LOCAL AND GLOBAL COMPARISONS OF THE AIRY DIFFERENCE PROFILE TO BROWNIAN LOCAL TIME

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ABSTRACT. There has recently been much activity within the Kardar-Parisi-Zhang universality class spurred by the construction of the canonical limiting object, the parabolic Airy sheet $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$ [DOV18]. The parabolic Airy sheet provides a coupling of parabolic Airy$_2$ processes—a universal limiting geodesic weight profile in planar last passage percolation models—and a natural goal is to understand this coupling. Geodesic geometry suggests that the difference of two parabolic Airy$_2$ processes, i.e., a difference profile, encodes important structural information. This difference profile $\mathcal{D}$, given by $\mathbb{R} \to \mathbb{R} : x \mapsto \mathcal{S}(1, x) - \mathcal{S}(-1, x)$, was first studied by Basu, Ganguly, and Hammond [BGH19a], who showed that it is monotone and almost everywhere constant, with its points of non-constancy forming a set of Hausdorff dimension $1/2$. Noticing that this is also the Hausdorff dimension of the zero set of Brownian motion leads to the question: is there a connection between $\mathcal{D}$ and Brownian local time? Establishing that there is indeed a connection, we prove two results. On a global scale, we show that $\mathcal{D}$ can be written as a *Brownian local time patchwork quilt*, i.e., as a concatenation of random restrictions of functions which are each absolutely continuous to Brownian local time (of rate four) away from the origin. On a local scale, we explicitly obtain Brownian local time of rate four as a local limit of $\mathcal{D}$ at a point of increase, picked by a number of methods, including at a typical point sampled according to the distribution function $\mathcal{D}$. Our arguments rely on the representation of $\mathcal{S}$ in terms of a last passage problem through the parabolic Airy line ensemble and an understanding of geodesic geometry at deterministic and random times.

CONTENTS

1. INTRODUCTION AND MAIN RESULT

The Kardar-Parisi-Zhang (KPZ) universality class refers to a broad family of models of one-dimensional random growth which are believed to exhibit certain common features, such as universal scaling exponents and limiting distributions. A plethora of models are believed to lie in this class, including asymmetric exclusion processes, first passage percolation, directed polymers in random environment, and the stochastic PDE known as the KPZ equation. Nonetheless, in spite of the breadth of models thought to lie in the class, only a handful have been rigorously shown to do so.

An important subclass of models believed to be members of the KPZ universality class is known as last passage percolation (LPP). While the microscopic details depend on the specification, in general terms they all consist of an environment of random noise through which *directed* paths travel, accruing the integral of the noise along it—a quantity known as energy or weight. Typically, two endpoints in the environment are fixed, and a maximization is done over the weights of all paths with these endpoints. A path which achieves the *maximum* weight is called a *geodesic*, although, traditionally, the latter is used to denote *shortest* paths in a metric space.

To facilitate our discussion here without getting into technical definitions, we imagine a limiting last passage percolation model defined on $\mathbb{R} \times [0,1]$, i.e., an infinite strip with height one; the height can be thought of as a *time* parameter. (The purpose of this imagined model is to explain the interpretations of certain limiting objects we will introduce, and is not meant to be one which is rigorously defined.) The precise distribution of the noise is not important for our expository purposes, but it can be thought of as translationally invariant and independent. We consider paths $\gamma : [0,1] \to \mathbb{R}$, where $\gamma(t)$ denotes the position of the path at time t; the directedness constraint on paths is implemented by the requirement that they are functions, and so cannot "backtrack" and have multiple values at any height, i.e., time. The weight of a given path can be thought of as the integral of the noise along the path in some sense. The last passage value from $(y,0)$ to $(x,1)$ is the maximum weight over all paths γ with $\gamma(0) = y$ and $\gamma(1) = x$.

