suppose $(a-b)^2 \geq 8(a+b)$ and the line $\ell_{a,b}$ connecting $(-\theta^{1/2}, a\theta)$ and $(\theta^{1/2}, b\theta)$ intersects the parabola $-x^2$ inside $[-\theta^{1/2}, \theta^{1/2}]$. Then, as $\theta \to \infty$ or as $a, b \to \infty$,*

$$\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) > a\theta, \mathfrak{h}_1^t(\theta^{1/2}) > b\theta\right) = \exp\left(-\frac{4}{3}\theta^{3/2}\left[(1+a)^{3/2} + (1+b)^{3/2}\right](1+o(1))\right),$$

i.e., the probability on the LHS is equal to $\mathbb{P}(\mathfrak{h}_1^t(-\theta^{1/2}) > a\theta) \cdot \mathbb{P}(\mathfrak{h}_1^t(\theta^{1/2}) > b\theta)$ up to first order in the exponent.

Proof. The case when $(a-b)^2 > 8(a+b)$ and $\ell_{a,b}$ intersects $-x^2$ inside $[-\theta^{1/2}, \theta^{1/2}]$ is asserted in Theorem 7.1, so we may assume $(a-b)^2 = 8(a+b)$ and the same condition on $\ell_{a,b}$.

We remind the reader from the proof of Lemma 6.1 that $\ell_{a,b}$ is given by

$$\ell_{a,b}(x) = \frac{(b-a)}{2} \cdot \theta^{1/2}(x + \theta^{1/2}) + a\theta,$$

and that the discriminant of the quadratic obtained by equating the right-hand side with $-x^2$ is equal to zero exactly when $(a-b)^2 = 8(a+b)$. This discriminant being zero is the condition for $\ell_{a,b}$ being a tangent to $-x^2$ somewhere on $\mathbb{R}$.

Now, for the tangency point to be inside $[-\theta^{1/2}, \theta^{1/2}]$, the slope $(b-a)\theta^{1/2}/2$ of L must lie inside the range of slopes of tangency lines of points inside $[-\theta^{1/2}, \theta^{1/2}]$, i.e., inside $[-2\theta^{1/2}, 2\theta^{1/2}]$. Thus, we get the condition

$$|a-b| \leq 4.$$

Let us set $a - b = 4z$ for $z \in [0,1]$, as $a \geq b$. Then from $(a-b)^2 = 8(a+b)$, we obtain

$$a = z^2 + 2z \quad \text{and} \quad b = z^2 - 2z.$$

Looking at the upper bound from Theorem 7.1 and using this parametrization of a and b, we see that

$$3(a-b)^2 + 24(a+b) + 32 = 48z^2 + 48z^2 + 32 = 96z^2 + 32,$$

while

$$16\left[(1+a)^{3/2} + (1+b)^{3/2}\right] = 16[(1-z)^3 + (1+z)^3] = 16(6z^2+2) = 96z^2 + 32.$$

Thus we see from Theorem 7.1 that

$$\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) > a\theta, \mathfrak{h}_1^t(\theta^{1/2}) > b\theta\right) = \exp\left(-\frac{32}{24}\theta^{3/2}\left[(1+a)^{3/2} + (1+b)^{3/2}\right](1+o(1))\right),$$

which simplifies to the claim. □

Remark 7.3 (Procedure for k-point asymptotics)**.** At this point the procedure for obtaining k-point asymptotics is clear: one finds pinning points (the number depends on the heights and locations of the individual point values, through their effect on the convex hull of the point values and the parabola, as in the three cases of Theorem 3) and uses Brownian bridge resamplings between pinning points to obtain the limit shape.

To obtain asymptotics, one essentially just has to calculate the Brownian bridge probabilities of achieving the desired heights when the pinning points are fixed, while taking into account the essentially parabolic lower boundary condition; for the lower bound, one handles the boundary condition by making use of the FKG inequality carefully (i.e., such that it satisfies the tangency conditions and so will be sharp) but otherwise ignoring it, while for the upper bound one makes use of Proposition 3.8 or Corollary 3.12.

Implementing this for higher values of k in general is quite tedious, as the resulting expressions get more complicated and the number of cases grows quite quickly; already from $k = 1$ to 2 it has increased from 1 case to 3. However, in practice one may be interested in only particular cases, in which case the procedure can be implemented for only those ones.

8. Extremal stationary ensembles

In this section we prove Theorem 2.8 for extremal stationary ensembles. More precisely, we show that extremal stationary line ensembles satisfy Assumption (iv) with $\beta = 1$, as the other aspects of Theorem 2.8, i.e., Assumptions (ii) and (iii), will be handled in Appendix A.

Theorem 8.1. *Suppose $\mathcal{L}$ is an extremal stationary line ensemble. Then there exist θ_0 and $c > 0$ such that, for $\theta > \theta_0$,*

$$\mathbb{P}\left(\mathcal{L}_1(0) > \theta\right) \leq \exp(-c\theta).$$

While we have stated the result for extremal stationary ensembles, it not be hard to check that the argument also applies to ensembles satisfying Assumptions (i)–(iii), if one changes applications of the Brownian Gibbs property to the H_t analogue. However, in the $t < \infty$ case, the constant θ_0 may depend on t; we do not have an argument to show that it can be taken uniform over $t \in [t_0, \infty]$, and it is for this reason that we rely on results from [CG20a] (recorded in Theorem A.7) to obtain such uniformity. See also Remark 8.4.

Before turning to the arguments for Theorem 8.1, we clarify a point of confusion concerning the meaning of extremal stationary ensembles that has arisen between two previous works which discuss them, namely [CH14] and [CS14].

8.1. Different notions of extremality.

The first work, [CH14], introduced extremal stationary ensembles in the context of Conjecture 5 that all such ensembles are the parabolic Airy line ensemble up to a deterministic vertical shift. The second, [CS14], proved that the parabolic Airy line ensemble is ergodic, and stated that this implies that it is the only candidate for the conjecture in [CH14]. However, there is a discrepancy in the meaning of an extremal stationary ensemble in the two works.

The discrepancy lies in the choice of convex set for which the extreme points are considered. Let Gibbs be the collection of measures of line ensembles which possess the Brownian Gibbs property, and Stat that of line ensembles which are stationary under horizontal shifts after an addition of the parabola $x \mapsto x^2$; by standard theory, both are convex sets. For a convex set A, let $\mathrm{Ex}(A)$ be the set of its extreme points.

In [CH14], the conjecture characterizes the elements of $\mathrm{Ex}(\mathsf{Gibbs}) \cap \mathsf{Stat}$ as being the parabolic Airy line ensemble up to a constant vertical shift, while [CS14] says the same about the elements of $\mathrm{Ex}(\mathsf{Gibbs} \cap \mathsf{Stat})$. Unfortunately, it is not obvious that these two sets coincide. However, it is easy to see that $\mathrm{Ex}(\mathsf{Gibbs}) \cap \mathsf{Stat} \subseteq \mathrm{Ex}(\mathsf{Gibbs} \cap \mathsf{Stat})$ as a general property of extreme points of convex sets (and indeed, [CS14] uses the similar fact that $\mathrm{Ex}(\mathsf{Stat}) \cap \mathsf{Gibbs} \subseteq \mathrm{Ex}(\mathsf{Gibbs} \cap \mathsf{Stat})$).

While it is not clear that the two sets are the same, we can say that the parabolic Airy line ensemble with any given deterministic vertical shift also belongs to $\mathrm{Ex}(\mathsf{Gibbs}) \cap \mathsf{Stat}$, just as [CS14] establishes their membership in $\mathrm{Ex}(\mathsf{Gibbs} \cap \mathsf{Stat})$. This follows from results on the triviality of the tail σ-algebra of determinantal point processes [OO18, Lyo18, BQS16][3] and the fact that the finite dimensional distributions of the parabolic Airy line ensemble are determinantal (with kernel the extended Airy kernel), combined with the abstract characterization of extremal measures as those which have trivial tail σ-algebras [Pre06, Theorem 2.1]. In particular, the parabolic Airy line ensemble is extremal stationary, and thus our arguments apply to it.

Having noted this discrepancy between [CH14] and [CS14], in the rest of this section, we will work with elements of $\mathrm{Ex}(\mathsf{Gibbs})\cap\mathsf{Stat}$; in particular, we will need the triviality of the σ-algebra containing boundary data as the domain goes to infinity, an equivalent condition to lying in $\mathrm{Ex}(\mathsf{Gibbs})$ as we just noted.

8.2. **The ingredients for the proof of Theorem 8.1.** To prove Theorem 8.1 we have several ingredients which are qualitative counterparts to similar steps in the proof of Theorem 5.1, as we explained earlier in Section 1.7. At a basic level, we need to be able to control the second curve to be below some decaying deterministic function, and have control over the top curve at two boundary points at some location.

We need to control the second curve on the event that $\mathcal{L}_1(0)$ is large; as we have seen many times before, by Assumption (ii), it is sufficient to control the top curve under no conditioning. The following proposition obtains this control, showing that the top curve lies below a linearly decaying function of any constant slope up to a random vertical shift.

Proposition 8.2. *Suppose $\mathcal{L}$ is an extremal stationary line ensemble. Then* $\sup_{x\in\mathbb{R}}\mathcal{L}_1(x)+K|x|$ *is almost surely finite for any $K>0$.*

The basic idea of the proof is to find a sequence of deterministic points where $\mathcal{L}_1(x) > -2K|x|$ only finitely often almost surely. But if there exist infinitely many (random) points where $\mathcal{L}_1(x) > -|K|x$, it would be unlikely that a Brownian bridge between those points goes below $-2K|x|$ at some location, especially since the lower boundary condition pushes the bridge upwards. This causes a contradiction since the random points must contain points from the deterministic sequence between them.

Proof of Proposition 8.2. Let K be given. We start by observing that $\mathbb{P}(\mathcal{L}_1(x) > -2K|x|) = \mathbb{P}(\mathcal{L}_1(0) > x^2-2K|x|)$, so, since $\mathcal{L}_1(0)$ is almost surely finite and $x^2-2K|x|\to\infty$ as $|x|\to\infty$, it follows that there exists a deterministic sequence of ordered points $\{x_n\}_{n\in\mathbb{Z}}$ with $x_n\to\infty, x_{-n}\to-\infty$ as $n\to\infty$ such that

$$\sum_{n\in\mathbb{Z}}\mathbb{P}\left(\mathcal{L}_1(x_n) > -2K|x_n|\right) < \infty. \tag{35}$$

It will be convenient to assume (without loss of generality) that $|x_n - x_{n-1}|\geq 2$, and we do so.

Consider the collection of intervals $\mathcal{I} = \{[x_{n-1}, x_n]\colon n\in\mathbb{Z}\}$. Note that $\cup_{I\in\mathcal{I}} I = \mathbb{R}$. For any interval $I\in\mathcal{I}$, let $S_I = \sup_{x\in I}\mathcal{L}_1(x)+K|x|$. Suppose that, for all $I\in\mathcal{I}$ with $\sup I\leq -1$ or $\inf I\geq 1$,

$$\sum_{\substack{J\in\mathcal{I}:\\ 0\to I\to J}}\mathbb{P}\left(S_I\geq 0, S_J\geq 0\right) < \infty; \tag{36}$$

here the second line below the sum is to indicate that the sum is over all J distinct from I which are further from 0 than I is, i.e., if I is to the right of zero then J is to the right of I, and analogously for the left.

[3]Technically, this is a statement about the triviality of the tail σ-algebra generated by finite collections of values of the line ensemble; further, the processes considered in [OO18, Lyo18, BQS16] take values in the space of probability measures as they are regarded as point processes. To get the statement made here about the tail σ-algebra of the line ensemble as defined on the space of infinite collections of continuous functions, one needs to change the space and approximate the process by the finite collections of values. We do not do the former here as this is a technical and not significant point for our purposes, while the latter follows from the almost sure continuity of the ensemble.

It is easy to see that (36) implies that $\sup_{x\in\mathbb{R}} \mathcal{L}_1(x) + K|x| < \infty$ almost surely. Indeed, given (36) for all I with $\inf I \geq 1$, it follows from the Borel-Cantelli lemma that almost surely, for each such I, either (i) $S_I < 0$ or (ii) $S_I \geq 0$ but there are only finitely many J with J to the right of I such that $S_J \geq 0$. Taking an intersection over these events yields a probability 1 event on which either $S_I < 0$ for all $I \in \mathcal{I}$ with $\inf I \geq 1$, or there exists some I with $S_I \geq 0$ and only finitely many J to the right of zero such that $S_J \geq 0$. In both cases, there is a random compact set $\mathcal{C} \subseteq [0,\infty)$ such that $\mathcal{L}_1(x) < -K|x|$ for all $x \in [0,\infty) \setminus \mathcal{C}$, and, by continuity, $\mathcal{L}_1(x) + K|x|$ is bounded inside $\mathcal{C}$ almost surely. So we obtain that $\sup_{x\geq 0} \mathcal{L}_1(x) + K|x| < \infty$ almost surely. Repeating the argument using (36) for all $I \in \mathcal{I}$ such that $\sup I \leq -1$ gives that $\sup_{x\leq 0} \mathcal{L}_1(x) + K|x| < \infty$ almost surely.

So it remains to prove (36). On the event that $S_I \geq 0$, let $\tau_I^\ell = \inf\{x \in I : \mathcal{L}_1(x) \geq -K|x|\}$ and $\tau_I^r = \sup\{x \in I : \mathcal{L}_1(x) \geq -K|x|\}$. If the corresponding sets are empty, i.e., when $S_I \leq 0$, let $\tau_I^\ell = \sup I + 1$ and $\tau_I^r = \inf I - 1$ (this definition is purely to make some future statements cleaner). We will use the analogous definitions for τ_J^ℓ and τ_J^r as well.

We will now argue (36) under the condition that $\inf I \geq 1$; the argument for the case that $\sup I \leq -1$ is analogous. We assume without loss of generality that $\sup I + 2 \leq \inf J$ as we are ignoring at most one term in (36) (recall $|x_n - x_{n-1}| \geq 2$). This is so that $\tau_I^\ell < \tau_J^r$.

We first observe that $[\tau_I^\ell, \tau_J^r]$ is a stopping domain (recall Definition 2.3). Let $\mathcal{F}_{\tau,I,J} = \mathcal{F}_{\mathrm{ext}}(1, [\tau_I^\ell, \tau_J^r])$ be the σ-algebra generated by the second and all lower curves on $\mathbb{R}$, and the top curve on $[\tau_I^\ell, \tau_J^r]$. Let n be such that $x_n = \inf J$, so that $x_n \in [\tau_I^\ell, \tau_J^r]$ on the event that $S_I \geq 0, S_J \geq 0$. Consider the event

$$A_{I,J}(\mathcal{L}_1) = \Big\{\mathcal{L}_1(x_n) \geq -2K|x_n|\Big\};$$

notice that the coefficient of $|x_n|$ is $-2K$ and not $-K$. Then we see that, since $\{S_I \geq 0, S_J \geq 0\} = \{\tau_I^\ell < \sup I + 1, \tau_J^r > -\inf J - 1\}$,

$$\begin{aligned}\mathbb{P}\big(S_I \geq 0, S_J \geq 0, A_{I,J}(\mathcal{L}_1)\big) &= \mathbb{P}\left(\tau_I^\ell < \sup I + 1, \tau_J^r > -\inf J - 1, A_{I,J}(\mathcal{L}_1)\right)\\ &= \mathbb{E}\left[\mathbb{P}_{\mathcal{F}_{\tau,I,J}}\left(A_{I,J}(\mathcal{L}_1)\right)\mathbb{1}_{\tau_I^\ell < \sup I+1, \tau_J^r > -\inf J - 1}\right].\end{aligned}$$

Now, $A_{I,J}(\mathcal{L}_1)$ is an increasing event. Further, on the event $\{\tau_I^\ell < \sup I + 1, \tau_J^r > -\inf J - 1\}$, $\mathcal{L}_1$ is distributed as a Brownian bridge from $(\tau_I^\ell, \mathcal{L}_1(\tau_I^\ell))$ to $(\tau_J^r, \mathcal{L}_1(\tau_J^r))$, conditioned on avoiding the second curve, and $\mathcal{L}_1(x) \geq -K|x|$ for $x \in \{\tau_I^\ell, \tau_J^r\}$. Let B be a rate two Brownian bridge between the points $(\tau_I^\ell, -K|\tau_I^\ell|)$ and $(\tau_I^r, -K|\tau_I^r|)$, but with no lower boundary conditioning. So, by monotonicity (Lemma 3.1),

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}_{\tau,I,J}}\left(A_{I,J}(\mathcal{L}_1)\right)\mathbb{1}_{\tau_I^\ell < \sup I+1, \tau_J^r > -\inf J - 1}\right] \geq \mathbb{E}\left[\mathbb{P}_{\mathcal{F}_{\tau,I,J}}\left(A_{I,J}(B)\right)\mathbb{1}_{\tau_I^\ell < \sup I+1, \tau_J^r > -\inf J - 1}\right].$$

Notice that $\mathbb{E}[B(x_n)] = -K|x_n|$. Thus, by symmetry of the normal distribution,

$$\mathbb{P}_{\mathcal{F}_{\tau,I,J}}\big(A_{I,J}(B)\big) = \mathbb{P}_{\mathcal{F}_{\tau,I,J}}\Big(B(x_n) \geq -2K|x_n|\Big) \geq \frac{1}{2}.$$

This yields, again since $\{S_I \geq 0, S_J \geq 0\} = \{\tau_I^\ell < \sup I + 1, \tau_J^r > -\inf J - 1\}$,

$$\mathbb{P}\big(S_I \geq 0, S_J \geq 0\big) \leq 2 \cdot \mathbb{P}\big(S_I \geq 0, S_J \geq 0, A_{I,J}(\mathcal{L}_1)\big).$$

Now, clearly $\mathbb{P}(S_I \geq 0, S_J \geq 0, A_{I,J}(\mathcal{L}_1)) \leq \mathbb{P}(A_{I,J}(\mathcal{L}_1)) = \mathbb{P}(\mathcal{L}_1(x_n) \geq -2K|x_n|)$. Note that, since $x_n = \inf J$, the n corresponding to distinct J are distinct. So, by our choice of $\{x_n\}_{n\in\mathbb{Z}}$ such that (35) holds, we obtain (36). ∎

Recall from Section 1.7.6, and as in the proof of Theorem 5.1, that to prove an upper bound on the upper tail of $\mathcal{L}_1(0)$, we first need to have control over the lower curve and know that the top curve is not too high at the boundary of an interval containing zero; then we will be able to resample on that interval and use these pieces of information to get a tail bound on the value at zero. In

Theorem 5.1 these two pieces of boundary information were provided by Theorem 4.1; here it is the next proposition.

Proposition 8.3. *Suppose $\mathcal{L}$ is a stationary extremal line ensemble. Then there exist $R > 0$ and θ_0 such that, for $\theta > \theta_0$,*

$$\mathbb{P}\Big(\mathcal{L}_1(\pm\theta) \le \frac{1}{2}\theta,\ \sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + 2|x| \le R \ \Big|\ \mathcal{L}_1(0) = \theta\Big) \ge \frac{1}{2}.$$

Remark 8.4. The arguments of Propositions 8.2 and 8.3 both apply essentially unchanged to the positive temperature analogue of extremal ensembles as well. However, the tail of the almost surely finite constant K in Proposition 8.2 may not be tight as t varies, i.e. over $t \ge t_0$ for some $t_0 > 0$, and this would result in the constants R and θ_0 in Proposition 8.3 depending on t as well. So the overall argument we are giving for zero temperature extremal ensembles would not yield Theorems 1–3 in the positive temperature case *with the uniformity in t*—though they will yield those results if the constants are allowed to depend on t.