A canonical limiting object in the KPZ universality class is known as the parabolic Airy$_2$ process, $\mathcal{P}_1 : \mathbb{R} \to \mathbb{R}$ [PS02]. (The subscript of 1 is due to $\mathcal{P}_1$ being the first in a family of random curves, the parabolic Airy line ensemble, which plays a central role in this paper.) In terms of our last passage percolation model, $\mathcal{P}_1(x)$ should be thought of as encoding the weight of the geodesic from $(0,0)$ to $(x,1)$. The term "parabolic" is included in the name because $\mathcal{P}_1$ is obtained by subtracting the parabola x^2 from the Airy$_2$ process, which is stationary.

Note that while the endpoint x is allowed to adopt any real value, the starting point is fixed to be zero in the interpretation of $\mathcal{P}_1(x)$. In our LPP model, of course, any starting point is also allowed (i.e., $\gamma(0)$ can take any real value), but the joint distributions of geodesic weights with differing starting and ending points is not captured in the one-dimensional object $\mathcal{P}_1$.

There is a richer universal object that encodes this larger class of joint distributions, providing a joint coupling of the geodesic weights as the endpoints are allowed to vary arbitrarily, known as the parabolic Airy sheet $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$. While first conjectured to exist in [CQR15], this was only recently proved in the important work [DOV18] (with assistance from [DV21]). The value of the parabolic Airy sheet $\mathcal{S}(y,x)$ should be thought of as the weight of the geodesic from $(y,0)$ to $(x,1)$.

The parabolic Airy sheet admits invariance under appropriate scalings guided by certain critical exponents, much like Brownian motion. Hence, as in the case of the latter, $\mathcal{S}$ is expected to exhibit various fractal or self-similar properties.

The inquiry into such aspects, including the existence of random fractal sets and their fractal dimensions, is indeed a well-established theme in probability theory and statistical mechanics. Particularly important examples include: (i) the zero set of Brownian motion (which has an important connection to our methods that we will return to), (ii) the set of exceptional times in dynamical critical planar percolation where an infinite cluster is present, which is proven to have Hausdorff dimension $\frac{31}{36}$ [GPS10] on the honeycomb lattice (and conjectured to have the same on the Euclidean lattice), and (iii) the study of Schramm-Loewner evolutions in connection to scaling limits of interfaces at criticality in various statistical mechanics models; here too Hausdorff dimensions are known, in this case of the curves themselves [RS05, Bef08].

The above has naturally led to the study of random fractal geometry within KPZ, which is still at a rather nascent stage, notwithstanding some important recent advances. The first such work is [BGH19a]. In deriving our main results, we will illuminate certain connections mooted by [BGH19a] and offer an alternative proof of the main result of that work. To make things more precise, let us define the object of study.

Fix $y_a, y_b \in \mathbb{R}$ with $y_b > y_a$. We consider the random function $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ given by

$$\mathcal{D}(x) = \mathcal{S}(y_b, x) - \mathcal{S}(y_a, x). \tag{1}$$

We will call $\mathcal{D}$ a *weight difference profile*; it encodes the difference in the weight of two geodesics with differing but fixed starting points and a common ending point as the latter varies. A first fact about $\mathcal{D}$ sets the stage for our work.

Lemma 1.1. *$\mathcal{D}$ is a continuous non-decreasing function almost surely.*

This has been proved several times in the literature, for example [DOV18, Lemma 9.1] or [BGH19a, Theorem 1.1(2)], but we will include the simple proof ahead in Section 2 for completeness. Lemma 1.1 implies that at any given point x, $\mathcal{D}$ is either constant in a neighbourhood of x or increasing at x on at least one side. The latter is defined precisely as follows.

Definition 1.2. A point $x \in \mathbb{R}$ is a *non-constant* point of a continuous function $f : \mathbb{R} \to \mathbb{R}$ if there does not exist any $\varepsilon > 0$ such that $f(y) = f(x)$ for all $y \in (x - \varepsilon, x + \varepsilon)$. The set of non-constant points of f is denoted by $\mathsf{NC}(f)$.

It turns out that $\mathcal{D}$ is almost everywhere constant, with probability one. Further, it is an easy argument that the non-constant points of a continuous non-decreasing function must form a perfect set, i.e., be closed and have no isolated points. Perfect sets must necessarily be uncountable. In particular, $\mathcal{D}$ can be interpreted as the distribution function of a random measure supported on the uncountable set $\mathcal{D}$. Canonical examples of a similar nature include the Cantor function or Brownian local time. Associated to any such set is its fractal dimension, which quantifies how "sparse" the set is. With the dimension of the mentioned two examples being classically known, one is led to ask: what is the fractal dimension of the set of non-constant points of $\mathcal{D}$?

This question was originally raised and answered in [BGH19a], yielding the following theorem, where the notion of fractal dimension adopted is the Hausdorff dimension. (The definition of the Hausdorff dimension of a set is recalled ahead in Definition 4.3 for the reader's convenience.)

Theorem 1 (Theorem 1.1 of [BGH19a]). *The Hausdorff dimension of $\mathsf{NC}(\mathcal{D})$ is equal to one-half almost surely.*

It is well-known that the set of non-constant points of Brownian local time (to be denoted by $\mathcal{L}$ henceforth), i.e, the set of zeroes of Brownian motion, also has Hausdorff dimension one-half, which might make one wonder if there is a stronger connection between the latter and $\mathsf{NC}(\mathcal{D})$. Indeed, this question was posed by Manjunath Krishnapur to the first author after a talk on [BGH19a] in 2019.

The aim of the present article is to illustrate that there is indeed a connection, and to develop it in a few forms. As one consequence, we obtain a second, shorter proof of Theorem 1.

Now, one can enquire about a comparison to $\mathcal{L}$ globally as well as locally. We obtain results for both scales and start by discussing the former.

1.1. A Brownian local time comparison on a global scale.

Comparisons of objects arising in KPZ to Brownian counterparts are by now a well-established theme. One form of comparison typically involves absolute continuity of the relevant objects to each other. For example, for the parabolic Airy$_2$ process $\mathcal{P}_1$, [CH14] proved that $x \mapsto \mathcal{P}_1(x) - \mathcal{P}_1(a)$ is absolutely continuous, as a process on a compact interval $[a, b]$, to Brownian motion; this was strengthened by getting bounds on a superpolynomial moment of the resulting Radon-Nikodym derivative in [Ham16] (though with an affinely shifted version of $\mathcal{P}_1$ being compared to Brownian bridge) and [CHH19] (comparing the originally introduced increment to Brownian motion).

As we mentioned earlier, the process $\mathcal{P}_1$ should be thought of as encoding the weight of geodesics with starting point fixed at $(0, 0)$. One can also consider other initial conditions, which are parametrized by functions $\mathfrak{h}_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. The interpretation of $\mathfrak{h}_0$ is that paths starting at $(y, 0)$ are given an extra (though possibly negative) weight $\mathfrak{h}_0(y)$ in addition to the usual weight they obtain on their journey through the environment. Maximizing this augmented weight over all paths with given endpoint $(x, 1)$ (with unconstrained starting point) gives the *KPZ fixed point* $\mathfrak{h}_1$.

(Here the subscript of 1 in $\mathfrak{h}_1$ indicates that we are at time 1, and we can analogously defined $\mathfrak{h}_t$ by making the purely notational modifications in our imagined continuum LPP model to allow all times in $(0, \infty)$. We will discuss fractal properties of the process $t \mapsto \mathfrak{h}_t$ ahead in our review of previous literature.)

Hammond used his results in [Ham16] along with a detailed study of geodesic geometry in a model known as Brownian LPP to obtain a comparison of $\mathfrak{h}_1$ with Brownian motion on a unit order, i.e., global, scale for quite a general class of initial data in [Ham19b] (and aided by two further works, [Ham19a, Ham20]). These works all relied on the Brownian Gibbs property, an invariance under resampling enjoyed by the parabolic Airy line ensemble, which shall play a central role in this paper as well. We will introduce it slightly later in Section 1.4, and formally in Definition 2.3.