As we outlined in Section 1.7.6, to prove Proposition 8.3 we will use stationarity, parabolic curvature, and one-point tightness to first find a faraway point x_θ so that $\mathcal{L}_1(x_\theta) \le -|x_\theta|$, i.e., below a line of slope -1. Next, we know from Proposition 8.2 that there exists a random constant $\tilde{R}$ such that $\mathcal{L}_2$ a.s. lies below $-2|x| + \tilde{R}$ on *all* of $\mathbb{R}$ (as usual, using the BK inequality from Assumption (ii) to say that $\mathcal{L}_2$ conditioned on any event of $\mathcal{L}_1$ is dominated by an unconditioned copy of $\mathcal{L}_1$).

Using the Gibbs property and monotonicity, we can dominate $\mathcal{L}_1$ on $[0, x_\theta]$ (and similarly for $[-x_\theta, 0]$) with a Brownian bridge B conditioned to stay above $-2|x|$, and so we have to understand the tail of $B(\theta)$. The next lemma says that the probability of this conditioning event is uniformly positive, and we prove it in Appendix C. The setup has been affinely transformed so that the lower line has slope zero.

Lemma 8.5. *For $K, r > 0$, let B^K be a rate two Brownian bridge between $(0,0)$ and (r, Kr). Let $\eta > 0$. Then there exists $C, c > 0$ (independent of K and r) such that, for $K \ge \frac{1}{2}\max(1, \eta^{-1})$,*

$$\mathbb{P}\left(\inf_{x\in[0,r]} B^K(x) < -\eta K\right) \le C\exp\left(-cK^2\right).$$

The form of the assumed lower bound on K is for technical convenience and as it suffices for the applications; but one could also assume $K \ge \delta\max(1, \eta^{-1})$ for some $\delta > 0$.

With this overview, we turn to the details of the proof of Proposition 8.3. It may help to recall the depiction of the setting from Figure 5.

Proof of Proposition 8.3. Let $\delta > 0$ be a constant to be fixed later. We first observe that, by Assumption (ii) (BK inequality) and Lemma 3.2 (monotonicity in conditioning), for any $R > 0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + 2|x| > R \ \Big|\ \mathcal{L}_1(0) = \theta\right) &\le \mathbb{P}\left(\sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + 2|x| > R \ \Big|\ \mathcal{L}_1(0) \ge \theta\right)\\ &\le \mathbb{P}\left(\sup_{x\in\mathbb{R}} \mathcal{L}_1(x) + 2|x| > R\right).\end{aligned}$$

By Proposition 8.2 with $K = 2$, this probability is less than δ for all sufficiently large R. We fix such an R for the remainder of the argument.

We now show that, for $x \in \{-\theta, \theta\}$,

$$\mathbb{P}\Big(\mathcal{L}_1(x) > \frac{1}{2}\theta, \sup_{y\in\mathbb{R}} \mathcal{L}_2(y) + 2|y| \le R \ \Big|\ \mathcal{L}_1(0) = \theta\Big) \le 2\delta.$$

This will clearly suffice as we may set δ a sufficiently small constant later. We focus on the $x=\theta$ case as the other is analogous.

We start by picking $x_\theta > \theta$ such that

$$\mathbb{P}\left(\mathcal{L}_1(x_\theta) > -x_\theta + \theta\right) \le \delta \cdot \mathbb{P}(\mathcal{L}_1(0) \ge \theta); \tag{37}$$

this is possible as the left-hand side is $\mathbb{P}(\mathcal{L}_1(0) > x_\theta^2 - x_\theta + \theta)$ by stationarity and $x^2 - x \to \infty$ as $x \to \infty$.

Next, by Lemma 3.2 (monotonicity in conditioning),

$$\mathbb{P}\Big(\mathcal{L}_1(x_\theta) > -x_\theta + \theta \;\Big|\; \mathcal{L}_1(0) = \theta\Big) \le \mathbb{P}\Big(\mathcal{L}_1(x_\theta) > -x_\theta + \theta \;\Big|\; \mathcal{L}_1(0) \ge \theta\Big) \le \delta.$$

Using this yields that

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{L}_1(\theta) > \frac{1}{2}\theta,\ \sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + 2|x| \le R \;\Big|\; \mathcal{L}_1(0) = \theta\Big) \\
&\qquad \le \mathbb{P}\Big(\mathcal{L}_1(\theta) > \frac{1}{2}\theta,\ \mathcal{L}_1(x_\theta) \le -x_\theta + \theta,\ \sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + 2|x| \le R \;\Big|\; \mathcal{L}_1(0) = \theta\Big) + \delta.
\end{aligned}$$

Let $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(1, [0, x_\theta])$ be the σ-algebra generated by the top curve outside $[0, x_\theta]$ and all the lower curves on $\mathbb{R}$. The first term on the RHS in the previous display is equal to

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\mathcal{L}_1(\theta) > \frac{1}{2}\theta \;\middle|\; \mathcal{L}_1(0) = \theta\right) \mathbb{1}_{\mathcal{L}_1(x_\theta) \le -x_\theta + \theta, \sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + 2|x| \le R}\right].$$

Under $\mathcal{F}$, $\mathcal{L}_1$ is distributed as a Brownian bridge from $(0, \theta)$ to $(x_\theta, -\mathcal{L}_1(x_\theta))$ conditioned to stay above $\mathcal{L}_2$. Let B be a Brownian bridge from $(0, \theta)$ to $(x_\theta, -x_\theta + \theta)$ without the lower boundary conditioning. Then, by monotonicity Lemma 3.1, the previous display is upper bounded by

$$\mathbb{P}\Big(B(\theta) > \frac{1}{2}\theta \;\Big|\; B(x) > -2|x| + R\ \forall x \in [0, x_\theta]\Big).$$

We want to uniformly lower bound the probability of the conditioning event using Lemma 8.5. For this we note that B is a Brownian bridge whose endpoints lie on a line of slope -1, while the conditioning is to stay above a line of slope -2; so the difference of slopes is 1. Further, by setting θ_0 large enough depending on R, the minimum distance between these lines is at least $\frac{1}{2}\theta$. Thus by Lemma 8.5 (with $K = 1$ and $\eta = 1/2$) the probability of the conditioning event is uniformly positive in θ, so the previous display is upper bounded by

$$C \cdot \mathbb{P}\left(B(\theta) > \frac{1}{2}\theta\right)$$

for some $C < \infty$ independent of θ. Now $B(\theta)$ is a normal random variable with mean $\theta - \theta = 0$ and variance $\sigma^2 = 2 \cdot \frac{\theta \times (x_\theta - \theta)}{x_\theta} \le 2\theta$ (recall B is of rate two). Thus using Gaussian tail bounds (Lemma 3.5) the previous display is upper bounded by

$$C \cdot \exp\left(-\frac{\frac{1}{4}\theta^2}{2\sigma^2}\right) \le C \cdot \exp\left(-\frac{\theta}{16}\right).$$

We set θ_0 such that this is less than δ for all $\theta > \theta_0$. This completes the proof after setting $\delta < 1/10$. ∎

With these statements in hand, we can now prove Theorem 8.1.

Proof of Theorem 8.1. As in the proof of the optimal one-point upper bound Theorem 5.1, we will in fact first bound $\mathbb{P}(\mathcal{L}_1(0) \in [\theta - 1, \theta])$.

Let $A_{R,\theta}$ be the event in the statement of Proposition 8.3 with the value of R in that statement, i.e.,

$$A_{R,\theta} = \left\{ \mathcal{L}_1(\pm\theta) \le \frac{1}{2}\theta, \sup_{x\in\mathbb{R}} \mathcal{L}_2(x) + |x| \le R \right\},$$

so that, by Proposition 8.3, Lemma 3.2 (monotonicity in conditioning), and since $A_{R,\theta}$ is a decreasing event,

$$\mathbb{P}(A_{R,\theta} \mid \mathcal{L}_1(0) \in [\theta-1,\theta]) \ge \mathbb{P}\left(A_{R,\theta} \mid \mathcal{L}_1(0) = \theta\right) \ge \frac{1}{2},$$

for $\theta > \theta_0$ with θ_0 as given in Proposition 8.3. We raise the value of θ_0 if needed so that $\theta_0 > 4R$, which we will need shortly. We observe that

$$\frac{1}{2} \cdot \mathbb{P}\left(\mathcal{L}_1(0) \in [\theta-1,\theta]\right) \le \mathbb{P}\left(\mathcal{L}_1(0) \in [\theta-1,\theta], A_{R,\theta}\right).$$

Let $\mathcal{F}$ be the σ-algebra generated by the top curve outside $[-\theta,\theta]$ and all the lower curves on $\mathbb{R}$. Then the previous display is equal to

$$\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\mathcal{L}_1(0) \in [\theta-1,\theta]\right) \mathbb{1}_{A_{R,\theta}}\right] \le \mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\mathcal{L}_1(0) \ge \theta-1\right) \mathbb{1}_{A_{R,\theta}}\right];$$

we performed this bound so as to make use of the increasing nature of the event $\{\mathcal{L}_1(0) > \theta\}$ on the right-hand side.

By the Brownian Gibbs property, conditionally on $\mathcal{F}$, $\mathcal{L}_1$ is a Brownian bridge from $(-\theta, \mathcal{L}_1(-\theta))$ to $(\theta, \mathcal{L}_1(\theta))$ conditioned on staying above $\mathcal{L}_2$. On $A_{R,\theta}$, by monotonicity Lemma 3.1, this Brownian bridge is stochastically dominated by the Brownian bridge B from $(-\theta, \frac{1}{2}\theta)$ to $(\theta, \frac{1}{2}\theta)$ conditioned to stay above $-|x| + R$. So, since $\{\mathcal{L}_1(0) > \theta - 1\}$ is an increasing event, the right-hand side of the previous display is upper bounded by

$$\frac{\mathbb{P}(B(0) \ge \theta - 1)}{\mathbb{P}\left(B(x) > -|x| + R \ \forall x \in [-\theta,\theta]\right)}.$$

Now, B has maximum deviation less than $\theta^{1/2}$ with positive probability independent of θ. Recall that the endpoints of B are at height $\frac{1}{2}\theta$. So, since $\theta > \theta_0 > 4R$ implies that $\frac{1}{2}\theta - \theta^{1/2} \ge R$, the denominator of the previous display is lower bounded by a constant. Now $B(0)$ is a normal random variable with mean $\theta/2$ and variance $2 \times \frac{\theta\times\theta}{2\theta} = \theta$, so, by standard normal tail bounds from Lemma 3.5, the numerator is upper bounded by

$$\exp\left(-\frac{(\theta - \frac{1}{2}\theta)^2}{2\theta}\right) = \exp\left(-\frac{1}{8}\theta\right).$$

Thus we have shown that $\mathbb{P}(\mathcal{L}_1(0) \in [s-1,s]) \le \exp(-cs)$ for some $c > 0$ and all $s > \theta_0$. Summing this $s = \theta + 1$ to ∞ gives that $\mathbb{P}(\mathcal{L}_1(0) > \theta) \le \exp(-c\theta)$ for any $\theta > \theta_0$. This completes the proof. □

9. General initial data

In this section we prove the one-point upper tail bounds for general initial data that was stated in Section 1 as Theorem 7. We separate the upper and lower bounds into the following two theorems to clarify which hypotheses on the initial data are needed for each. Recall the definition of the class of initial data $\mathrm{Hyp}(K, L, M, \delta)$ from Definition 1.1.

In Theorem 9.1 (upper bound), we assume that the initial condition $f^{(t)}$ satisfies certain growth conditions, but do not quantify any lower bounds. For Theorem 9.2 (lower bound), we need to assume both growth conditions as well as a lower bound on $f^{(t)}$ on some non-trivial set.

Theorem 9.1. *Let $T_0 \in (t_0, \infty]$ and $f^{(t)} \in \mathrm{Hyp}(K, L, M = \infty, \delta = 0)$ for some fixed K and $L > 0$ for all $t \in [t_0, T_0)$. There exist $\theta_0 > 0$ and $C < \infty$ such that, for $t \in [t_0, T_0)$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(\mathfrak{h}^{t,f}(0) \geq \theta\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

Theorem 9.2. *Let $T_0 \in (t_0, \infty]$ and $f^{(t)} \in \mathrm{Hyp}(K, L, M, \delta)$ for some fixed positive K, L, M, and δ for all $t \in [t_0, T_0)$. There exist $\theta_0 > 0$ and $C < \infty$ such that, for $t \in [t_0, T_0)$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(\mathfrak{h}^{t,f}(0) \geq \theta\right) \geq \exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{3/4}\right).$$

Both the upper and lower bounds' proofs relies on relating the upper tail for $\mathfrak{h}^{t,f}(0)$ to the upper tails of spatial maximums and minimums of $\mathfrak{h}^t_1$. This is done via the following distributional identity.

Lemma 9.3 (Lemma 4.3 of [CH16])**.** *Let $\mathcal{H}(t, x)$ be the solution to the KPZ equation started from general initial data $\mathcal{H}(0, \cdot)$. For fixed time $t > 0$, the following distributional equality holds:*

$$\mathcal{H}(t, 0) \stackrel{d}{=} \log\left(\int_{-\infty}^{\infty} \exp\left\{t^{1/3}\big(\mathfrak{h}^t_1(y) + t^{-1/3}\mathcal{H}(0, t^{2/3}y)\big)\right\} \mathrm{d}y\right) - \frac{t}{12} + \frac{2}{3}\log t,$$

so that, if $\mathcal{H}(0, y) = t^{1/3} f^{(t)}(t^{-2/3}y)$ for a function $f^{(t)}$,

$$\mathfrak{h}^{t,f}(0) \stackrel{d}{=} t^{-1/3}\log\left(\int_{-\infty}^{\infty} \exp\left\{t^{1/3}\big(\mathfrak{h}^t_1(y) + f^{(t)}(y)\big)\right\} \mathrm{d}y\right).$$

Remark 9.4. Using the above convolution formula and the sharp upper tail bounds, one can try to mimic the proof of Proposition 5.6 to obtain a degree of regularity on the density of $\mathfrak{h}^{t,f}(0)$, and thereby obtain a sharp bound on this density as in the proof of Theorem 5.5.

To do this one would again condition on $\mathcal{F}_{\mathrm{ext}}([-\theta^{1/2}, \theta^{1/2}], 1)$ as well as on the bridges of $\mathfrak{h}^t_1$ on $[-\theta^{1/2}, 0]$ and $[0, \theta^{1/2}]$. One would then consider the map G (itself a function of the conditioned data) which, under the coupling of $\mathfrak{h}^t_1$ and $\mathfrak{h}^{t,f}$ given by the convolution formula, takes $\mathfrak{h}^t_1(0)$ to $\mathfrak{h}^{t,f}(0)$. To make use of this representation, one would then need to understand properties of this map: in particular, it would suffice to know that it and its inverse are Lipschitz, and that $G^{-1}(\theta)$ equals θ up to some error term. (It is easy to see that G is increasing and so has an inverse, and in fact also that G is 1-Lipschitz, from the convolution formula and the formula for $\mathfrak{h}^t_1$ in terms of $\mathfrak{h}^t_1(0)$ and the bridges on either side.)

However, to obtain that the inverse is Lipschitz and that $G^{-1}(\theta) \approx \theta$, one needs to control the entire profile of $\mathfrak{h}^t_1$ conditional on $\mathfrak{h}^{t,f}(0) = \theta$, unlike in the narrow-wedge case, as the entire profile is involved in the convolution formula. This requires more work which we refrain from pursuing here, but we expect it should not be too difficult given the ideas and techniques developed in this paper.

9.1. **Upper bound for general initial data.** For the upper bound we will need to control the supremum of $\mathfrak{h}^t_1$ over $\mathbb{R}$. More precisely we need the following.

Proposition 9.5. *Let $L > 0$. There exist θ_0 and $C < \infty$ such that, for all $t \in [t_0, \infty]$,*

$$\mathbb{P}\left(\sup_{x \in \mathbb{R}} \mathfrak{h}^t_1(x) + x^2 - L|x| > \theta\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4}\right).$$

We include the growth of x^2 in the supremum because we allow the initial data to grow like x^2; however, because $\mathfrak{h}^t_1$ decays like $-x^2$, an extra lower order decay needs to be included with $+x^2$ to ensure that the solution does not immediately blow up, which is the role of $-L|x|$.

With Proposition 9.5 the proof of Theorem 9.1 is straightforward.

Proof of Theorem 9.1. Using the distributional identity from Lemma 9.3 and the bound on $f^{(t)}$ for $t \in [t_0, T_0)$ coming from Definition 1.1 and the hypotheses,

$$\begin{aligned}
\mathfrak{h}^{t,f}(0) &\stackrel{d}{=} t^{-1/3} \log \int_{-\infty}^{\infty} \exp\left\{t^{1/3}\left(\mathfrak{h}_1^t(y) + f^{(t)}(y)\right)\right\} \mathrm{d}y \\
&\le t^{-1/3} \log \int_{-\infty}^{\infty} \exp\left\{t^{1/3}\left(\mathfrak{h}_1^t(y) + y^2 - L|y| + K\right)\right\} \mathrm{d}y \\
&= t^{-1/3} \log \int_{-\infty}^{\infty} \exp\left\{t^{1/3}\left(\mathfrak{h}_1^t(y) + y^2 - \tfrac{1}{2}L|y|\right)\right\} \cdot e^{-t^{1/3}L|y|/2}\, \mathrm{d}y + K \\
&\le \sup_{x\in\mathbb{R}} \left[\mathfrak{h}_1^t(x) + x^2 - \tfrac{1}{2}L|x|\right] + t^{-1/3} \log \int_{-\infty}^{\infty} e^{-t^{1/3}L|y|/2}\, \mathrm{d}y + K \\
&= \sup_{x\in\mathbb{R}} \left[\mathfrak{h}_1^t(x) + x^2 - \tfrac{1}{2}L|x|\right] + t^{-1/3} \log[4L^{-1}t^{-1/3}] + K.
\end{aligned}$$

Since $t \ge t_0$, the second term in the final line is bounded uniformly in t. With this the proof is complete by invoking Proposition 9.5. □

Next we turn to proving Proposition 9.5. It relies on splitting up the supremum over $\mathbb{R}$ as a countable number of supremums over unit intervals and performing a union bound. For this we make use of Proposition 3.3, which gives a sharp upper tail estimate on the supremum of $\mathfrak{h}_1^t(x) + x^2$ over a unit interval; we will prove this latter statement in Section B.2.

Proof of Proposition 9.5. By a union bound,

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}} \mathfrak{h}_1^t(x) + x^2 - L|x| > \theta\right) \le \sum_{k=-\infty}^{\infty} \mathbb{P}\left(\sup_{x\in[k,k+1]} \mathfrak{h}_1^t(x) + x^2 - L|x| > \theta\right).$$

Next we see that, by stationarity of $\mathfrak{h}_1^t(x) + x^2$, for $k \in \mathbb{Z}$,

$$\begin{aligned}
\sup_{x\in[k,k+1]} \mathfrak{h}_1^t(x) + x^2 - L|x| &\le \sup_{x\in[k,k+1]} \mathfrak{h}_1^t(x) + x^2 - L(|k|-1) \\
&\stackrel{d}{=} \sup_{x\in[0,1]} \mathfrak{h}_1^t(x) + x^2 - L(|k|-1).
\end{aligned}$$

Thus, by Proposition 3.3 and Theorem 5.1, for $k \in \mathbb{Z}$ and $\theta > \theta_0$

$$\begin{aligned}
\mathbb{P}\left(\sup_{x\in[k,k+1]} \mathfrak{h}_1^t(x) + x^2 - L|x| > \theta\right) &\le \mathbb{P}\left(\sup_{x\in[0,1]} \mathfrak{h}_1^t(x) + x^2 > \theta + L(|k|-1)\right) \\
&\le \exp\left(-\frac{4}{3}\theta^{3/2} - \frac{4}{3}L^{3/2}(|k|-1)^{3/2} + C\theta^{3/4}\right).
\end{aligned}$$

This is clearly summable in k to give $c(L)\exp(-\frac{4}{3}\theta^{3/2} + C\theta^{3/4})$, completing the proof by modifying the value of C to absorb $c(L)$. □

9.2. **Lower bound for general initial data.** The estimate on $\mathfrak{h}_1^t$ that we need to obtain a lower bound on the upper tail for $\mathfrak{h}^{t,f}(0)$ is the following.