[Ham19b]'s result on a unit-order comparison of $\mathfrak{h}_1$ to Brownian motion proved to be quite subtle and had a delicate quantification. In brief, it was roughly the following for a given compact interval $[a, b]$. First, there exists an infinite sequence of random functions $\{Y_i\}_{i\in\mathbb{N}}$ defined on $[a, b]$, called a *fabric* sequence, with each Y_i absolutely continuous to Brownian bridge (with a certain number of moments of the Radon-Nikodym derivative finite). Then, $[a, b]$ can be divided up into a finite (but random) number of random *patches* $\{[x_i, x_{i+1}]\}$ such that the restriction of Y_i to $[x_i, x_{i+1}]$ can be shifted vertically by a random amount y_i to equal $\mathfrak{h}_1$ on that patch. He introduced the terminology of a *Brownian patchwork quilt* to describe this setup. Later, the comparison of Y_i to Brownian bridge was improved to one with Brownian motion, with the same regularity guarantees, in [CHH19]. More recently, [SV21] obtained a comparison with a *single* patch, but with no further regularity information than absolute continuity. We will say more about [SV21] and their proof approach later in Sections 1.4 and 2.

Let $\mathcal{L} : [0, \infty) \to [0, \infty)$ be the local time at zero associated to Brownian motion of rate σ^2 begun at the origin (the definition of local time is recalled in Definition 3.1). We will always explicitly mention the relevant diffusivity σ, and so we will drop the dependence on the same in the notation $\mathcal{L}$.

For our first comparison result for the weight difference profile $\mathcal{D}$ to $\mathcal{L}$, we will rely on an analogous patchwork quilt framework as well. One difference with [Ham19b] is that we work on all of $\mathbb{R}$ instead of a compact interval, which necessitates an extra horizontal shift μ_k in addition to the vertical shifts y_k. We now proceed to setting up the precise definition tailored to our purposes.

Definition 1.3. A stochastic process $X : \mathbb{R} \to \mathbb{R}$ is said to be a *Brownian local time patchwork quilt* of rate σ^2 if there exist random $\{\mu_k, y_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, random functions $\{Y_k\}_{k\in\mathbb{Z}}$, with $Y_k : [0, \infty) \to [0, \infty)$ for all $k \in \mathbb{Z}$, and random points $\{x_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, with $x_k > \mu_k$ and $x_k < x_{k+1}$ for all $k \in \mathbb{Z}$, such that the following hold (with (ii) and (iii) on an almost sure event):

(i) For any $\delta \in (0, 1)$ and $k \in \mathbb{Z}$, the law of $Y_k|_{[\delta,\delta^{-1}]}$ is absolutely continuous to that of $\mathcal{L}|_{[\delta,\delta^{-1}]}$.
(ii) For any compact set $\mathcal{K} \subseteq \mathbb{R}$, at most finitely many x_k lie in $\mathcal{K}$.
(iii) $Y_k(z - \mu_k) + y_k = X(z)$ for all $z \in [x_k, x_{k+1}]$ and $k \in \mathbb{Z}$.

The functions $Y_k|_{[x_k-\mu_k, x_{k+1}-\mu_k]}$ again should be thought of as *fabric pieces* which are stitched together (after a vertical shift of y_k to ensure continuity) at the boundaries of the intervals $[x_k, x_{k+1}]$ to form a patchwork quilt which equals X.

Note that in point (i) the comparison of Y_k to $\mathcal{L}$ avoids the origin; this is because the Y_ks we will define for $X = \mathcal{D}$ are expected to possess a singularity at zero, which we will discuss later in Remark 3.11. However, since $x_k > \mu_k$, this singularity at the origin for the Y_k is not seen at any point for X.

We also point out that (ii) implies that the intervals $[x_k, x_{k+1}]$ cover all of $\mathbb{R}$. Finally, we observe that devices which play the roles of y_k and μ_k are necessary to make (iii) possible; the former since $\{Y_k\}_{k\in\mathbb{Z}}$ are non-negative but X may adopt any real value (as well as to ensure continuity across patches), and the latter to handle the domain of Y_k being $[0,\infty)$ while that of X is $\mathbb{R}$.

We now come to the first main result of this paper.

Theorem 2. *$\mathcal{D}$ is a Brownian local time patchwork quilt of rate four.*

We in fact a prove a technically stronger version of Theorem 2 which contains more information on the quantities μ_k, y_k, x_k, etc., as Theorem 4.1.