Proposition 9.6. *Let $M > 0$. There exist θ_0 and $C < \infty$ such that, for all $t \in [t_0, \infty]$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(\min_{x\in[-M,M]} \mathfrak{h}_1^t(x) \ge \theta\right) \ge \exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{1/2}\log\theta\right).$$

Similar to the upper bound, given Proposition 9.6, the proof of Theorem 9.2 is almost immediate.

Proof of Theorem 9.2. As in the upper bound, by the distributional identity from Lemma 9.3,

$$\begin{aligned}
\mathfrak{h}^{t,f}(0) &\stackrel{d}{=} t^{-1/3}\log\int_{-\infty}^{\infty}\exp\left\{t^{1/3}\left(\mathfrak{h}_1^t(y)+f^{(t)}(y)\right)\right\}\mathrm{d}y\\
&\geq \min_{x\in[-M,M]}\mathfrak{h}_1^t(x)+t^{-1/3}\log\int_{\{y\in[-M,M]:f^{(t)}(y)\geq -K\}}\exp(t^{1/3}f^{(t)}(y))\,\mathrm{d}y\\
&\geq \min_{x\in[-M,M]}\mathfrak{h}_1^t(x)-K-t^{-1/3}\log\delta^{-1},
\end{aligned}$$

the last inequality since $f^{(t)}\in\mathrm{Hyp}(K,L,M,\delta)$ for all $t\in[t_0,T_0)$ implies that $\mathrm{Leb}\{x\in[-M,M]: f^{(t)}(x)\geq -K\}\geq\delta$. With this, and noting that the last term is bounded since $t\geq t_0$, the proof of Theorem 9.2 is complete by invoking Proposition 9.6. □

It only remains to prove Proposition 9.6. One approach to proving it is to observe that, by Theorem 4.1, on the event that $\mathfrak{h}_1^t(0)>\theta+3M\theta^{1/2}$, it holds with probability at least $\frac{1}{2}$ that

$$\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq(\theta+3M\theta^{1/2})-2M(\theta+3M\theta^{1/2})^{1/2}-K\theta^{1/4}\geq\theta$$

by choosing K appropriately. The probability that $\mathfrak{h}_1^t(0)>\theta+3M\theta^{1/2}$ is at least $\exp(-\frac{4}{3}(\theta+3M\theta^{1/2})^{3/2})\geq\exp(-\frac{4}{3}\theta^{3/2}-C\theta)$, which gives our claim with a slightly worse lower order error term.

We instead give a different argument making use of the explicit two-point upper tail asymptotic from Theorem 7.1 to lower bound the probability that $\mathfrak{h}_1^t(\pm M)\geq\theta+R$ for a large constant R; on this event the fluctuation of $\mathfrak{h}_1^t$ on $[-M,M]$ can be easily controlled via the Gibbs property. This approach gives a smaller error term as R is a constant.

Proof of Proposition 9.6. Let R be a large constant to be fixed later. Trivially,

$$\mathbb{P}\left(\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq\theta\right)\geq\mathbb{P}\left(\mathfrak{h}_1^t(\pm M)\geq\theta+R,\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq\theta\right).$$

We apply the H_t-Brownian Gibbs property to $[-M,M]$, letting $\mathcal{F}$ be the σ-algebra generated by $\mathfrak{h}_1^t$ on $[-M,M]^c$ and the lower curves on $\mathbb{R}$, to see that

$$\mathbb{P}\left(\mathfrak{h}_1^t(\pm M)\geq\theta+R,\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq\theta\right)=\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq\theta\right)\mathbb{1}_{\mathfrak{h}_1^t(\pm M)\geq\theta+R}\right]$$

Let B be a Brownian bridge from $(-M,\theta+R)$ to $(M,\theta+R)$. By monotonicity (Lemma 3.1), the inner conditional probability in the previous display is lower bounded by the same probability with B in place of $\mathfrak{h}_1^t$, so we obtain

$$\mathbb{P}\left(\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq\theta\right)\geq\mathbb{P}\left(\min_{x\in[-M,M]}B(x)\geq\theta\right)\cdot\mathbb{P}\left(\mathfrak{h}_1^t(-M)>\theta+R,\mathfrak{h}_1^t(M)>\theta+R\right).$$

By Lemma 3.6 (on the tail of the supremum of Brownian bridge), $\mathbb{P}\left(\min_{x\in[-M,M]}B(x)\geq\theta\right)=1-\exp(-R^2/2)$, and we may set R a large enough constant, depending only on M, so that this term is at least $\frac{1}{2}$.

So we have obtained that

$$\mathbb{P}\left(\min_{x\in[-M,M]}\mathfrak{h}_1^t(x)\geq\theta\right)\geq\frac{1}{2}\mathbb{P}\left(\mathfrak{h}_1^t(-M)>\theta+R,\mathfrak{h}_1^t(M)>\theta+R\right)$$

We lower bound the final term using Theorem 7.1. The parameters in that theorem are set as $\theta\mapsto M^2$, $a=b\mapsto(\theta+R)/M^2$. This yields that

$$\mathbb{P}\left(\mathfrak{h}_1^t(-M)>\theta+R,\mathfrak{h}_1^t(M)>\theta+R\right)\geq\exp\left(-\frac{4}{3}\theta^{3/2}-C\theta^{1/2}\log\theta\right),$$

as can be checked by substituting the mentioned values of θ, a, and b into the asymptotic expression in Theorem 7.1. □

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Appendix A. Monotonicity proofs & assumption verification

In this appendix we prove that the KPZ, parabolic Airy, and extremal stationary ensembles all satisfy Assumptions (ii) and (iii). Recall that the former says that the line ensembles are positively associated and that the conditioned $\mathfrak{h}_2^t$ is dominated by an unconditioned $\mathfrak{h}_1^t$ (BK inequality), and the latter says that the conditional law of $\mathfrak{h}_1^t$ given its values at one or two points is stochastically increasing in those values.

In each of our proofs we will first prove an analogous statement for collections of Brownian bridges or motions which are reweighted by an appropriate Radon-Nikodym derivative. To then obtain the statements for the ensembles of interest we make use of the triviality of the tail σ-algebra for extremal stationary ensembles and the parabolic Airy line ensemble, for which this triviality is known. This extremality is not known for the KPZ line ensemble, and so for that we prove the statement for the prelimiting O'Connell-Yor diffusion and then pass to a limit.

The proof of the BK inequality is simpler than the other two and relies only on the monotonicity properties recorded in Lemma 3.1.

The proofs of the remaining two are in spirit similar to the proofs of Lemma 3.1 (monotonicity) given in [CH14] and [CH16] (and expanded on in [DM21, Dim21]): they construct a pair of Markov chains which each have as stationary distribution the law in question (one chain for each of the boundary datas for which the equilibrium measures are being stochastically compared), and find a coupling of the chains which exhibits the claimed monotonicity.

One difference, however, is that the proofs in the aforementioned works construct chains on discrete random walks and rely on diffusive scaling to their continuous analogues in order to obtain the desired statements. In contrast, we construct Markov chains directly on the space of k-tuples of continuous functions. While this introduces some additional complexities in ensuring the chains converge to their equilibrium measures, we avoid technical statements regarding diffusive scaling limits. Further, though the arguments appear slightly longer than the earlier ones, we believe this continuum framework may prove useful in future applications.

We start by introducing the O'Connell-Yor diffusion that we will need to prove our statements for the KPZ line ensemble.

A.1. **O'Connell-Yor diffusion.** The O'Connell-Yor free energy line ensemble is a diffusion $X^N : \{1,\ldots,N\}\times[0,\infty)\to\mathbb{R}$ whose infinitesimal generator is given by

$$\frac{1}{2}\Delta + \nabla\log\psi_0\cdot\nabla,$$

where Δ is the Laplacian on $\mathbb{R}^N$, ∇ is the gradient on $\mathbb{R}^N$, and ψ_0 is the class one $\mathfrak{gl}_N$-Whittaker function. See [O'C12, Theorem 3.1 and Corollary 4.1].

It follows from the discussion in [CH16, Section 8.3] that $X^N|_{[0,x]}$ has an equivalent description, for any $x>0$, as a collection of N independent rate one Brownian motions on $[0,x]$ reweighted by the Radon-Nikodym derivative (where $H_1(x) = 2\exp(x)$ is as defined in Proposition 2.6)

$$\frac{1}{Z}\exp\left\{\int_0^x\sum_{i=0}^N \exp\left(X_{i+1}^N(u)-X_i^N(u)\right)\,\mathrm{d}u\right\} = \frac{1}{Z}\exp\left\{\int_0^x\sum_{i=0}^N \tfrac{1}{2}H_1\left(X_{i+1}^N(u)-X_i^N(u)\right)\,\mathrm{d}u\right\}, \tag{38}$$

where X_0 and X_{N+1} are interpreted as $+\infty$ and $-\infty$ respectively, but can in general be taken to be upper and lower boundary data functions, as we will need later. If the diffusion is defined on an interval $[a,b]$, the analogous expression holds.

It is proven in [Nic21, Corollary 1.7] that, after appropriate scaling, X^N converges in the sense of finite dimensional distributions to an ensemble whose lowest-indexed line is the narrow-wedge solution to the KPZ equation at time t, i.e., the KPZ line ensemble; the KPZ equation considered in [Nic21] has slightly different coefficients than in this paper, but one can go between the two by simple scaling relations, as we discuss in Section B.1 ahead. The arguments for [CH16, Theorem 3.10] then upgrades this convergence to hold on the level of processes. This yields the following theorem.

Theorem A.1. *Fix $t>0$. Let $C(n,t,x) = N + \frac{\sqrt{Nt}+z}{2} + zt^{-1/2}N^{1/2} + \frac{N}{2}\log(tN^{-1})$ and $c_{i,t}^{-1} = t^{-i(i-1)/2}\prod_{j=0}^{i-1} j!$. Then, in the topology of uniform convergence on compact sets, as $N\to\infty$ and* *for each $i \in \mathbb{N}$,*

$$X_i^N(\sqrt{Nt}+x) - i\cdot C(N,t,x) + \log c_{i,t} \xrightarrow{d} \widetilde{\mathcal{H}}_k(t,x),$$

where $\widetilde{\mathcal{H}}(t,x) = \mathcal{H}(t/2,x/2)$ is the narrow-wedge solution to $\partial_t\widetilde{\mathcal{H}} = \frac{1}{2}\partial_x^2\widetilde{\mathcal{H}} + \frac{1}{2}(\partial_x\widetilde{\mathcal{H}})^2 + \xi$ (see Appendix B.1).

In particular, for the convergence of X^N to the KPZ line ensemble, it is enough to consider X^N on the interval $[0,N]$. Thus our approach to proving Assumptions (ii) and (iii) for $\mathfrak{h}^t$ will be to establish these properties for $X^N|_{[0,N]}$ and then appeal to the mentioned weak convergence. Further, by the above discussions, $X^N_{[0,N]}$ is a reweighted collection of Brownian motions. Thus we only need to prove that the assumptions hold for ensembles of such reweighted Brownian motions on finite intervals; this is in contrast to the proofs of the assumptions for extremal stationary and parabolic Airy line ensembles, where we first prove the assumptions for suitably reweighted Brownian *bridges*. However, the proofs are essentially the same for the bridges and motions cases, and so we will only present the proofs for the bridge case.

A.2. **Reweighted Brownian bridge/motion ensembles.** Next we state the results we will prove. Introducing some notation will be helpful. For $k \in \mathbb{N}$, let $\mathbb{R}^k_>$ be the set of decreasing k-vectors, i.e., $\mathbb{R}^k_> = \{\vec{x} \in \mathbb{R}^k : x_1 > x_2 > \ldots > x_k\}$.

For $k \in \mathbb{N}$, vectors $\vec{w},\vec{z} \in \mathbb{R}^k$, $\vec{x},\vec{y} \in \mathbb{R}^2$, and $g : [0,T] \to \mathbb{R}\cup\{-\infty\}$, let $\mu^{\vec{x},\vec{y},\vec{w},\vec{z},g}_{k,H_t}$ be the law of k independent Brownian bridges $B_1,\ldots,B_k$ on $[0,T]$ reweighted by the Radon-Nikodym derivative given by (8) with lower boundary data g, where $B_j(a) = w_j$ and $B_j(b) = z_j$ for $j = 1,\ldots,k$, and additionally $B_1(x_i)$ is conditioned on equaling y_i for $i = 1,2$. Similarly, $\mu^{\vec{w},\vec{z},g}_{k,H_t}$ will be the same without the final mentioned conditioning.

Theorem A.2 (Monotonicity in conditioning of reweighted Brownian bridges)**.** *Let $[a,b] \subseteq \mathbb{R}$, $y_{\uparrow,i} > y_{\downarrow,i}$ for $i = 1,2$ and $g : [a,b] \to \mathbb{R}\cup\{-\infty\}$ measurable. There is a coupling of $\mu^{\vec{x},\vec{y}_\uparrow,\vec{w},\vec{z},g}_{k,H_t}$ and $\mu^{\vec{x},\vec{y}_\downarrow,\vec{w},\vec{z},g}_{k,H_t}$ under which $B^\uparrow_j(x) > B^\downarrow_j(x)$ almost surely for all $x \in [a,b]$ and $j = 1,\ldots,k$, where $B^\uparrow = (B^\uparrow_1,\ldots,B^\uparrow_k)$ and $B^\downarrow = (B^\downarrow_1,\ldots,B^\downarrow_k)$ are distributed according to the respective measures.*

The same holds for Brownian motions $(B_1,\ldots,B_k)$ on compact intervals $[a,b]$ reweighted by (38), where initial data consists of ordered starting values at a, i.e., $B_i(a) = w_i$. The same also holds for the zero temperature (i.e., non-intersecting) cases of Brownian motion and Brownian bridge if additionally $\vec{w},\vec{z} \in \mathbb{R}^k_>$ and $g(a) < w_k$ and $g(b) < z_k$.

Theorem A.3 (Positive association of reweighted Brownian bridges)**.** *Let $[a,b] \subseteq \mathbb{R}$, $\vec{w},\vec{z} \in \mathbb{R}^k_>$, and $g : [a,b] \to \mathbb{R}\cup\{-\infty\}$ measurable. Let $B = (B_1,\ldots,B_k)$ be a collection of k Brownian bridges on $[a,b]$ distributed according to $\mu^{\vec{w},\vec{z},g}_{k,H_t}$. Let $A,E \subseteq \mathcal{C}^k([a,b],\mathbb{R})$ be increasing events under the natural order induced from the pointwise order on $\mathcal{C}([a,b],\mathbb{R})$. Then*

$$\mathbb{P}(B \in A, B \in E) \geq \mathbb{P}(B \in A)\cdot\mathbb{P}(B \in E).$$

The same holds for Brownian motions $(B_1,\ldots,B_k)$ on compact intervals $[a,b]$ reweighted by (38), where initial data consists of starting values at a, i.e., $B_i(a) = w_i$. The same also holds for the zero temperature (i.e., non-intersecting) cases of Brownian motion and Brownian bridge if additionally $\vec{w},\vec{z} \in \mathbb{R}^k_>$ and $g(a) < w_k$ and $g(b) < z_k$.

Theorem A.4 (BK inequality for reweighted Brownian bridge ensemble)**.** *Let $[a,b] \subseteq \mathbb{R}$, $\vec{w},\vec{z} \in \mathbb{R}^k_>$. Let $B = (B_1,\ldots,B_k)$ be a collection of k Brownian bridges on $[a,b]$ distributed according to $\mu^{\vec{w},\vec{z},-\infty}_{k,H_t}$. Let $A,E \subseteq \mathcal{C}([a,b],\mathbb{R})$ with A an increasing event and $\mathbb{P}(B_1 \in E) > 0$. Then*

$$\mathbb{P}(B_2 \in A \mid B_1 \in E) \leq \mathbb{P}(B_1 \in A).$$

The same holds for Brownian motions $(B_1, \dots, B_k)$ on compact intervals $[a, b]$ reweighted by (38), where initial data consists of starting values at a, i.e., $B_i(a) = w_i$. The same also holds for the zero temperature (i.e., non-intersecting) cases of Brownian motion and Brownian bridge if additionally $\vec{w}, \vec{z} \in \mathbb{R}^k_>$ and with lower boundary condition $g : [a, b] \to \mathbb{R} \cup \{-\infty\}$ measurable if $g(a) < w_k$ and $g(b) < z_k$.

As we mentioned, the proof of Theorem A.4 is simpler than those of the other two as it does not rely on Markov chain couplings. The Brownian motion case of it (as well as of the others) will need a monotonicity result for reweighted Brownian motion ensembles (analogous to the one for the Brownian bridge ensembles in Lemma 3.1), which we state next, after which we present the proof of Theorem A.4.

Lemma A.5 (Monotonicity in boundary data for reweighted Brownian motions)**.** *Fix $k \in \mathbb{N}$, $T > 0$, a pair of vectors $x^{(i)} \in \mathbb{R}^k$ and two pairs of measurable functions $(f^{(i)}, g^{(i)})$ for $i \in \{1, 2\}$ such that $x^{(1)}_j \le x^{(2)}_j$ for all $j = 1, \dots, k$ and $f^{(i)} : [0, T] \to \mathbb{R} \cup \{\infty\}$, $g^{(i)} : [0, T] \to \mathbb{R} \cup \{-\infty\}$ such that for all $0 \le s \le T$, $f^{(1)}(s) \le f^{(2)}(s)$ and $g^{(1)}(s) \le g^{(2)}(s)$.*

For $i \in \{1, 2\}$, let $\mathcal{Q}^{(i)} = \{\mathcal{Q}^{(i)}_j\}^k_{j=1}$ be a collection of k independent rate two Brownian motions $B_1, \dots, B_k$ on $[0, T]$ with $B_i(0) = x_i$, reweighted by the Radon-Nikodym derivative (38) associated with lower and upper boundary data $f^{(i)}, g^{(i)}$ respectively.

There exists a coupling of the laws of $\{\mathcal{Q}^{(1)}_j\}$ and $\{\mathcal{Q}^{(2)}_j\}$ such that almost surely $\mathcal{Q}^{(1)}_j(s) \le \mathcal{Q}^{(2)}_j(s)$ for all $j \in \{1, \dots, k\}$ and all $0 \le s \le T$.

Proof sketch. The proof follows the same lines as that of [CH16, Lemmas 2.6 and 2.7]. We consider a pair of Markov chains on the space of random walk paths (one for each set of boundary data) which do local updates in exactly the same way as in [CH16], so that monotonicity relations that hold for the initial state are preserved by the dynamic and the equilibrium measure is a discrete analogue for random walk ensembles of the desired one of reweighted Brownian motions. We choose initial states which indeed are monotonically ordered, which is possible by the ordering of the boundary datas. By taking a limit of the chain to the equilibrium measure, we thus obtain a monotone coupling of the two measures. The coupling is transferred to the reweighted Brownian motions measure by taking a diffusive limit of the discrete equilibrium measures via the Komlós–Major–Tusnády embedding [KMT75, KMT76], analogous to what is done to go from random walk bridges to Brownian bridges in [Dim21]. □

Next we prove Theorem A.4. Observe that the statement allows a lower boundary condition for the zero-temperature case. This is because a lower boundary will be needed when we apply this theorem to prove that extremal stationary ensembles satisfy the BK inequality.

Proof of Theorem A.4. We give the details of the positive temperature, Brownian bridge case first. The other cases are analogous and mentioned at the end.

Let $B = (B_1, \dots, B_k)$ be distributed according to $\mu^{\vec{w},\vec{z},-\infty}_{k,H_t}$. We have to prove that, for $A, E \subseteq \mathcal{C}([a, b], \mathbb{R})$ with A an increasing event,

$$\mathbb{P}\left(B_2 \in A \mid B_1 \in E\right) \le \mathbb{P}\left(B_1 \in A\right), \tag{39}$$

To prove (39), we make use of the H_t-Brownian Gibbs property and monotonicity (Lemma 3.1). For $k \ge 1$, $\vec{w}, \vec{z} \in \mathbb{R}^k$, and $f, g : [a, b] \to \mathbb{R}$ measurable, let $\mu^{k,\vec{w},\vec{z}}_{f,g}$ be the law of k Brownian bridges $(W_1, \dots, W_k)$ on $[a, b]$, with $W_i(a) = w_i$ and $W_i(b) = z_i$, reweighted by (8) with lower boundary curve f and upper boundary curve g (we omit the t-dependence in the notation for convenience). Thus $(B_1, \dots, B_k)$ is distributed is $\mu^{k,\vec{w},\vec{z}}_{-\infty,\infty}$.

Let $\mathcal{F}$ be the σ-algebra generated by B_1 on $[a,b]$. Then we see that

$$\mathbb{P}\left(B_2 \in A, B_1 \in E\right) = \mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(B_2 \in A\right)\mathbb{1}_{B_1\in E}\right] = \mathbb{E}\left[\mu_{-\infty,B_1}^{k-1,\vec{w}',\vec{z}'}\left(B_1' \in A\right)\mathbb{1}_{B_1\in E}\right],$$

where $\vec{w}' = (w_1',\ldots,w_{k-1}')$ and $\vec{z}' = (z_1',\ldots,z_{k-1}')$ are given by $w_i' = w_{i+1}$ and $z_i' = z_{i+1}$, and $B' = (B_1',\ldots,B_{k-1}')$ is distributed as $\mu_{-\infty,B_1}^{k-1,\vec{w}',\vec{z}'}$ conditionally on $\mathcal{F}$. By monotonicity (Lemma 3.1) and since A is an increasing event, by raising the upper boundary B_1 to $+\infty$,

$$\mu_{-\infty,B_1}^{k-1,\vec{w}',\vec{z}'}\left(B_1' \in A\right) \leq \mu_{-\infty,\infty}^{k-1,\vec{w}',\vec{z}'}\left(B_1' \in A\right).$$

Thus

$$\mathbb{P}\left(B_2 \in A, B_1 \in E\right) \leq \mathbb{P}\left(B_1 \in E\right)\cdot \mu_{-\infty,\infty}^{k-1,\vec{w}',\vec{z}'}\left(B_1' \in A\right). \tag{40}$$

Now it is easy to see that B_1' (now distributed as $\mu_{-\infty,\infty}^{k-1,\vec{w}',\vec{z}'}$) is dominated by the lowest indexed curve of an ensemble distributed as $\mu_{-\infty,\infty}^{k,\vec{w},\vec{z}}$, as the latter ensemble is obtained from the first by increasing all the endpoint values and including an extra curve at the bottom (which can be regarded as a lower boundary and thus, by Lemma 3.1, pushes the top curve up). Thus, again since A is an increasing event, and since the dominating ensemble is the same in law as the original ensemble $(B_1,\ldots,B_k)$,

$$\mu_{-\infty,\infty}^{k-1,\vec{w}',\vec{z}'}\left(B_1' \in A\right) \leq \mathbb{P}\left(B_1 \in A\right),$$

thus completing the proof in the positive temperature, Brownian bridge case. The other Brownian motion case is analogous; we just uses Lemma A.5 on monotonicity of reweighted Brownian motion ensembles instead of Lemma 3.1.

The zero temperature cases of Brownian motion is analogous to that for Brownian bridge and we indicate the changes needed for the latter. First, we need to replace the $-\infty$ lower boundary with g throughout. The only place where a change is needed is in the justification of the stochastic domination claim made right after (40), that the top curve of $\mu_{g,\infty}^{k-1,\vec{w}',\vec{z}'}$ is dominated by the top curve of $\mu_{g,\infty}^{k,\vec{w},\vec{z}}$. This is clear by regarding the k^{th} curve of the latter ensemble as a lower boundary for the top $k-1$ curves and then making the comparison via Lemma 3.1. (Note that this would not directly work in positive temperature, as the k^{th} curve need not be point-wise higher than g; this is why we assume a lower boundary of $-\infty$ in positive temperature, though we do expect the stochastic domination we are discussing to hold with a lower boundary condition even in positive temperature.)

This completes the proof of Theorem A.4. □

In the next two sections we give the proofs of Lemma 3.2 (monotonicity in conditioning) and Theorem 2.8, for the latter using Theorems A.2–A.4. After that, the rest of Appendix A will be devoted to proving Theorems A.2 and A.3.

A.3. **Proof of Lemma 3.2.**

Proof of Lemma 3.2. We prove the first inequality, in which $\inf E_i = y_i$ and F is increasing; the other is proved analogously.

Observe that

$$\begin{aligned}
&\mathbb{E}\left[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) \in E_1, \mathfrak{h}_1^t(\theta^{1/2}) \in E_2\right] \\
&\qquad = \frac{\mathbb{E}\left[F\cdot \mathbb{1}_{\mathfrak{h}_1^t(-\theta^{1/2})\in E_1,\mathfrak{h}_1^t(\theta^{1/2})\in E_2}\right]}{\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) \in E_1, \mathfrak{h}_1^t(\theta^{1/2}) \in E_2\right)} \\
&\qquad = \mathbb{E}\left[\mathbb{E}\Big[F \mid \mathfrak{h}_1^t(-\theta^{1/2}), \mathfrak{h}_1^t(\theta^{1/2})\Big]\frac{\mathbb{1}_{\mathfrak{h}_1^t(-\theta^{1/2})\in E_1,\mathfrak{h}_1^t(\theta^{1/2})\in E_2}}{\mathbb{P}\left(\mathfrak{h}_1^t(-\theta^{1/2}) \in E_1, \mathfrak{h}_1^t(\theta^{1/2}) \in E_2\right)}\right].
\end{aligned} \tag{41}$$

The expression on the last line is an average of the function

$$(z_1, z_2) \mapsto \mathbb{E}\left[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) = z_1, \mathfrak{h}_1^t(\theta^{1/2}) = z_2\right].$$

against a certain probability measure supported on a subset of $[y_1, \infty) \times [y_2, \infty)$ (recall that $\inf E_i = y_i$). By Assumption (iii), this function is increasing in (z_1, z_2). Thus the right-hand side of (41) is lower bounded by $\mathbb{E}[F \mid \mathfrak{h}_1^t(-\theta^{1/2}) = y_1, \mathfrak{h}_1^t(\theta^{1/2}) = y_2]$, which completes the proof. □

Next we give the proof of Theorem 2.8 by verifying that the KPZ, parabolic Airy, and extremal stationary line ensembles satisfy Assumptions (ii)–(iii).

A.4. **Verifying Assumptions (i)–(iv) for the KPZ, parabolic Airy, and extremal stationary line ensembles.** To verify Assumptions (i) (Brownian Gibbs and stationarity) and (iv) (a priori upper tail bounds) for the KPZ and parabolic Airy line ensembles we first start with references to the literature.

We begin with Assumption (i).

Proposition A.6 (Stationarity and Brownian Gibbs)**.** *For each $t > 0$, there exists a line ensemble $\mathfrak{h}^t$ with the H_t-Brownian Gibbs property such that $\mathfrak{h}_1^t$ is distributed as the narrow-wedge solution to the KPZ equation (1). Additionally, for each $t > 0$ and $k \in \mathbb{N}$, adding x^2 to $\mathfrak{h}_k^t$ gives a stationary process.*

Similarly, the parabolic Airy line ensemble possesses the Brownian Gibbs property and each curve is stationary after the addition of x^2.

Note that in fact we need stationarity only of the top curve to meet Assumption (i).

Proof of Proposition A.6. For $t > 0$, the existence of the line ensemble and that it possesses the H_t-Brownian Gibbs property is stated in [CH16, Theorem 2.15]. [ACQ11] gives that $x \mapsto \mathfrak{h}_1^t(x) + x^2$ is stationary, but not the lower curves (or the whole ensemble). The latter assertion is proved in [Nic21, Corollary 1.13].

For the parabolic Airy line ensemble $\mathcal{P}$, that it possesses the Brownian Gibbs property is [CH14, Theorem 3.1], and the stationarity of $\mathcal{P}_1(x) + x^2$ is [PS02, Theorem 4.3]. For stationarity of the lower curves, we could not find an explicit reference in the literature, but it follows from the same stationarity for the KPZ line ensemble and its convergence to the parabolic Airy line ensemble as proven in [QS22, Theorem 2.2(4)] or [Vir20, Theorem 3]. □

Next we verify that the KPZ line ensemble and the parabolic Airy line ensemble satisfy Assumption (iv) with $\beta = \frac{3}{2}$.

Theorem A.7 (Theorem 1.11 of [CG20a] and Theorem 1.3 of [RRV11])**.** *For any $t_0 > 0$, there exist $c_1 \geq c_2 > 0$ and θ_0 such that, for any $t \in [t_0, \infty]$, $x \in \mathbb{R}$, and $\theta \geq \theta_0$,*

$$\exp\bigl(-c_1\theta^{3/2}\bigr) \leq \mathbb{P}\bigl(\mathfrak{h}_1^t(x) + x^2 > \theta\bigr) \leq \exp\bigl(-c_2\theta^{3/2}\bigr)$$

With this we have verified Assumptions (i) and (iv) for the KPZ and parabolic Airy line ensembles. Note that Assumption (i) holds for extremal stationary line ensembles by definition, and Assumption (iv) is verified for them via Theorems 5.3 and 8.1. So it remains only to prove Assumptions (ii) and (iii) hold for the three ensembles.

Proof of Theorem 2.8: Assumptions (ii) and (iii). There are three cases that we must handle: for the KPZ line ensemble, for the parabolic Airy line ensemble, and for extremal zero-temperature ensembles. For the first, while we expect it to be true, we do not know that the σ-algebra at infinity is trivial; showing this would be an interesting result.

For this reason for the KPZ line ensemble we prove the needed statements for the prelimiting model of the O'Connell-Yor free energy line ensemble introduced in Appendix A.1, and take the edge scaling limit to obtain the result for the KPZ line ensemble. For general extremal stationary ensembles, in contrast, we do not have access to any such prelimiting model, and so we must work with the ensemble directly; in place of the prelimiting model we use the extremality. This approach also works for the parabolic Airy line ensemble using its extremality as noted in Section 8.1.

We will prove Assumption (iii) here; the proofs for the two parts (FKG and BK) of Assumption (ii) are analogous with Theorems A.3 and A.4 respectively used in place of Theorem A.2.

We have to show that, for any interval $[a,b] \subseteq \mathbb{R}$ containing x_1 and x_2, any bounded increasing function $F : \mathcal{C}([a,b],\mathbb{R}) \to \mathbb{R}$, and any $y_i^* \in \mathbb{R}$ with $i = 1,2$ and $* \in \{\uparrow,\downarrow\}$ with $y_i^\uparrow > y_i^\downarrow$,

$$\mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) = y_1^\uparrow, \mathfrak{h}_1^t(x_2) = y_2^\uparrow\right] \geq \mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) = y_1^\downarrow, \mathfrak{h}_1^t(x_2) = y_2^\downarrow\right].$$

In the prelimiting situation, this statement is exactly what is provided by Theorem A.2. Now, we cannot take $n \to \infty$ directly in Theorem A.2 to obtain the previous display because we are conditioning on a zero probability event. Thus we will first show the same with the conditionings equal to $\mathfrak{h}^t(x_i) \in [y_i^\uparrow - \varepsilon, y_i^\uparrow]$ for all small enough ε, as these have positive probability, by proving it for finite n and taking $n \to \infty$. Finally taking $\varepsilon \to 0$ will complete the proof.

KPZ line ensemble: For the O'Connell-Yor free energy line ensemble, scaled such that it converge to $\mathfrak{h}^t$, which we denote by $\mathfrak{h}^{t,n} = (\mathfrak{h}_1^{t,n}, \ldots, \mathfrak{h}_n^{t,n})$, it follows from the Brownian motion case of Theorem A.2 along with an averaging argument as in the proof of Lemma 3.2 above that

$$\begin{aligned}\mathbb{E}\Big[F \mid \mathfrak{h}_1^{t,n}(x_1) \in [y_1^\uparrow - \varepsilon, y_1^\uparrow], \mathfrak{h}_1^{t,n}(x_2) \in [y_2^\uparrow - \varepsilon, y_2^\uparrow]\Big]& \\ \geq \mathbb{E}\Big[F \mid \mathfrak{h}_1^{t,n}(x_1) \in [y_1^\downarrow - \varepsilon, y_1^\downarrow], \mathfrak{h}_1^{t,n}(x_2) \in [y_2^\downarrow - \varepsilon, y_2^\downarrow]\Big]&.\end{aligned} \tag{42}$$

(We point out that the scalings imposed on X^N will translate to scalings on the Brownian ensemble in Theorem A.2, but we have neglected to be explicit about this as they are just affine transformations and scalings by constants which can, for example, be absorbed into F.)

We take $n \to \infty$ and use that the conditioning events are positive probability to conclude that (42) holds with $\mathfrak{h}^t$ in place of $\mathfrak{h}^{t,n}$. Taking $\varepsilon \to 0$ completes the proof in the cases of the KPZ line ensemble.

Extremal zero-temperature and parabolic Airy ensembles: We need to prove (42) with $\mathcal{L}$ or $\mathcal{P}$ in place of $\mathfrak{h}_1^{t,n}$; we will stick with the $\mathcal{P}$ notation. For an event A and σ-algebra $\mathcal{F}$, we will use the notation $\mathbb{P}(\cdot \mid A, \mathcal{F}) = \mathbb{P}(\cdot \cap A \mid \mathcal{F})/\mathbb{P}(A \mid \mathcal{F})$.

We condition on the σ-algebra $\mathcal{F}_n = \mathcal{F}_{\mathrm{ext}}(n, [-n,n])$. By commutativity of conditionings (see [Kal21, Theorem 8.15]), we have a collection of n non-intersecting Brownian bridges with the top one conditioned to pass through $[y_i^* - \varepsilon, y_i^*]$ at x_i for $i = 1,2$, one collection for each of $* \in \{\uparrow,\downarrow\}$, and conditioned to stay above $\mathcal{P}_{n+1}$. By the Brownian bridge and zero-temperature case of Theorem A.2, we obtain that the $\uparrow$-ensemble stochastically dominates the $\downarrow$ one, so that

$$\begin{aligned}\mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) \in [y_1^\uparrow - \varepsilon, y_1^\uparrow], \mathfrak{h}_1^t(x_2) \in [y_2^\uparrow - \varepsilon, y_2^\uparrow], \mathcal{F}_n\right]& \\ \geq \mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) \in [y_1^\downarrow - \varepsilon, y_1^\downarrow], \mathfrak{h}_1^t(x_2) \in [y_2^\downarrow - \varepsilon, y_2^\downarrow], \mathcal{F}_n\right]&.\end{aligned}$$

Now we take $n \to \infty$ and use that the limiting σ-algebra is trivial. Since $\mathfrak{h}_1^t(x_1) \in [y_1^* - \varepsilon, y_1^*], \mathfrak{h}_1^t(x_2) \in [y_2^* - \varepsilon, y_2^*]$ are positive probability events, we see that

$$\begin{aligned}&\mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) \in [y_1^\uparrow - \varepsilon, y_1^\uparrow], \mathfrak{h}_1^t(x_2) \in [y_2^\uparrow - \varepsilon, y_2^\uparrow]\right] \\ &\qquad = \lim_{n\to\infty} \mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) \in [y_1^\uparrow - \varepsilon, y_1^\uparrow], \mathfrak{h}_1^t(x_2) \in [y_2^\uparrow - \varepsilon, y_2^\uparrow], \mathcal{F}_n\right] \\ &\qquad \geq \lim_{n\to\infty} \mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) \in [y_1^\downarrow - \varepsilon, y_1^\downarrow], \mathfrak{h}_1^t(x_2) \in [y_2^\downarrow - \varepsilon, y_2^\downarrow], \mathcal{F}_n\right]\end{aligned}$$

$$= \mathbb{E}\left[F \mid \mathfrak{h}_1^t(x_1) \in [y_1^{\downarrow} - \varepsilon, y_1^{\downarrow}], \mathfrak{h}_1^t(x_2) \in [y_2^{\downarrow} - \varepsilon, y_2^{\downarrow}]\right].$$

Taking $\varepsilon \to 0$ completes the proof. □

In the rest of the appendix we will prove Theorems A.2 and A.3.

A.5. **Definition of the Markov chain.** Next we move to defining the Markov chain that will be used to prove Theorem A.2; the chain for Theorem A.3 is similar and we will indicate the changes required later.

We will describe a Markov chain on $\mathcal{C}^k([a,b],\mathbb{R})$ and denote it by $(\mathcal{X}_n)_{n\in\mathbb{N}}$; it will depend on parameters $\varepsilon > 0$, $x_1 < x_2 \in [a,b]$, $y_1, y_2 \in \mathbb{R}$, $M > 0$, and $g : [a,b] \to \mathbb{R}$ measurable with $g(x_i) < y_i$ for $i = 1, 2$, but we suppress these in the notation. The chain is defined by the following.

At time $n+1$, each of the k curves are resampled in increasing order of the index. The i^{th} curve is resampled as a Brownian bridge from (a, w_i) to (b, z_i) such that it is

(i) reweighted by the Radon-Nikodym derivative from (8) associated with upper and lower boundary data $\mathcal{X}_{n+1}(i-1,\cdot)$ and $\mathcal{X}_n(i+1,\cdot)$ respectively (here we take $\mathcal{X}_n(k+1,\cdot) = g$ for all n) if $2 \le i \le k$; note that the top curve is the one at time $n+1$ and the below one is from time n, and that in the notation of (8) we take $k = 1$ since we are resampling a single curve. For $i = 1$, the curve is resampled in the same with with lower boundary $\mathcal{X}_n(2,\cdot)$ and no upper boundary, with the additional conditioning that it passes through $[y_1 - \varepsilon, y_1 + \varepsilon]$ at x_1 and $[y_2 - \varepsilon, y_2 + \varepsilon]$ at x_2, and

(ii) conditioned on lying inside $[-M, M]$ on all of $[a,b]$ and with Hölder-$\frac{1}{4}$ norm at most M.

The second condition is to ensure that $\mathcal{X}$ takes values in a *compact* subset of $\mathcal{C}^k([a,b],\mathbb{R})$, by the Azela-Ascoli theorem; the value of $\frac{1}{4}$ is not significant, with the only important feature that it is smaller than $\frac{1}{2}$. We assume that M is large enough that the set of functions meeting condition (ii) and being greater than g is non-empty (which will be needed only in the zero-temperature case).