The source of the rate of four in Theorem 2 is that $\mathcal{S}$ (recall the definition from the discussion preceding (1)), roughly speaking, locally has diffusion rate two. This is simply the normalization adopted in the definition of the Airy$_2$ process to ensure that $\mathcal{P}_1$, or equivalently $\mathcal{S}(0,\cdot)$, is obtained from the latter by simply subtracting off the parabola x^2 without any multiplicative factor. This local diffusivity rate is reflected in the fact that the distributional limit of $\varepsilon^{-1/2}(\mathcal{S}(0,x+\varepsilon t)-\mathcal{S}(0,x))$ is two-sided Brownian motion of rate two [Häg08, QR13, CP15]. The aforementioned Brownian Gibbs property of the yet-to-be-introduced parabolic Airy line ensemble also makes this evident (see Section 2 for further elaboration). Our proof will express $\mathcal{D}$ as the running maximum of a difference of two processes which enjoy a joint comparison to independent Brownian motions of rate two. The difference of these processes is like a rate four Brownian motion. We then use Lévy's identity to relate the running maximum to $\mathcal{L}$.

Given Theorem 2, it is easy to prove Theorem 1 using that the support of $\mathcal{L}$ almost surely has Hausdorff dimension one-half, along with the countable stability property of Hausdorff dimension. However, to maintain the flow of exposition and properly recall these properties, we defer the proof of Theorem 1 to Section 4.2.

1.2. Brownian local time in the local limit.

In contrast to the global scale result, our main result on the local scale is stronger: we explicitly obtain $\mathcal{L}$ as a local limit. Before stating it precisely, we make a few remarks.

We will be considering limits of the form $\varepsilon^{-1/2}(\mathcal{D}(\tau+\varepsilon t)-\mathcal{D}(\tau))$ where τ is a random time. First, note that τ indeed needs to be random: at a deterministic time, $\mathcal{L}$ is almost surely constant, and the local limit will be trivial. Further, for the same reason, τ needs to be almost surely a point of increase of $\mathcal{D}$ for there to be any hope of obtaining $\mathcal{L}$ in the limit.

Now, there are a number of ways of choosing a point of increase of $\mathcal{D}$. For instance, we may fix $\lambda\in\mathbb{R}$ and consider the first point of increase τ_λ following λ. This would, in some sense, be a choice size-biased by the length of the flat portion preceding τ_λ; also, it excludes any of the "interior" points of increase of $\mathcal{D}$ from being considered. Alternately, we may consider the first time ρ^h that $\mathcal{D}$ hits a given level $h\in\mathbb{R}$; intuitively (and as we will prove), this is a point of increase of $\mathcal{D}$ and will typically be an "interior" one. A third method of choice, which addresses the size-biasing issue, could be to choose, for an interval $[c,d]$, a point $\xi_{[c,d]}$ *uniformly* from all the non-constant points of $\mathcal{D}$ on $[c,d]$, by sampling from the probability measure on $[c,d]$ with distribution function $(\mathcal{D}(\cdot)-\mathcal{D}(c))/(\mathcal{D}(d)-\mathcal{D}(c))$, conditionally on the event that this is a non-zero measure, which we will show has positive probability in Lemma 5.11. (Indeed, distributions at random times sampled according to such local times have been studied, for example, in the context of dynamical critical percolation [HPS15].)

We prove that the local limit at any of these three types of random times is Brownian local time:

Theorem 3. *Let τ be equal to either τ_λ, ρ^h, or $\xi_{[c,d]}$ (the last conditionally on $\mathsf{NC}(\mathcal{D})\cap[c,d]\neq\varnothing$) as above. Then,*

$$\lim_{\varepsilon\to0}\varepsilon^{-1/2}\big(\mathcal{D}(\tau+\varepsilon t)-\mathcal{D}(\tau)\big)=\mathcal{L}(t),$$

where, as before, $\mathcal{L} : [0, \infty) \to [0, \infty)$ *is the local time at the origin of rate four Brownian motion, and the limit is in distribution in the topology of uniform convergence on compact sets of* $[0, \infty)$*.*

For completeness, we will also prove that each of the mentioned random points are almost surely points of increase of $\mathcal{D}$, and, as mentioned, that $\mathsf{NC}(\mathcal{D}) \cap [c, d] \neq \varnothing$ with positive probability.