We will verify two things. First, that $\mathcal{X}$ has as stationary measure the law of k independent Brownian bridges with appropriate endpoints reweighted by the Radon-Nikodym derivative from (8) and conditioned on passing through $[y_i - \varepsilon, y_i + \varepsilon]$ at x_i for $i = 1, 2$ and on $\sup_{1\le i\le k} |\mathcal{X}_n(i,\cdot)| \le M$, $\max_{1\le i\le k} \|\mathcal{X}_n(i,\cdot)\|_{\frac{1}{4}}$, where for a function $f : [a,b] \to \mathbb{R}$, $\|f\|_{\frac{1}{4}}$ denotes its Hölder-$\frac{1}{4}$ norm. Second, that $\mathcal{X}_n$ converges in law to the stationary distribution in an averaged sense. The latter will allow us to transfer properties of any coupling of two versions of $\mathcal{X}_n$ (eg. with different parameters) to a corresponding coupling of the respective stationary measures.

To verify these things, we will need several abstract results about Markov chains on general state spaces. We collect these, along with the necessary definitions, in the next subsection, and are taken from [MT12].

A.6. **Results on Markov chains on abstract state spaces.** In this subsection, the Markov chain will be denoted $\mathcal{X}$ and the state space S. The state space is assumed to be locally compact, separable, metrizable, and equipped with the Borel σ-algebra. (Observe that the state space we have chosen above satisfies these conditions, and is in fact compact.)

Following the definition from [MT12, Section 6.1.1], a Markov process $\mathcal{X}$ with transition kernel P is strongly Feller if, for any bounded measurable function f, $x \mapsto P(x, f) = \int f(y)P(x, \mathrm{d}y)$ is a bounded continuous function. This condition is equivalent to the statement that, for any sequence x_n converging to any x, $P(x_n,\cdot) \to P(x,\cdot)$ in the total variation norm.

As in Markov chains on countable state spaces, to have convergence to an equilibrium measure, the chain needs to have an irreducibility property. This is known as ψ-irreducibility in the general case; while we do not define it here, we have the following statement which gives a sufficient condition for it to hold. The statement holds for a class of Markov chains known as T-chains; again we do not define what these are, but in fact strongly Feller chains are T-chains, as can be easily verified by referring to the definitions on [MT12, pages 127–128].

Proposition A.8 (Proposition 6.2.1 of [MT12])**.** *Suppose $\mathcal{X}$ is a T-chain and there exists $x_0 \in S$ such that, for every open set $O \subseteq S$ containing x_0, it holds that, for all $y \in S$,*

$$\mathbb{P}\left(\mathcal{X}_n \in O \text{ for some } n \mid \mathcal{X}_0 = y\right) > 0.$$

Then $\mathcal{X}$ is ψ-irreducible.

We also need to know that there exists an invariant measure and that it is unique. For existence we will exhibit the invariant measure explicitly (which recall is our target distribution for which we wish to demonstrate a coupling); to show uniqueness we will need a recurrence condition, again analogous to countable state space Markov chains. The following is a sufficient condition which ensures recurrence.

Theorem A.9 (Theorems 9.2.2 and 10.0.1 of [MT12])**.** *If $\mathcal{X}$ is a ψ-irreducible T-chain, and there exists a compact set A such that*

$$\mathbb{P}\left(\mathcal{X}_n \in A \text{ i.o.}\right) > 0,$$

then $\mathcal{X}$ is recurrent. Further, if $\mathcal{X}$ is recurrent, then it has a unique (up to multiplication by a constant) invariant measure.

Since our chain takes values in a compact space, the condition of Theorem A.9 will hold automatically.

A chain $\mathcal{X}$ is said to be bounded in probability on average if, for every initial condition x, the family of probability measures $\{\sum_{j=1}^n n^{-1} P^j(x,\cdot)\}_{n\in\mathbb{N}}$ is tight. This will be guaranteed for our chain again since the state space is compact, and any family of probability measures on a compact space is tight.

Proposition A.10 (Proposition 12.1.4 of [MT12])**.** *Suppose $\mathcal{X}$ is Feller, is bounded in probability on average, and has a unique invariant measure π. Then, for every $x \in S$, as $n \to \infty$,*

$$\frac{1}{n}\sum_{i=1}^{n} P^n(x,\cdot) \xrightarrow{d} \pi.$$

A.7. **Applying the abstract results to our Markov chain.** We warn the reader that this subsection is somewhat long and technical, and amounts to verifying that our Markov chain of interest satisfies the conditions of the abstract statements collected in the previous subsection.

We start by verifying that our chain is strongly Feller. In this subsection $\mathcal{X}$ refers to the chain defined in Section A.5.

A.7.1. *$\mathcal{X}$ is strongly Feller.* We will verify the condition recalled in the previous subsection for a Markov chain to be strongly Feller, namely that the transition kernel of satisfies $P(x_n,\cdot) \to P(x,\cdot)$ in total variation norm whenever $x_n \to x$. For this we will need the following two simple lemmas.

Lemma A.11. *For every $\varepsilon > 0$, let μ_1^ε and μ_2^ε be probability measures on a space Ω, and suppose they respectively have densities $W_1^\varepsilon/Z_1^\varepsilon$ and $W_2^\varepsilon/Z_2^\varepsilon$ with respect to a common probability measure μ, where $W_1^\varepsilon, W_2^\varepsilon : \Omega \to [0,1]$ and $Z_i^\varepsilon = \int W_i^\varepsilon \,\mathrm{d}\mu$. Suppose also that $\int |W_1^\varepsilon - W_2^\varepsilon| \,\mathrm{d}\mu \to 0$ as $\varepsilon \to 0$ and $\inf_{\varepsilon>0}\{Z_1^\varepsilon, Z_2^\varepsilon\} > 0$. Then it holds that, as $\varepsilon \to 0$,*

$$\|\mu_1^\varepsilon - \mu_2^\varepsilon\|_{\mathrm{TV}} \to 0.$$

Proof. Follows from the well-known fact that $\|\mu_1^\varepsilon - \mu_2^\varepsilon\|_{\mathrm{TV}} = \frac{1}{2}\int |W_1^\varepsilon/Z_1^\varepsilon - W_2^\varepsilon/Z_2^\varepsilon| \,\mathrm{d}\mu$. ☐

Lemma A.12. *Let $\mathbb{P}_{\mathrm{free}} = \mathbb{P}_{\mathrm{free}}^{1,[a,b],w,z}$ be the law of Brownian bridge $B : [a,b] \to \mathbb{R}$ with $B(a) = w$ and $B(b) = z$. Let $\{f_\varepsilon\}_{\varepsilon>0}, \{g_\varepsilon\}_{\varepsilon>0}$ be families of continuous functions on $[a,b]$ with $\sup_{x\in[a,b]} |f_\varepsilon(x) - f(x)|, \sup_{x\in[a,b]} |g_\varepsilon(x) - g(x)| \le \varepsilon$ for some $f, g : [a,b] \to \mathbb{R}$ continuous. Then, for any $t \in (0,\infty)$, as $\varepsilon \to 0$,*

$$\mathbb{E}_{\mathrm{free}}\left[\left|W_{H_t}^{f_\varepsilon,g_\varepsilon} - W_{H_t}^{f,g}\right|\right] \to 0 \quad \text{and} \quad \inf_{\varepsilon>0} \mathbb{E}_{\mathrm{free}}[W_{H_t}^{f_\varepsilon,g_\varepsilon}] > 0$$

where $W_{H_t}^{f_\varepsilon,g_\varepsilon}$ is the Boltzmann factor from Definition 2.2 associated to lower and upper boundary conditions $f_{\varepsilon,i}$ and $g_{\varepsilon,i}$ respectively, and similarly for $W_{H_t}^{f,g}$. The same is true for $t = \infty$ under the additional assumption that the endpoints and the functions are ordered for all $\varepsilon > 0$, i.e., $f_\varepsilon(a) < w < g_\varepsilon(a)$, $f_\varepsilon(b) < z < g_\varepsilon(b)$, and $f_\varepsilon(x) < g_\varepsilon(x)$ for all $x \in [a,b]$, and the same with f, g in place of $f_\varepsilon, g_\varepsilon$.

In particular, from Lemma A.11, $\|\mathbb{P}_{H_t}^{1,[a,b],w,z,f_\varepsilon,g_\varepsilon} - \mathbb{P}_{H_t}^{1,[a,b],w,z,f,g}\|_{\mathrm{TV}} \to 0$ as $\varepsilon \to 0$.

The proof follows from inspecting the formula for $W_{H_t}^{f_\varepsilon,g_\varepsilon}$ and we omit it.

Lemma A.13. *$\mathcal{X}$ is a strongly Feller chain.*

Proof. Note that the step of the Markov chain represented by P can be broken up into k-substeps, with the i^{th} curve being resampled in the i^{th} substep. It is sufficient if we show that the transition kernel associated to each of these substeps is strongly Feller. Since each substep is a resampling of a Brownian bridge conditioned to avoid upper and lower boundary data defined by $\mathcal{X}_0^{(n)}$ and the previous substep, this follows from Lemma A.12. □

A.7.2. *$\mathcal{X}$ is ψ-irreducible.* By Proposition A.8, we have to verify that there is an open set O such that for every initial condition f, $\mathcal{X}_n$ will lie in O for some n with positive probability. This is implied by the following.

Lemma A.14. *Let $k \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$, $\vec{w}, \vec{z} \in \mathbb{R}^k$, and $g \in \mathcal{C}([a,b],\mathbb{R})$ be given, additionally with $w_k > g(a)$, $z_k > g(b)$, and $w, z \in \mathbb{R}^k_>$ in the zero temperature case. Then there exists $h \in \mathcal{C}^k([a,b],\mathbb{R})$ such that, for every $\delta > 0$ and every valid initial condition $f \in \mathcal{C}^k([a,b],\mathbb{R})$ (i.e., correct end points and, in the zero temperature case, ordered)*

$$\mathbb{P}(\mathcal{X}_n \in B(h,\delta) \text{ for some } n \mid \mathcal{X}_0 = f) > 0.$$

Proof. This is immediate in the positive temperature case: indeed, since in any substep the Markov chain, where an individual curve is resampled, the conditional distribution of that curve given the upper and lower boundaries is still the entire state space, we can fix $h \in \mathcal{C}^k([a,b],\mathbb{R})$ arbitrarily and the Markov chain will lie in $B(h,\delta)$ with positive probability after one step (i.e., k substeps).

This does not work in zero temperature since the non-intersection condition restricts the support of the conditional distribution in each substep. The following is the argument for zero temperature.

Let $\eta > 0$ be such that $\eta < \frac{1}{2}\min_{i=1,\ldots,k}(w_i - w_{i+1}, z_i - z_{i+1})$ and $\eta < \frac{1}{2k}\min_{i=1,2} y_i - g(x_i)$, where we write $w_{k+1} = g(a)$ and $z_{k+1} = g(b)$ (this condition is needed only to ensure non-intersection).

Define $h \in \mathcal{C}^k([a,b],\mathbb{R})$ to be a collection of k curves such that

- $h_j(a) = w_j$, $h_j(b) = z_j$ for $j = 1,\ldots,k$.
- $h_j(\cdot) > h_{j+1}(\cdot) + \eta$ on $[a,b]$ for $j = 1,\ldots,k$ (with $h_{k+1} = g$).
- $h_1(x_i) = y_i$ for $i = 1,2$.

Our definition of η ensures that such a choice exists, in particular that there is enough space for the second and third bullet points to hold.

We may assume without loss of generality that $\delta < \eta$. Thus it follows that $B(h_i,\delta)$ are pairwise disjoint sets for $i = 1,\ldots,k$.

We will now specify a sequence of subsets of states that the Markov chain can occupy with positive probability that will ensure that it lies inside $B(h, \delta)$ at time k. Essentially, we will moved up all the upper curves one-by-one so that the bottom curve has space to come close to h_k. We then iterate this basic procedure of raising the upper curves to make space for the lower ones to adopt the desired locations inductively until, after k steps, each is close to the respective h_i.

First we let $\beta^{(1)} \in \mathcal{C}^k([a,b],\mathbb{R})$ be such that $\max_{1\leq i\leq k} \|\beta^{(1)}(i,\cdot)\|_{L^\infty}, \max_{1\leq i\leq k} \|\beta^{(1)}(i,\cdot)\|_{\frac{1}{4}} \leq M$ which is valid (correct endpoints and ordered) such that

$$\beta^{(1)}(i,\cdot) > \mathcal{X}_0(i,\cdot)$$

for $i = 1,\ldots,k-1$, with $\beta^{(1)}(k,\cdot) = h_k$. Let $\eta^{(1)} > 0$ be such that $\eta^{(1)} < \delta$ and $B(\beta^{(1)}, \eta^{(1)})$ consists of disjoint subsets, which is possible by the above displayed condition on $\beta^{(1)}$. Then we see, by the definition of the transition of the chain and the fact that Brownian bridge occupies any open ball of $\mathcal{C}([a,b],\mathbb{R})$ with positive probability, that

$$\mathbb{P}\left(\mathcal{X}_1 \in B(\beta^{(1)}, \eta^{(1)}) \mid \mathcal{X}_0 = f\right) > 0.$$

In particular, $\mathcal{X}_1(k,\cdot)$ lies within $B(h_k, \delta)$. We will make it so that, at time j, $\mathcal{X}_j(i,\cdot) \in B(h_i,\delta)$ for $i = k-j+1,\ldots,k$. In more detail, let $\beta^{(j)} \in \mathcal{C}^k([a,b],\mathbb{R})$ be valid and such that

$$\beta^{(j)}(i,\cdot) > \beta^{(j-1)}(i,\cdot)$$

for $i = 1,\ldots,k-j$, and with $\beta^{(j)}(i,\cdot) = h_i$ for $i = k-j+1,\ldots,k$. Let $\eta^{(j)} > 0$ be such that $\eta^{(j)} < \delta$ and $B(\beta^{(j)}, \eta^{(j)})$ consists of disjoint subsets. Then it is easy to see by construction and the same reasoning as above that

$$\inf_{\lambda\in B(\beta^{(j-1)},\eta^{(j-1)})} \mathbb{P}\left(\mathcal{X}_j \in B(\beta^{(j)}, \eta^{(j)}) \mid \mathcal{X}_{j-1} = \lambda\right) > 0.$$

Multiplying these inequalities for $j = 1,\ldots,k$ gives that

$$\mathbb{P}\left(\mathcal{X}_k \in B(\beta^{(k)}, \eta^{(k)}) \mid \mathcal{X}_0 = f\right) > 0.$$

Since $\beta^{(k)} = h$ and $\eta^{(k)} < \delta$, we are done. □

A.7.3. *The stationary distribution of $\mathcal{X}$.*

Proposition A.15. *The unique stationary distribution of $\mathcal{X}$ is the distribution of k independent Brownian bridges $B_1,\ldots,B_k : [a,b] \to \mathbb{R}$, with $B_i(a) = w_i$, $B_i(b) = z_i$ reweighted by the Radon-Nikodym derivative (8) with lower boundary curve g and further conditioned on the following positive probability event:*

$$\Big\{B_1(x_i) \in [y_i - \varepsilon, y_i + \varepsilon] \text{ for } i = 1,2\Big\} \cap \Big\{\max_{i=1,\ldots,k} \sup_{x\in[a,b]} |B_i(x)| \leq M, \max_{i=1,\ldots,k} \|B_i\|_{\frac{1}{4}} \leq M\Big\},$$

where $\|f\|_{\frac{1}{4}}$ is the Hölder-$\frac{1}{4}$ norm of f. The analogous statement with non-intersection in place of the Radon-Nikodym derivative holds in the zero temperature case.

Proof. That $\mathcal{X}$ has a unique stationary distribution follows from Theorem A.9 and the fact that the state space is compact. So we only have to prove that the claimed distribution is stationary. We do this for the positive temperature case as the zero temperature case is analogous.

We have to show that if $B_1,\ldots,B_k$ with the distribution stated above are the initial state $\mathcal{X}_0$ for $\mathcal{X}$, the distribution is unchanged on proceeding with one step of the chain. To show this it is sufficient to show two things.

First, the conditional distribution of B_1 conditionally on $B_2, \ldots, B_k$ is that of a Brownian bridge $\tilde{B}$ from (a, w_1) to (b, w_1) reweighted by (8) (with lower boundary data B_2 and $k = 1$) conditioned on $\tilde{B}(x_i) \in [y_i - \varepsilon, y_i + \varepsilon]$ for $i = 1, 2$, on $\sup_{x\in[a,b]} |\tilde{B}(x)| \leq M$, and on $\|\tilde{B}\|_{\frac{1}{4}} \leq M$.

Second, that the conditional distribution of B_j given $B_1, \ldots, B_{j-1}$ and $B_{j+1}, \ldots, B_k$ is that of a Brownian bridge $\tilde{B}$ from (a, w_j) to (b, z_j) reweighted by (8) with lower and upper boundary curves B_{j-1} and B_{j+1} respectively, conditioned on $\sup_{x\in[a,b]} |\tilde{B}(x)| \leq M$, and on $\|\tilde{B}\|_{\frac{1}{4}} \leq M$.

The second proof is along the same lines of the first, and so we only give the details for the first here. Let $\tilde{B}_1, \ldots, \tilde{B}_k : [a, b] \to \mathbb{R}$ be independent Brownian bridges with the appropriate endpoints and reweighted by (8). Let $A_{\varepsilon,M}$ be the event given by

$$A_{\varepsilon,M} := \Big\{\tilde{B}_1(x_i) \in [y_i - \varepsilon, y_i + \varepsilon] \ \forall i = 1, 2, \max_{1\leq i\leq k} \sup_{x\in[a,b]} |\tilde{B}_i(x)| \leq M, \max_{i=1,\ldots,k} \|\tilde{B}_i\|_{\frac{1}{4}} \leq M\Big\}.$$

Let B^{cond} be $\tilde{B}$ conditioned on $A_{\varepsilon,M}$. We have to show that, for any bounded continuous function $F : \mathcal{C}^k([a, b], \mathbb{R}) \to \mathbb{R}$, almost surely,

$$\mathbb{E}\left[F(B_1^{\mathrm{cond}}) \mid B_2^{\mathrm{cond}}, \ldots, B_k^{\mathrm{cond}}\right] = \mathbb{E}_{H_t}^{\vec{x},\vec{y},B_2^{\mathrm{cond}},\varepsilon,M}\left[F(B')\right] =: h_F(B_2^{\mathrm{cond}}), \tag{43}$$

where $\mathbb{E}_{H_t}^{\vec{x},\vec{y},g,\varepsilon,M}$ is the expectation operator associated to the probability measure $\mathbb{P}_{H_t}^{\vec{x},\vec{y},g,\varepsilon,M}$ under which B' is distributed as a Brownian bridge from (a, w_1) to (b, z_1) reweighted by (8) with lower boundary g, on lying inside $[y_i - \varepsilon, y_i + \varepsilon]$ at location x_i for $i = 1, 2$, and on $\sup_{x\in[a,b]} |B'(x)| \leq M, \|B'\|_{\frac{1}{4}} \leq M$.

To show (43), we need to show that, for any event $E \in \sigma(B_2^{\mathrm{cond}}, \ldots, B_k^{\mathrm{cond}})$,

$$\mathbb{E}\left[F(B_1^{\mathrm{cond}})\mathbb{1}_E\right] = \mathbb{E}\left[h_F(B_2^{\mathrm{cond}})\mathbb{1}_E\right] \tag{44}$$

We observe that, by definition,

$$\mathbb{E}\left[F(B_1^{\mathrm{cond}})\mathbb{1}_E\right] = \mathbb{E}\left[F(\tilde{B}_1)\frac{\mathbb{1}_{A_{\varepsilon,M}}}{\mathbb{P}(A_{\varepsilon,M})}\mathbb{1}_E\right].$$

Let $A_{\varepsilon,M}^{\mathrm{top}} = \{\tilde{B}_1(x_i) \in [y_i - \varepsilon, y_i + \varepsilon] \ \forall i = 1, 2, \sup_{x\in[a,b]} |\tilde{B}_1(x)| \leq M, \|\tilde{B}_1\|_{\frac{1}{4}} \leq M\}$ (i.e., imposing the L^∞ and modulus of continuity conditions on only the top curve) and $\mathcal{F}$ be the σ-algebra generated by $\mathbb{1}_{A_{\varepsilon,M}^{\mathrm{top}}}, \tilde{B}_2, \ldots, \tilde{B}_k$. Observe that $A_{\varepsilon,M} \in \mathcal{F}$. Then we see

$$\mathbb{E}\left[F(\tilde{B}_1)\frac{\mathbb{1}_{A_{\varepsilon,M}}}{\mathbb{P}(A_{\varepsilon,M})}\mathbb{1}_E\right] = \mathbb{E}\left[\mathbb{E}\left[F(\tilde{B}_1) \mid \mathcal{F}\right]\frac{\mathbb{1}_{A_{\varepsilon,M}}}{\mathbb{P}(A_{\varepsilon,M})}\mathbb{1}_E\right]. \tag{45}$$

Now, $\mathbb{E}[F(\tilde{B}_1) \mid \mathcal{F}] = g_F(\mathbb{1}_{A_{\varepsilon,M}^{\mathrm{top}}}, \tilde{B}_2, \ldots, \tilde{B}_k)$ for some measurable function g_F.

We focus on the situation where $\mathbb{1}(A_{\varepsilon,M}^{\mathrm{top}}) = 1$. Since $\tilde{B}$ is a collection of (reweighted) Brownian bridges and $\mathbb{1}(A_{\varepsilon,M}^{\mathrm{top}}) = 1$ is a positive probability event involving only the top curve, on this event, by the H_t-Brownian Gibbs property, the conditional distribution of $\tilde{B}_1$ given $\tilde{B}_2, \ldots, \tilde{B}_k$ depends only on $\tilde{B}_2$. So we may write $\mathbb{E}[F(\tilde{B}_1) \mid \mathcal{F}] = g_F(\mathbb{1}_{A_{\varepsilon,M}}, \tilde{B}_2)$; it also follows from the Gibbs property that

$$g_F(1, \tilde{B}_2) = \mathbb{E}_{H_t}^{\vec{x},\vec{y},\tilde{B}_2,\varepsilon,M}[F(B)] = h_F(\tilde{B}_2),$$

where, under $\mathbb{E}_{H_t}^{\vec{x},\vec{y},g,\varepsilon,M}$, B is a Brownian bridge from (a, w_1) to (b, z_1) reweighted by (8) with lower boundary g and conditioned on $B(x_i) \in [y_i - \varepsilon, y_i + \varepsilon]$ for $i = 1, 2$, on $\sup_{[a,b]} B \leq M$, and on $\|B\|_{\frac{1}{4}} \leq M$.

Now since B_2^{cond} is distributed as $\tilde{B}_2$ conditioned on $\mathbb{1}_{A_{\varepsilon,M}} = 1$, we see from (45) and the just concluded discussion that

$$\mathbb{E}\left[F(\tilde{B}_1)\frac{\mathbb{1}_{A_{\varepsilon,M}}}{\mathbb{P}(A_{\varepsilon,M})}\mathbb{1}_E\right] = \mathbb{E}\left[h_F(\tilde{B}_2)\frac{\mathbb{1}_{A_{\varepsilon,M}}}{\mathbb{P}(A_{\varepsilon,M})}\mathbb{1}_E\right] = \mathbb{E}\left[h_F(B_2^{\mathrm{cond}})\mathbb{1}_E\right].$$

This is exactly (44) so the proof is complete. □

A.7.4. *Proofs of Theorems A.2 and A.3.* We start with the proof of Theorem A.2 before turning to the proof of Theorem A.3.

Proof of Theorem A.2. We will give the details for the proof of Theorem A.2 for the positive temperature and Brownian bridge case in detail. The cases of zero temperature and Brownian motion are analogous, and the required changes are mentioned at the end.

Let $\mu_{k,H_t}^{\vec{x},\vec{y},\vec{w},\vec{z},g,\varepsilon,M}$ be the distribution of k independent Brownian bridges, the j^{th} one from (a, w_j) to (b, z_j) reweighted by (8) with lower boundary g, and conditioned on $B(x_i) \in [y_i - \varepsilon, y_i + \varepsilon]$ for $i = 1, 2$, as well as $\max_{1\leq i\leq k}\sup_{x\in[a,b]}|B_i(x)| \leq M$ and $\max_{1\leq i\leq k}\|B_i\|_{\frac{1}{4}} \leq M$. Let $\mu_k^{\vec{x},\vec{y},\vec{w},\vec{z},g,\varepsilon}$ be the same with $M = \infty$, and $\mu_k^{\vec{x},\vec{y},\vec{w},\vec{z},g}$ with $M = \infty$ $\varepsilon = 0$, i.e., with only the conditioning that $B(x_i) = y_i$ for $i = 1, 2$.

It is enough to prove the statement for $\mu^{\uparrow,\varepsilon,M} = \mu_{k,H_t}^{\vec{x},\vec{y}^{\uparrow},\vec{w},\vec{z},g,\varepsilon,M}$ and $\mu^{\downarrow,\varepsilon,M} = \mu_{k,H_t}^{\vec{x},\vec{y}^{\downarrow},\vec{w},\vec{z},g,\varepsilon,M}$ and then take $\varepsilon \to 0$ and $M \to \infty$, appealing to the weak convergence of each measure to their $\varepsilon = 0$, $M = \infty$ versions, which follows from standard arguments used to make sense of zero-probability conditional measures for Brownian objects.

To prove the statement for $\mu^{\uparrow,\varepsilon,M}$ and $\mu^{\downarrow,\varepsilon,M}$, we will show the existence of a coupling for the respective Markov chains which have these measures as their respective stationary measures. In particular, we will show a coupling of the dynamics which will preserve this ordering, assuming the initial conditions are correspondingly ordered. This will show that the monotonicity property holds for the induced coupling of the stationary measures by appealing to Proposition A.10.

We assume $\varepsilon < \frac{1}{2}\min_{i=1,2}(y_i^{\uparrow} - y_i^{\downarrow})$, so that $y_i^{\mathrm{t}} - \varepsilon > y_i^{\downarrow} + \varepsilon$ for each i.

More precisely, we consider the Markov chains $\mathcal{X}^{\uparrow}$ and $\mathcal{X}^{\downarrow}$ with data $\vec{y}^{\uparrow}$ and $\vec{y}^{\downarrow}$ respectively. We assume that the initial conditions $\mathcal{X}_0^{\uparrow}$ and $\mathcal{X}_0^{\downarrow}$ are such that $\mathcal{X}_{0,j}^{\uparrow}(x) \geq \mathcal{X}_{0,j}^{\downarrow}(x)$ for all $x \in [a, b]$ and $1 \leq j \leq k$. We describe the coupling of the transition of the chain from time n to time $n + 1$ next; the coupling will be such that monotonicity holds with probability $1 - o(1)$, where $o(1) \to 0$ as $M \to \infty$, which will suffice as ultimately we take that limit.

We assume that at time n the states $\mathcal{X}_n^{\uparrow}$ and $\mathcal{X}_n^{\downarrow}$ are such that $\mathcal{X}_{n,j}^{\uparrow}(x) \geq \mathcal{X}_{n,j}^{\downarrow}(x)$ for all $x \in [a, b]$ and $1 \leq j \leq k$, and the dynamics will be such that this is preserved. The proof will be complete once we describe these dynamics.

For $j = 1$: for $* \in \{\uparrow, \downarrow\}$, we independently sample a value (Y_1^*, Y_2^*) with the distribution of $(B^*(x_1), B^*(x_2))$ where B^* is a Brownian bridge from (a, w_1) to (b, z_1) conditioned to stay above $\mathcal{X}_{n,2}^*$, satisfy $B^*(x_i) \in [y_i^* - \varepsilon, y_i^* + \varepsilon]$ for $i = 1, 2$, $\sup_{x\in[a,b]} B^*(x) \leq M$, and $\|B^*\|_{\frac{1}{4}} \leq M$.

Clearly $Y_i^{\uparrow} > Y_i^{\downarrow}$ for $i = 1, 2$ by our smallness assumption on ε. Next for each of $* \in \{\uparrow, \downarrow\}$, we sample three Brownian bridges which are monotonically coupled across the two values of $*$. That is, for each $* \in \{\uparrow, \downarrow\}$, we sample three independent Brownian bridges conditioned to lie above $\mathcal{X}_{n,2}^*$, the first from (a, w_1) to (x_1, Y_1^*), the second from (x_1, Y_1^*) to (x_2, Y_2^*), and the third from (x_2, Y_2^*) to (b, z_1), along with the L^{∞} and Hölder-$\frac{1}{4}$ norms being at most M. These bridges are coupled across $* \in \{\uparrow, \downarrow\}$ such that the bridges corresponding to $* =\uparrow$ are higher than the respective ones corresponding to $* =\downarrow$. This can be done using Lemma 3.1, since the boundary datas are ordered, on an event of probability $1 - o(1)$ (to account for the L^{∞} and Hölder norm conditioning), where

$o(1) \to 0$ as $M \to \infty$, since the L^∞ and Hölder norm events have probability $1 - o(1)$ under the marginal measures. The three sampled bridges are concatenated in the obvious way to yield $\mathcal{X}_{n+1,1}$.

For $2 \le j \le k$: We do the analogous monotonically coupled sampling of bridges, but without the conditioning to pass through specified heights and x_1 and x_2. Here we also have an upper boundary condition, but as these are also monotonically ordered by assumption, we may still sample $\mathcal{X}^{\uparrow}_{n,j}$ and $\mathcal{X}^{\downarrow}_{n,j}$ in a monotonically coupled manner.

It is easy to see that the chains have the correct marginal distributions, so we have the desired coupling with probability $1 - o(1)$.

The Brownian motion case of Theorem A.2 follows by considering a similar Markov chain, but with Brownian motions instead of Brownian bridges. We will need the monotonicity statement for reweighted Brownian motions, Lemma A.5, instead of Lemma 3.1. The zero temperature cases follow by considering the zero-temperature analogues of the same chains. This completes the proof of Theorem A.2. ☐

Proof of Theorem A.3. Theorem A.3 on positive association is also an easy consequence of monotonicity (Lemma 3.1) and a similar Markov chain argument. Let $\mathcal{X}$ be the Markov chain used in the above argument, but without the requirement that $\mathcal{X}_n(1, x_i) \in [y_i - \varepsilon, y_i + \varepsilon]$ is maintained for each n and $i = 1, 2$. Similar to the earlier arguments this chain will have as stationary measure $\mu^{\vec{w},\vec{z},g,M}_{k,H_t}$ and will converge to it in an averaged sense as in Proposition A.10. Further, as we saw in the above proof, this measure also satisfies a version of Lemma 3.1 even with the extra conditioning on the sup and Hölder norms being at most M, on an event of probability $1 - o(1)$.

Let A and B be increasing events, and let $\mathcal{X}_{n,i}$ refer to the i^{th} substep of the n^{th} step of the chain. By the definition of the chain, we can write $\mathcal{X}_{n,i} = f_i(\mathcal{X}_{n,i-1}, \mathcal{X}_{n-1}, U)$ for each n and $1 \le i \le k$, with U a uniform random variable on $[0, 1]$ independent of everything else, and f a measurable function. Now, the stochastic monotonicity in boundary data means that f_i is non-decreasing in the first two arguments, and we may assume without loss of generality that it is also non-decreasing in the third argument (by using the well-known distributional representation of any random variable as the generalized inverse CDF, which is non-decreasing, applied to a uniform random variable). Then we see that

$$\begin{aligned}
\mathbb{P}(\mathcal{X}_{n,i} \in A, \mathcal{X}_{n,i} \in B) &= \mathbb{P}\Big(f_i(\mathcal{X}_{n,i-1}, \mathcal{X}_{n-1}, U) \in A, f_i(\mathcal{X}_{n,i-1}, \mathcal{X}_{n-1}, U) \in B\Big) \\
&\ge \mathbb{P}\Big(f_i(\mathcal{X}_{n,i-1}, \mathcal{X}_{n-1}, U) \in A\Big) \cdot \mathbb{P}\Big(f_i(\mathcal{X}_{n,i-1}, \mathcal{X}_{n-1}, U) \in B\Big),
\end{aligned}$$

by the Harris inequality, using the independence of the first two arguments from the third. Now invoking the averaged convergence to equilibrium from Proposition A.10 completes the proof. A similar argument works for the zero temperature and Brownian motion cases, using Lemma A.5 for monotonicity in the Brownian motion cases. ☐

Appendix B. Calculations involving $\mathfrak{h}^t_1$

In this appendix we give the proofs of Proposition 2.6 (the form of the Hamiltonian H_t for the Brownian Gibbs property enjoyed by the scaled narrow-wedge solution $\mathfrak{h}^t$) in Section B.1 and Proposition 3.3 (upper bound on the tail of $\sup_{[-1,1]} \mathfrak{h}^t_1(x) + x^2$ in terms of the one-point tail) in Section B.2.

B.1. **Calculation of the Hamiltonian.** Let the height function $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(t, x)$ be the Cole-Hopf solution to the KPZ equation

$$\partial_t \widetilde{\mathcal{H}} = \frac{1}{2}\partial_x^2 \widetilde{\mathcal{H}} + \frac{1}{2}(\partial_x \widetilde{\mathcal{H}})^2 + \xi,$$

where $\xi : (0,\infty)\times\mathbb{R}\to\mathbb{R}$ is rate 1 space-time white noise. It is known [CH16, Theorem 2.15(ii)] that $\widetilde{\mathcal{H}}(t,\cdot)$ can be embedded as the lowest-indexed curve of a line ensemble satisfies the H-Brownian-Gibbs property with respect to rate *one* Brownian bridges where $H(x) = \exp(x)$. This is a consequence of the (non-trivial) fact that the O'Connell-Yor polymer free energy line ensemble has the same resampling property and converges to the KPZ line ensemble corresponding to $\widetilde{\mathcal{H}}(t,\cdot)$ [CH16, Nic21].

The process $\mathcal{H}$ we consider from (1) can be related to $\widetilde{\mathcal{H}}$ by

$$\mathcal{H}(t,x) = \widetilde{\mathcal{H}}(2t,2x).$$

Next recall that the rescaled height function $\mathfrak{h}^t(x) = \mathfrak{h}(t,x)$ is given by $\mathfrak{h}^t(x) = t^{-1/3}\mathcal{H}(t,t^{2/3}x)$. We can now provide the proof of Proposition 2.6 that $\mathfrak{h}^t$ satisfies the H_t-Brownian Gibbs property with $H_t(x) = 2t^{2/3}\exp(t^{1/3}x)$.

Proof of Proposition 2.6. Our starting point is the Gibbs property enjoyed $\widetilde{\mathcal{H}}(t,\cdot)$ recalled above. We consider the following more abstract setup. Suppose $(\Omega_1,\mathcal{F}_1)$ and $(\Omega_2,\mathcal{F}_2)$ are measure spaces, and let μ,ν be probability measures on Ω_1 with $\mu \ll \nu$ (i.e., μ is absolutely continuous to ν). Suppose $T:\Omega_1\to\Omega_2$ is measurable, and let μ^*, ν^* be the pushforwards under T of the respective measures. Then it is easy to show that $\mu^* \ll \nu^*$ and, for all $x\in\Omega_2$,

$$\frac{\mathrm{d}\mu^*}{\mathrm{d}\nu^*}(x) = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(T^{-1}(x)).$$

In our setting, $\Omega_1 = \mathcal{C}(\llbracket 1,k\rrbracket\times[\ell,r])$; $\Omega_2 = \mathcal{C}(\llbracket 1,k\rrbracket\times[2^{-1}t^{-2/3}\ell, 2^{-1}t^{-2/3}r])$; $(T(g_i))(x) = t^{-1/3}g_i(2t^{2/3}x)$ for all $i\in\llbracket 1,k\rrbracket$, and $x\in[\ell,r]$ and for all $(g_1,\ldots,g_k)\in\Omega_1$; $\mu = \mathbb{P}_H^{k,a,b,\vec{x},\vec{y},f}$ with $H(x) = e^x$ is the conditional law of $(\widetilde{\mathcal{H}}_1(t,\cdot),\ldots,\widetilde{\mathcal{H}}_k(t,\cdot))$ given $\mathcal{F}_{\mathrm{ext}}(k,\ell,r)$, where $t>0$ is fixed; and $\nu = \mathbb{P}_{\mathrm{free}}^{k,a,b,\vec{x},\vec{y}}$ is the law of k independent rate one Brownian bridges on $[\ell,r]$.

By Brownian scaling, ν^* is the law of k independent rate *two* Brownian bridges with appropriately transformed boundary values. By definition μ^* is the law of $(\mathfrak{h}_1^t(\cdot),\ldots,\mathfrak{h}_k^t(\cdot))$ for the same t and conditionally on the analogously T-transformed boundary data.

We have to show that $\mathrm{d}\mu^*/\mathrm{d}\nu^*$ is given by the Radon-Nikodym derivative described by the H_t-Brownian Gibbs property. By Definition 2.2 and the above fact, for any $(g_1,\ldots,g_k)\in\Omega_2$,

$$\begin{aligned}\frac{\mathrm{d}\mu^*}{\mathrm{d}\nu^*}(g_1,\ldots,g_k) &\propto \exp\left\{-\sum_{i=0}^{k}\int_\ell^r H\Big(\big(T^{-1}(g_{i+1})\big)(x) - \big(T^{-1}(g_i)\big)(x)\Big)\,\mathrm{d}x\right\}\\ &= \exp\left\{-\sum_{i=0}^{k}\int_\ell^r H\Big(t^{1/3}g_{i+1}(2^{-1}t^{-2/3}x) - t^{1/3}g_i(2^{-1}t^{-2/3}x)\Big)\,\mathrm{d}x\right\}\\ &= \exp\left\{-\sum_{i=0}^{k}\int_{2^{-1}t^{-2/3}\ell}^{2^{-1}t^{-2/3}r} 2t^{2/3}\cdot H\Big(t^{1/3}(g_{i+1}(y)-g_i(y))\Big)\,\mathrm{d}y\right\}\end{aligned}$$

by making the transformation $y = 2^{-1}t^{-2/3}x$ in the last line. Now $2t^{2/3}H(t^{1/3}x)$ is exactly $H_t(x)$ and the final line is exactly the form of the H_t-Brownian Gibbs property on Ω_2 so the proof is complete. □

B.2. **The proof of Proposition 3.3 via the no big max argument.** Here we prove Proposition 3.3. As mentioned earlier, the proof is a refinement of that of the "no big max" argument given in [Ham22, Proposition 2.27] and [CH14, Proposition 4.4].

Proof of Proposition 3.3. Since $x^2\le 1$ on $[-1,1]$, we may bound $\mathbb{P}(\sup_{x\in[-1,1]}\mathfrak{h}_1^t(x)\ge\theta)$ and replace θ by $\theta-1$ at the end.

Let $\chi = \sup\{x \in [-1,1] : \mathfrak{h}_1^t(x) \geq \theta\}$, with $\sup \varnothing = -\infty$. Then we are trying to bound $\mathbb{P}(\chi \in [-1,1])$; this formulation will help in applying the strong Gibbs property.

Let $x_0, x_2, \ldots, x_N$ be given by $x_i = -1 + \frac{i}{2}\lfloor \theta^{-1} \rfloor$ with $N = 4(\lfloor \theta^{-1} \rfloor)^{-1}$. Consider the event

$$A = \bigcup_{i=0}^{N-1} \left\{ \mathfrak{h}_1^t(x_i) \geq \theta - 1 \right\} \tag{46}$$

and let $E = \{\mathfrak{h}_1^t(-2), \mathfrak{h}_1^t(2) \geq -\theta/2\}$.

We can bound $\mathbb{P}(A)$ in terms of the one-point tail of $\mathfrak{h}_1^t(0)$ using stationarity; so if we can show that $\mathbb{P}(A \mid \chi \in [-1,1])$ is not small, we can use this to show that $\mathbb{P}(\chi \in [-1,1])$ must be small. So we start by lower bounding the probability (for the final term we write $[x_{N-1}, x_N)$ for notational ease, but understand it to be $[x_{N-1}, x_N]$)

$$\begin{aligned}
\mathbb{P}\left(A \;\middle|\; \chi \in [-1,1], E\right) &\geq \sum_{i=0}^{N-1} \mathbb{P}\left(\mathfrak{h}_1^t(x_i) \geq \theta - 1, \chi \in [x_i, x_{i+1}) \;\middle|\; \chi \in [-1,1], E\right) \\
&= \sum_{i=0}^{N-1} \mathbb{P}\left(\mathfrak{h}_1^t(x_i) \geq \theta - 1 \;\middle|\; \chi \in [x_i, x_{i+1}), E\right) \\
&\qquad \times \mathbb{P}\left(\chi \in [x_i, x_{i+1}) \;\middle|\; \chi \in [-1,1], E\right).
\end{aligned}$$

We claim that the right-hand side is lower bounded by the constant $\frac{1}{2}$ for large enough θ. Since $\sum_{i=0}^{N-1} \mathbb{P}\left(\chi \in [x_i, x_{i+1}] \;\middle|\; \chi \in [-1,1], E\right) = 1$, it is sufficient to show that, for some θ_0 and all $\theta > \theta_0$ and $i = 0, \ldots, N$,

$$\mathbb{P}\left(\mathfrak{h}_1^t(x_i) < \theta - 1 \;\middle|\; \chi \in [x_i, x_{i+1}), E\right) \leq \frac{1}{2}. \tag{47}$$

To prove this, we first note that $[-2, \chi]$ is a stopping domain (recall from Definition 2.3), so we may apply the strong H_t-Brownian Gibbs property to it. By the strong H_t-Brownian Gibbs and monotonicity (Lemma 3.1), we see that the previous probability is upper bounded by

$$\mathbb{P}\left(B(x_i) < \theta - 1\right),$$

where B is a Brownian bridge on $[-2, \chi]$ with boundary values $-\theta/2$ and θ. (Note that by monotonicity we have ignored the interaction with the lower curve $\mathfrak{h}_2^t$, and brought down the boundary values as far as possible.) Since $\chi - x_i \leq \frac{1}{2}\theta^{-1}$ on the event that $\chi \in [x_i, x_{i+1}]$ and $\chi \geq -1$ on the same event, we see that

$$\mathbb{E}[B(x_i)] = \frac{\chi - x_i}{\chi + 2}(-\theta/2) + \frac{x_i + 2}{\chi + 2}\theta = \theta - \frac{\chi - x_i}{\chi + 2} \cdot \frac{3\theta}{2} \geq \theta - \frac{3}{4}.$$

Letting $\sigma^2 = \mathrm{Var}(B(x_i))$, we see that

$$\mathbb{P}\left(B(x_i) < \theta - 1\right) \leq \mathbb{P}\left(N\left(\theta - \tfrac{3}{4}, \sigma^2\right) < \theta - 1\right) \leq \frac{1}{2}.$$

What we have shown above is that, with A as in (46),

$$\mathbb{P}\Big(A \mid \chi \in [-1,1], E\Big) \geq \frac{1}{2}.$$

Recall our ultimate goal is to show $\mathbb{P}(\chi \in [-1,1])$ is small. We will break this probability up based on whether E occurs or not. We have an a priori bound on $\mathbb{P}(A)$ in terms of the one-point upper tail (along with a union bound), and the previous display says $\mathbb{P}(A \mid \chi \in [-1,1], E)$ is large, which can be combined to control $\mathbb{P}(\chi \in [-1,1], E)$ using the inequality

$$\mathbb{P}(A) \geq \mathbb{P}(A, \chi \in [-1,1], E) = \mathbb{P}(\chi \in [-1,1], E) \cdot \mathbb{P}(A \mid \chi \in [-1,1], E) \geq \frac{1}{2}\mathbb{P}(\chi \in [-1,1], E).$$

Thus we see that

$$\mathbb{P}\left(\chi \in [-1,1]\right) \le \mathbb{P}(\chi \in [-1,1], E) + \mathbb{P}(\chi \in [-1,1], E^c) \le 2\cdot \mathbb{P}(A) + \mathbb{P}(\chi \in [-1,1])\cdot \mathbb{P}(E^c),$$

the second term bounded in the last inequality by the FKG inequality from Assumption (ii) since both E and $\{\chi \in [-1,1]\} = \{\sup_{x\in[-1,1]} \mathfrak{h}_1^t(x) \ge \theta\}$ are increasing events; note that this is an important step, since we do not have access to quantitative upper bounds on the lower tail to usefully upper bound $\mathbb{P}(\chi \in [-1,1], E^c)$ by $\mathbb{P}(E^c)$ directly. The above argument via FKG allows us to rely on tightness instead.

Indeed, since $\{\mathfrak{h}_1^t(-2), \mathfrak{h}_1^t(2)\}_{t\ge t_0}$ is a tight collection of random variables as we assumed, we may pick θ_0 such that $\theta > \theta_0$ implies $\mathbb{P}(E^c) \le \frac{1}{2}$ for all $t \ge t_0$. Substituting this into the previous display shows that, for $\theta > \theta_0$,

$$\mathbb{P}(\chi \in [-1,1]) \le 4\cdot \mathbb{P}(A) \le 4\theta \cdot \mathbb{P}(\mathfrak{h}_1^t(0) \ge \theta - 1).$$

the final inequality by a union bound. Replacing θ by $\theta - 1$ as mentioned in the beginning of the proof completes the argument. □

Appendix C. Proofs of Brownian estimates

In this appendix we provide the proofs of a number of Brownian estimates from the main paper. We prove them in the following sections:

- In Section C.1, Lemma 3.7 (the tail of the supremum of Brownian bridge over a subinterval)
- In Section C.2, Proposition 3.8 (lower bound on the probability of a Brownian bridge avoiding a parabola with endpoints near the parabola)
- In Section C.3, Proposition 3.14 (same as previous with endpoints along a tangent) and
- In Section C.4, Lemma 8.5 (the probability of a Brownian bridge avoiding a line of lower slope, uniformly in the length of the interval).

C.1. **The tail of the supremum of Brownian bridge over a subinterval.**

Proof of Lemma 3.7. We condition upon $B(\inf J)$ and $B(\sup J)$ to see that

$$\mathbb{P}\left(\sup_{x\in J} B(x) \ge M\sigma_J\right) = \mathbb{E}\left[\mathbb{P}\left(\sup_{x\in J} B(x) \ge M\sigma_J \;\middle|\; B(\inf J), B(\sup J)\right)\right].$$

For $x \in J$, let $X(x) = \mathbb{E}[B(x) \mid B(\inf J), B(\sup J)]$ and $\overline{X} = \max(B(\inf J), B(\sup J))$; note that $X(x) \le \overline{X}$ for all $x \in J$. Then the right-hand side of the previous display is upper bounded by

$$\mathbb{E}\left[\mathbb{P}\left(\sup_{x\in J} B(x) - X(x) \ge M\sigma_J - \overline{X} \;\middle|\; B(\inf J), B(\sup J)\right)\right].$$

We note that, conditionally on $B(\inf J)$ and $B(\sup J)$, $B(x) - X(x)$ is a Brownian bridge on J with boundary values zero. Thus by Lemma 3.6, we have the bound

$$\begin{aligned}\mathbb{P}\left(\sup_{x\in J} B(x) \ge M\sigma_J\right) &\le \mathbb{E}\left[\exp\left\{-\frac{(M\sigma_J - \overline{X})^2}{2\cdot |J|/4}\right\}\mathbb{1}_{\overline{X}\le \frac{1}{2}M\sigma_J}\right] + \mathbb{P}(\overline{X} \ge \tfrac{1}{2}M\sigma_J)\\ &\le \exp\left\{-\frac{2(\frac{1}{2}M\sigma_J)^2}{|J|}\right\} + \mathbb{P}(\overline{X} \ge \tfrac{1}{2}M\sigma_J) \qquad (48)\end{aligned}$$

For the second term, at least one of $B(\inf J)$ and $B(\sup J)$ must exceed $\frac{1}{2}M\sigma_J$; these are mean zero normal variables, and each have variance at most σ_J^2 by definition of σ_J. Thus by a union bound and Lemma 3.5 the second term is at most $2\exp(-M^2/8)$. Next we bound the first term, which amounts to lower bounding $\sigma_J^2/|J|$. We break into two cases for this.

In the first case, the midpoint m_I of I lies inside J. In this case a computation shows that $\sigma_J^2 = \mathrm{Var}(B(m_I)) = |I|/4 \geq |J|/4$, so we obtain a lower bound on $\sigma_J^2/|J|$ of $1/4$.

The second case is when $m_I \notin J$. We may assume $\sup J < m_I$, as the other case of $\inf J > m_I$ is symmetric. Here a computation shows that $\sigma_J^2 = \mathrm{Var} B(\sup J) = (\sup J - \inf I)(\sup I - \sup J)/|I| \geq |J| \times (\sup I - m_I)/|I| = \frac{1}{2}|J|$.

Substituting $\sigma_J^2/|J| \geq \frac{1}{4}$ into the right-hand side of (48) yields that it is upper bounded by $3\exp(-M^2/8)$. This completes the proof of Lemma 3.7. □

C.2. **Lower bound on parabolic avoidance probability.** Here we prove Proposition 3.8. The proof is straightforward but somewhat long. Essentially, we define a fine mesh and consider the probability that the Brownian bridge B remains at least distance 1 above the parabola at all these points; this is calculated using the covariance formulas for Brownian bridge and Gaussian tail bounds. The mesh is chosen fine enough that with high probability B has fluctuations less than $\frac{1}{2}$ on all the intervals between mesh points, and thus avoids the parabola throughout.

Proof of Proposition 3.8. First we note that, by monotonicity (Lemma 3.1) it is enough to prove the statement for the case that endpoints are equal to $(z_1, -z_1^2 + 1)$ and $(z_2, -z_2^2 + 1)$ instead of higher than them.

For an interval $[a, b] \subseteq [z_1, z_2]$, we define $B^{[a,b]} : [a, b] \to \mathbb{R}$ by

$$B^{[a,b]}(x) = B(x) - \frac{x-a}{b-a}B(b) - \frac{b-x}{b-a}B(a), \tag{49}$$

which is distributed as a rate two standard Brownian bridge on $[a, b]$.

For $\varepsilon > 0$ to be specified later (for simplicity we assume $(z_2 - z_1)\varepsilon^{-1}$ is an integer), and for $j = 0, \ldots, (z_2 - z_1)\varepsilon^{-1}$, let $x_j = z_1 + \varepsilon j$. Consider the event

$$\bigcap_{j=1}^{(z_2-z_1)\varepsilon^{-1}-1} \left\{B(x_j) > -x_j^2 + 1\right\} \cap \left\{\inf B^{[x_j,x_{j+1}]} \geq -\frac{1}{2}\right\}. \tag{50}$$

By Lemma C.1 ahead, we see that this event is contained in $\{B(x) > -x^2 \text{ for all } x \in [z_1, z_2]\}$ if $\varepsilon < 2^{1/2}$, and we will indeed ultimately set ε to satisfy this constraint.

Thus we need to lower bound the probability of (50). By properties of Brownian bridge, $B^{[x_j,x_{j+1}]}$ are independent across j and are independent of $B(x_j)$ for all j, and are also identically distributed as rate two standard Brownian bridges on an interval of size ε. Letting

$$p := \mathbb{P}\left(\inf B^{[x_j,x_{j+1}]} \geq -\frac{1}{2}\right) > 1 - \exp\left(-c\varepsilon^{-1}\right)$$

(using Lemma 3.6 with $\sigma_I^2 = \varepsilon/2$), we see that

$$\mathbb{P}\left(\bigcap_{j=1}^{(z_2-z_1)\varepsilon^{-1}-1} \left\{B(x_j) > -x_j^2 + 1\right\} \cap \left\{\inf B^{[x_j,x_{j+1}]} \geq -\frac{1}{2}\right\}\right) \tag{51}$$

$$= p^{(z_2-z_1)\varepsilon^{-1}-1} \cdot \mathbb{P}\left(\bigcap_{j=1}^{(z_2-z_1)\varepsilon^{-1}} \left\{B(x_j) > -x_j^2 + 1\right\}\right). \tag{52}$$

Observe also that $p^{(z_2-z_1)\varepsilon^{-1}} \to 1$ as $\varepsilon \to 0$. Next we lower bound the second factor. We use the property of Brownian bridges that, conditional on $B(x_0)$ for any given x_0, the distribution of B on

$[x_0, z_2]$ is a Brownian bridge from $(x_0, B(x_0))$ to $(z_2, B(z_2))$. This, along with monotonicity of the probabilities of increasing events in the endpoint values, implies that

$$\mathbb{P}\left(\bigcap_{j=1}^{(z_2-z_1)\varepsilon^{-1}-1}\left\{B(x_j) > -x_j^2+1\right\}\right)$$
$$\geq \prod_{j=1}^{(z_2-z_1)\varepsilon^{-1}-1} \mathbb{P}\left(B(x_j) > -x_j^2+1 \;\middle|\; B(x_{j-1}) = -x_{j-1}^2+1\right). \tag{53}$$

Now, the earlier mentioned property of Brownian bridges implies that the distribution of $B(x_j)$ conditional on $B(x_{j-1})$ is a normal distribution with mean μ and variance σ^2, whose values are given by

$$\mu := \frac{z_2 - x_j}{z_2 - x_{j-1}} B(x_{j-1}) + \frac{x_j - x_{j-1}}{z_2 - x_{j-1}} B(z_2) = (1-\lambda_j)B(x_{j-1}) + \lambda_j B(z_2).$$
$$\sigma^2 := 2 \times \frac{(x_j - x_{j-1}) \times (z_2 - x_j)}{z_2 - x_{j-1}} = 2\varepsilon \cdot (1-\lambda_j) \leq 2\varepsilon,$$

where $\lambda_j = \frac{x_j - x_{j-1}}{z_2 - x_{j-1}} = \frac{\varepsilon}{z_2 - z_1 - \varepsilon(j-1)}$, and recalling that we are working with rate two Brownian bridges.

Let $\tilde{\mu}$ be μ with $-x_{j-1}^2+1$ in place of $B(x_{j-1})$. Thus we see

$$\begin{aligned}
&\mathbb{P}\left(B(x_j) > -x_j^2+1 \;\middle|\; B(x_{j-1}) = -x_{j-1}^2+1\right)\\
&\qquad \geq \mathbb{P}\left(N(\tilde{\mu}, \sigma^2) \geq -x_j^2+1\right)\\
&\qquad \geq \frac{1}{4\sqrt{\pi}} \cdot \frac{\varepsilon^{1/2}(1-\lambda_j)^{1/2}}{|-x_j^2+1-\tilde{\mu}|} \exp\left(-\frac{1}{4\varepsilon(1-\lambda_j)}(-x_j^2+1-\tilde{\mu})^2\right),
\end{aligned} \tag{54}$$

where we used Lemma 3.5 for the final inequality, assuming $-x_j^2+1-\tilde{\mu} > (4/3)^{1/2}\sigma$ for now, which we will soon verify. First, we can simplify $-x_j^2+1-\tilde{\mu}$ as

$$\begin{aligned}
-x_j^2+1-\tilde{\mu} &= -\left[(x_{j-1}+\varepsilon)^2 - x_{j-1}^2 + \lambda_j(x_{j-1}^2 - z_2^2)\right]\\
&= -\left[2\varepsilon x_{j-1} + \varepsilon^2 + \lambda_j(x_{j-1}^2 - z_2^2)\right]\\
&= -\left[2\varepsilon(z_1 + \varepsilon(j-1)) - \varepsilon(z_2+z_1) - \varepsilon^2(j-1) + \varepsilon^2\right]\\
&= \varepsilon\left[z_2 - z_1 - \varepsilon j\right].
\end{aligned}$$

Substituting this final expression for $-x_j^2+1-\tilde{\mu}$ into (54) shows that

$$\begin{aligned}
&\mathbb{P}\left(B(x_j) > -x_j^2+1 \;\middle|\; B(x_{j-1}) = -x_{j-1}^2+1\right)\\
&\qquad \geq \frac{1}{4\sqrt{\pi}} \cdot \frac{\varepsilon^{-1/2}(1-\lambda_j)^{1/2}}{z_2 - z_1 - \varepsilon j} \exp\left(-\frac{\varepsilon}{4(1-\lambda_j)}(z_2 - z_1 - \varepsilon j)^2\right)\\
&\qquad = \frac{1}{4\sqrt{\pi}} \cdot \frac{\varepsilon^{-1/2}(1-\lambda_j)^{1/2}}{z_2 - z_1 - \varepsilon j} \exp\left(-\frac{\varepsilon}{4}(z_2 - z_1 - \varepsilon j)(z_2 - z_1 - \varepsilon(j-1))\right),
\end{aligned}$$

where for the first inequality we assumed that $\varepsilon(z_2 - z_1 - \varepsilon j) \geq (4/3)^{1/2}\sigma = (8/3)^{1/2}\varepsilon^{1/2}(1-\lambda_j)^{1/2}$. We have to verify this inequality holds. Recalling that $1-\lambda_j = (z_2-z_1-\varepsilon j)/(z_2-z_1-\varepsilon(j-1))$ and squaring both sides reduces it to showing that $\varepsilon(z_2-z_1-\varepsilon j)(z_2-z_1-\varepsilon(j-1))^2 \geq 8/3$. Substituting $j \leq (z_2-z_1)\varepsilon^{-1} - 1$ on the left side, it reduces to $2\varepsilon^3 \geq 8/3$, which is equivalent to $\varepsilon \geq (4/3)^{1/3}$. We will now set ε to satisfy this inequality. Recall we previously also assumed $\varepsilon < 2^{1/2}$, and note that $(4/3)^{1/3} < 2^{1/2}$. We now set ε to be such that both the upper and lower bounds are satisfied (eg. $\varepsilon = 6/5$).

By substituting in the same expression for $1-\lambda_j$ and using $j \geq 1$ we also see that the factor in front of the exponential in the previous display is bounded below by $c\varepsilon^{-1/2}(z_2-z_1)^{-1}$ some $c > 0$. Substituting the previous bound into (53) and then using (52) shows that

$$\begin{aligned}
&\mathbb{P}\Big(B(x) > -x^2 \quad \forall x \in [z_1, z_2]\Big)\\
&\geq p^{(z_2-z_1)\varepsilon^{-1}}\\
&\quad\times \exp\left\{-\frac{\varepsilon}{4}\sum_{j=1}^{(z_2-z_1)\varepsilon^{-1}-1}(z_2-z_1-\varepsilon j)(z_2-z_1-\varepsilon(j-1)) - 2(z_2-z_1)\varepsilon^{-1}\log[\varepsilon(z_2-z_1)]\right\},
\end{aligned}$$

implicitly absorbing the term $-(z_2-z_1)\varepsilon^{-1}\log c^{-1}$ into the $(z_2-z_1)\varepsilon^{-1}\log(\varepsilon(z_2-z_1))$ term for all large enough $z_2 - z_1$ by raising the latter's coefficient. Expanding yields sums of powers of j, and using standard formulas shows that

$$\varepsilon \sum_{j=1}^{(z_2-z_1)\varepsilon^{-1}-1}(z_2-z_1-\varepsilon j)(z_2-z_1-\varepsilon(j-1)) = \frac{(z_2-z_1)^3}{3} - \varepsilon^2\frac{(z_2-z_1)}{3} \leq \frac{(z_2-z_1)^3}{3}.$$

Thus, since $\varepsilon = \frac{6}{5}$ and $p > 0$ uniformly, we obtain

$$\begin{aligned}
\mathbb{P}\left(B(x) > -x^2 \quad \forall x \in [z_1, z_2]\right) &\geq \exp\left\{-\frac{(z_2-z_1)^3}{12} - (z_2-z_1)\varepsilon^{-1}\left(2\log[\varepsilon(z_2-z_1)] + \log p^{-1}\right)\right\}\\
&\geq \exp\left\{-\frac{(z_2-z_1)^3}{12} - 2(z_2-z_1)\log(z_2-z_1)\right\}.
\end{aligned}$$

for $z_2 - z_1$ sufficiently large. This completes the proof. □

Lemma C.1. *If $\varepsilon < 2^{1/2}$, then (50) is contained in $\{B(x) > -x^2 \text{ for all } x \in [z_1, z_2]\}$.*

Proof. From (49), it is sufficient to verify that, for every $j = 1, \ldots, (z_2-z_1)\varepsilon^{-1}-1$ and $x \in [x_j, x_{j+1}]$,

$$-\frac{1}{2} + \frac{x-x_j}{x_{j+1}-x_j}(-x_{j+1}^2) + \frac{x_{j+1}-x}{x_{j+1}-x_j}(-x_j^2) + 1 > -x^2,$$

which we do now. Letting $x = x_j + \varepsilon y$ (so that $y \in [0,1]$), using that $x_{j+1} - x_j = \varepsilon$, and multiplying throughout by -1, the inequality can be simplified to

$$-\frac{1}{2} + yx_{j+1}^2 + (1-y)x_j^2 < x_j^2 + \varepsilon^2 y^2 + 2\varepsilon x_j y.$$

Cancelling x_j^2 gives

$$-\frac{1}{2} + y(x_{j+1}^2 - x_j^2) < \varepsilon^2 y^2 + 2\varepsilon x_j y.$$

Since $x_{j+1}^2 - x_j^2 = (x_{j+1}-x_j)(x_{j+1}+x_j) = \varepsilon(2x_j+\varepsilon)$, we can further simplify to

$$\varepsilon^2 y(1-y) < \frac{1}{2}.$$

Noting that $y(1-y) \leq \frac{1}{4}$, we see that the previous inequality is satisfied under our hypothesis on ε. □

C.3. **Parabola avoidance probability along a tangent.**

Proof of Proposition 3.14. We assume without loss of generality that $x_{\tan} \le \frac{\inf I + \sup I}{2}$.
Let $p(x) = -x^2$ and let

$$\mathsf{NonInt}_{\theta,M} = \{B(x) > p(x) + \varepsilon M \sigma_{\tan}\ \forall x \in I\}.$$

Observe that the probability in (9) is lower bounded by $\mathbb{P}(\mathsf{NonInt}_{\theta,M})$, and that $p(x) \le \ell^{\tan}(x)$ for all $x \in \mathbb{R}$.

We see that $\mathbb{P}\big(\mathsf{NonInt}_{\theta,M}\big) = \mathbb{P}\left(\inf_{x\in I} B(x) - p(x) \ge \varepsilon M\sigma_{\tan}\right)$. Now, if we define $\tilde{B} : [0,1] \to \mathbb{R}$ by

$$\tilde{B}(x) = |I|^{-1/2}\Big[B(\inf I + |I|x) - \ell^{\tan}(\inf I + |I|x)\Big],$$

then $\tilde{B}$ is a rate two Brownian bridge on $[0,1]$ with $\tilde{B}(0) = \tilde{B}(1) = 0$. This means that

$$\mathbb{P}\big(\mathsf{NonInt}_{\theta,M}\big) = \mathbb{P}\left(\inf_{x\in[0,1]} |I|^{1/2}\tilde{B}(x) + \ell^{\tan}(\inf I + |I|x) - p(\inf I + |I|x) \ge \varepsilon M\sigma_{\tan}\right).$$

For notational convenience, define $\tilde{x}_{\tan}$ via $x_{\tan} = \inf I + |I|\tilde{x}_{\tan}$ and $\tilde{\sigma}_{\tan} = |I|^{-1/2}\sigma_{\tan}$; note that $\tilde{x}_{\tan} \in [0, 1/2]$ by the assumption on $x_{\tan}$ we made at the beginning of the proof.

Since $\ell^{\tan}$ is tangent to $p(x)$ at $x_{\tan}$, it is an easy calculation that $\ell^{\tan}(w) - p(w) = (w - x_{\tan})^2$ for any w, so the previous displayed probability is equal to

$$\begin{aligned}\mathbb{P}&\left(\inf_{x\in[0,1]} \tilde{B}(x) + |I|^{-1/2}(\inf I + |I|x - x_{\tan})^2 \ge \frac{1}{2}\varepsilon M\tilde{\sigma}_{\tan}\right)\\ &\qquad = \mathbb{P}\left(\inf_{x\in[0,1]} \tilde{B}(x) + |I|^{3/2}(x - \tilde{x}_{\tan})^2 \ge \frac{1}{2}\varepsilon M\tilde{\sigma}_{\tan}\right)\end{aligned}$$

To lower bound this probability we break up $[0,1]$ into three intervals given by $I_1 = [0, \frac{1}{2}\tilde{x}_{\tan}]$, $I_2 = [\frac{1}{2}\tilde{x}_{\tan}, 2\tilde{x}_{\tan}]$, and $I_3 = [2\tilde{x}_{\tan}, 1]$, and use the positive association property of Brownian bridges. The previous display is lower bounded by

$$\prod_{i=1}^{3} \mathbb{P}\left(\inf_{x\in I_i} \tilde{B}(x) + |I|^{3/2}(x - \tilde{x}_{\tan})^2 \ge \frac{1}{2}\varepsilon M\tilde{\sigma}_{\tan}\right). \tag{55}$$

We will show that the $i = 2$ factor is lower bounded by $\exp(-c\varepsilon^2 M^2)$, and the $i = 1$ and 3 factors by a constant factor independent of M and $|I|$.

For $i = 1$, we will reduce the calculation to that of lower bounding the probability that a Brownian bridge on an interval of size δ, with starting and ending points at height at least a constant times $\delta^{1/2}$, stays above $-c\delta^{1/2}$, where $c > 0$ is a constant and $\delta = \frac{1}{2}\tilde{x}_{\tan}$; this probability is of course uniformly positive. Indeed, we can lower bound $(x - \tilde{x}_{\tan})^2$ on I_1 by $\tilde{x}_{\tan}^2/4$, and we note that

$$\frac{1}{2}\varepsilon M\tilde{\sigma}_{\tan} < \frac{1}{8}|I|^{3/2}\tilde{x}_{\tan}^2 \tag{56}$$

by our assumption that $M \le C^{-1}(x_{\tan} - \inf I)^{3/2} \le C^{-1}|I|^{3/2}$ for a C to be specified, since $\tilde{\sigma}_{\tan} \le \tilde{x}_{\tan}^{1/2}$ and

$$\frac{|I|^{3/2}\tilde{x}_{\tan}^2}{x_{\tan}^{1/2}} = |I|^{3/2}\frac{(x_{\tan} - \inf I)^{3/2}}{|I|^{3/2}} = (x_{\tan} - \inf I)^{3/2},$$

using the definition of $\tilde{x}_{\tan}$ (to be explicit, we take $C^{-1} = \frac{1}{4}\varepsilon^{-1}$, where $\varepsilon > 0$ is a constant still to be set). So by Lemma 3.7 for the second inequality,

$$\mathbb{P}\left(\inf_{x\in I_1} \tilde{B}(x) + |I|^{3/2}(x - \tilde{x}_{\tan})^2 \ge \frac{1}{2}\varepsilon M\tilde{\sigma}_{\tan}\right) \ge \mathbb{P}\left(\inf_{x\in I_1} \tilde{B}(x) \ge -\frac{1}{8}|I|^{3/2}\tilde{x}_{\tan}^2\right)$$

$$\geq 1 - \exp\left(-c' \frac{|I|^3 \tilde{x}_{\tan}^4}{\tilde{x}_{\tan}}\right),$$

the second inequality also using that $\mathrm{Var}(\tilde{B}(x)) \leq x$ for all $x \in [0,1]$. The final expression is strictly positive independent of $|I|$ as by hypothesis $\tilde{x}_{\tan} \geq |I|^{-1}$.

For $i = 2$ in (55) we lower bound the parabolic term by 0 and consider the event that the values of $\tilde{B}$ at the endpoints of I_2 are at height $\varepsilon M \tilde{\sigma}_{\tan}$; so the $i = 2$ term is lower bounded by

$$\mathbb{P}\left(\inf_{I_2} \tilde{B} > \frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan} \;\middle|\; \tilde{B}(\tfrac{1}{2}\tilde{x}_{\tan}), \tilde{B}(2\tilde{x}_{\tan}) \geq \varepsilon M \tilde{\sigma}_{\tan}\right) \cdot \mathbb{P}\left(\tilde{B}(\tfrac{1}{2}\tilde{x}_{\tan}), \tilde{B}(2\tilde{x}_{\tan}) \geq \varepsilon M \tilde{\sigma}_{\tan}\right).$$

The first probability is at least that of a Brownian bridge on an interval of size $|I_2| = \frac{3}{2}\tilde{x}_{\tan}$ with zero endpoints staying above $-\frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan}$, which is $1 - \exp(-2 \times \frac{1}{4}\varepsilon^2 M^2 \tilde{\sigma}_{\tan}^2 / (\frac{3}{2}\tilde{x}_{\tan}))$ by Lemma 3.6. Since $\tilde{\sigma}_{\tan}^2 \geq \tilde{x}_{\tan}/2$ (by our assumption that $\tilde{x}_{\tan} \leq \frac{1}{2}$), the expression in the previous sentence is at least $1 - \exp(-\varepsilon^2 M^2/6)$.

Turning to the second probability in the previous display, using the positive association of $\tilde{B}(\frac{1}{2}\tilde{x}_{\tan})$ and $\tilde{B}(2\tilde{x}_{\tan})$, the fact that they have variances $2 \times \frac{1}{2}\tilde{x}_{\tan}(1 - \frac{1}{2}\tilde{x}_{\tan}) \leq \tilde{x}_{\tan}$ and $2 \times 2\tilde{x}_{\tan}(1 - 2\tilde{x}_{\tan}) \leq 4\tilde{x}_{\tan}$ respectively, and the lower bound on Gaussian tails from Lemma 3.5, we obtain that

$$\begin{aligned}
\mathbb{P}\left(\tilde{B}\left(\tfrac{1}{2}\tilde{x}_{\tan}\right), \tilde{B}\left(2\tilde{x}_{\tan}\right) \geq \varepsilon M \tilde{\sigma}_{\tan}\right) &\geq \left[\frac{1}{\sqrt{8\pi \tilde{x}_{\tan}}} \exp\left(-\frac{(\varepsilon M \tilde{\sigma}_{\tan})^2}{2\tilde{x}_{\tan}(1 - \frac{1}{2}\tilde{x}_{\tan})}\right)\right]^2 \\
&\geq (8\pi)^{-1} \exp\left(-2\varepsilon^2 M^2\right),
\end{aligned}$$

the last inequality since $\tilde{x}_{\tan} \leq 1$ and $\tilde{\sigma}_{\tan}^2 \leq \tilde{x}_{\tan}$.

Finally we handle the $i = 3$ term in the product in (55). Here we will analyse the fluctuations on dyadic scales, which is needed because the interval is of unit order and the increasing fluctuations need to be balanced against the increasing decay of the parabola.

Let k_0 be the smallest k such that $2^{k+1}\tilde{x}_{\tan} \geq 1$. For simplicity of notation let us interpret $2^{k_0+1}\tilde{x}_{\tan}$ as 1 in the following. Making use of the positive association of $\tilde{B}$, we see that

$$\begin{aligned}
&\mathbb{P}\left(\inf_{x \in I_3} \tilde{B}(x) + |I|^{3/2}(x - \tilde{x}_{\tan})^2 \geq \frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan}\right) \\
&\qquad \geq \prod_{k=1}^{k_0} \mathbb{P}\left(\inf_{x \in [2^k \tilde{x}_{\tan}, 2^{k+1}\tilde{x}_{\tan}]} \tilde{B}(x) + |I|^{3/2}(x - \tilde{x}_{\tan})^2 > \frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan}\right) \\
&\qquad \geq \prod_{k=1}^{k_0} \mathbb{P}\left(\inf_{x \in [2^k \tilde{x}_{\tan}, 2^{k+1}\tilde{x}_{\tan}]} \tilde{B}(x) > -|I|^{3/2}(2^k - 1)^2 \tilde{x}_{\tan}^2 + \frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan}\right).
\end{aligned} \tag{57}$$

Now since $\frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan} < \frac{1}{8}|I|^{3/2}\tilde{x}_{\tan}^2$ by (56),

$$\frac{1}{2}\varepsilon M \tilde{\sigma}_{\tan} - |I|^{3/2}(2^k - 1)^2 \tilde{x}_{\tan}^2 \leq -\frac{1}{2}|I|^{3/2}(2^k - 1)^2 \tilde{x}_{\tan}^2,$$

so the k^{th} term in the product in (57) is lower bounded by

$$\begin{aligned}
\mathbb{P}\left(\inf_{x \in [2^k \tilde{x}_{\tan}, 2^{k+1}\tilde{x}_{\tan}]} \tilde{B}(x) > -\frac{1}{2}|I|^{3/2}(2^k - 1)^2 \tilde{x}_{\tan}^2\right) &\geq 1 - \exp\left(-c' \frac{|I|^3 (2^k - 1)^4 \tilde{x}_{\tan}^4}{2^k \tilde{x}_{\tan}}\right) \\
&\geq 1 - \exp\left(-c' 2^{3k-4}\right)
\end{aligned}$$

for an absolute constant $c' > 0$ by Lemma 3.7, using that $\mathrm{Var}(\tilde{B}(x)) \leq x$ for all $x \in [0,1]$. Substituting the previous display into (57) shows that the latter expression is lower bounded by a constant.

Overall we have shown that, for some $c > 0$,

$$\mathbb{P}(\mathsf{NonInt}_{\theta,M}) \geq c \cdot \exp(-2\varepsilon^2 M^2).$$

Thus, setting $\varepsilon = \frac{1}{5}$, we obtain that, for $0 < M < C^{-1}(x_{\tan} - \inf I)^{3/2}$,

$$\mathbb{P}\left(\mathsf{NonInt}_{\theta,M}\right) \geq c\exp(-\tfrac{2}{25}M^2).$$

This completes the proof of the first (zero-temperature) part of Proposition 3.14.

To obtain a similar lower bound on Z_{H_t}, we make use of Lemma 3.11 with $[z_1, z_2] = I$, $x = y_{\inf I}$, $y = y_{\sup I}$, $p(x) = -x^2 + \varepsilon M\sigma_{\tan}$, and $g(x) = \ell^{\tan}(x) + x^2 = (x - x_{\tan})^2$, which is non-negative. This yields, with the above lower bound on $\mathbb{P}(\mathsf{NonInt}_{\theta,M})$,

$$\begin{aligned} Z_{H_t} &\geq \exp\left[-2t^{2/3}e^{-t^{1/6}}\int_I \exp\left(-\frac{t^{1/3}}{2}(u - x_{\tan})^2\right)\,\mathrm{d}u\right] \cdot e^{-c'\varepsilon^2M^2} \\ &\geq \exp\left[-2t^{2/3}e^{-t^{1/6}}\int_{-\infty}^{\infty} \exp\left(-\frac{t^{1/3}}{2}(u - x_{\tan})^2\right)\,\mathrm{d}u\right] \cdot e^{-c'\varepsilon^2M^2} \\ &= \exp\left[-2t^{2/3}e^{-t^{1/6}} \cdot \sqrt{2\pi}t^{-1/6}\right] \cdot e^{-c'\varepsilon^2M^2}. \end{aligned}$$

The proof is complete by noting that $t^{1/2}\exp(-t^{1/6})$ is upper bounded by a uniform constant for all $t > 0$, and by observing that we can set ε independent of t so that $c'\varepsilon^2 = \frac{2}{25}$. □

C.4. **Avoiding a line of more extreme slope.**

Proof of Lemma 8.5. We may assume $\eta = 1$ without loss of generality. Indeed, if $\eta > 1$, then

$$\mathbb{P}\left(\inf_{x\in[0,r]} B^K(x) < -\eta K\right) \leq \mathbb{P}\left(\inf_{x\in[0,r]} B^K(x) < -K\right).$$

If $\eta < 1$, then B^K stochastically dominates $B^{\eta K}$, and so

$$\mathbb{P}\left(\inf_{x\in[0,r]} B^K(x) < -\eta K\right) \leq \mathbb{P}\left(\inf_{x\in[0,r]} B^{\eta K}(x) < -\eta K\right).$$

So we are in the case $\eta = 1$ and $K \geq \frac{1}{2}$. Let B be a rate two Brownian bridge between $(0, 0)$ and $(r, 0)$, where we will assume r is an integer for notational convenience. We also assume $r \geq 2$ as the claim is trivial for $r \in [0, 2]$ (indeed, for any fixed bounded interval). We may define B^K as $B^K(x) = B(x) + Kx$. Now, the right-hand side of the previous display can be upper bounded by

$$\begin{aligned} \sum_{j=1}^{r} \mathbb{P}\left(\inf_{x\in[j-1,j]} B(x) + Kx < -K\right) &\leq \sum_{j=1}^{r} \mathbb{P}\left(\inf_{x\in[j-1,j]} B(x) < -Kj\right) \\ &\leq 2 \cdot \sum_{j=1}^{\lceil r/2\rceil} \mathbb{P}\left(\inf_{x\in[j-1,j]} B(x) < -Kj\right), \end{aligned} \tag{58}$$

the last line by the equality in distribution between $x \mapsto B(x)$ and $x \mapsto B(r - x)$. Let $\sigma_j = \max_{x\in[j-1,j]} \mathrm{Var}(B(x)) = \mathrm{Var}(B(j)) = 2j(r-j)/r$, the second equality as we have assumed $j \leq \lceil r/2\rceil$ (this may fail for $j = \lceil r/2\rceil$, but it is easy to see that this case can be handled separately, as we will have $\sigma_{\lceil r/2\rceil} = O(r)$). Now by Lemma 3.7,

$$\mathbb{P}\left(\inf_{x\in[j-1,j]} B(x) < -Kj\right) \leq 3\exp\left(-\frac{K^2j^2}{8\sigma_j^2}\right) \leq 3\exp\left(-\frac{K^2jr}{8(r-j)}\right) \leq 3\exp\left(-\frac{1}{8}K^2j\right),$$

as $r/(r-j) \geq r/(r-1) \geq 1$ since $j \geq 1$. Putting this bound back in (58) yields that, for $K \geq \frac{1}{2}$,

$$\mathbb{P}\left(\inf_{x\in[0,r]} B^K(x) < -K\right) \leq C\exp\left(-\frac{1}{8}K^2\right). \qquad \square$$

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