As was said earlier, we expect that the fabric pieces Y_k in Theorem 2 have a singularity with Brownian local time at the origin. Given this, the fact that Theorem 3 holds is initially somewhat surprising. The proof indeed has a few subtleties and we will explain how the apparent tension is resolved in the proof overview in Section 1.4.

Several recent papers have proved an analogous statement that the local limit of the KPZ fixed point at a fixed location is two-sided Brownian motion. In the case of the parabolic Airy$_2$ process, this was first done in various senses in [Häg08, CP15, QR13]. Under general initial conditions, this is proven in the sense of convergence of finite dimensional distributions in [MQR21, Theorem 4.14]. In the stronger topology of uniform convergence on compact sets, the convergence is implied by combining the statement of absolute continuity to Brownian motion on compact intervals ([SV21, Theorem 1.2]) with a statement that a local limit of a process which is absolutely continuous to Brownian motion is itself Brownian motion (see [DSV20, Lemma 4.3], also cited here as Lemma 5.3).

That the local limit is a given process generally does not logically follow from a patchwork quilt description. Indeed, that the local limit of the KPZ fixed point is Brownian motion is not an immediate implication of its Brownian patchwork quilt description as proved in [Ham19b, CHH19], essentially because of the randomness inherent to which of the fabric functions Y_k is in operation at a given point, and the possibility that the given point is an endpoint of a patch. (However, it might well be possible to overcome these issues without requiring any significantly new ideas.) Nonetheless, all of this is avoided in [SV21] because it is a single patch.

In the same vein, Theorem 3 is not a direct logical consequence of Theorem 2. However, an enhancement of the techniques used to prove Theorem 2 suffices to yield Theorem 3 as well. We will discuss the proof techniques in greater detail in Section 1.4.

We mention another interesting recent work investigating a local limit, though in a slightly different sense. [DSV20] considers the local limit of the environment around the geodesic, in the continuum LPP model where the environment is given by the *directed landscape*, constructed in [DOV18], which is a richer scaling limit than $\mathcal{S}$ and, in terms of our earlier imagined continuum LPP model, encodes the joint distribution of LPP weights when even the starting and ending heights, fixed to be 0 and 1 for $\mathcal{S}$, are allowed to adopt any values $s < t$. [DSV20] shows that the local limit of the environment around an interior point of the geodesic in this random environment as well as the local limits of the geodesic and its weight can be described in terms of an LPP problem driven by $\mathcal{S}$ and boundary data involving classical objects such as two-sided Brownian motion and the Bessel-3 process.

(It is this continuum model of LPP defined by the directed landscape which is a rigorously realized version of the continuum model we have been using for expository purposes, in the sense that limiting objects such as $\mathcal{P}_1$ and $\mathfrak{h}_t$ are as described above with the directed landscape LPP model. However, the environment defined by the directed landscape does not consist of independent noise. A continuum LPP model with some sort of independent noise whose weights are encoded by the directed landscape, i.e., a zero temperature analogue to the continuum directed random polymer [AKQ14] whose free energy is encoded by the KPZ equation, has not yet been constructed.)

1.3. **Prior work on fractal aspects of KPZ.** As we said, [BGH19a] initiated the study of fractal geometry within KPZ by identifying the Hausdorff dimension of the set $\mathsf{NC} := \mathsf{NC}(\mathcal{D})$. At a high level, the argument for the upper bound on the Hausdorff dimension relied on showing that NC is a subset of the set of points x admitting geodesics from $(y_a, 0)$ and $(y_b, 0)$ to $(x, 1)$ that are disjoint except for the common endpoint. We will elaborate on this slightly when we contrast our proof strategy with that of [BGH19a] in Section 1.4; see also Figure 1.

An exact form of this correspondence between geodesic disjointness and the non-constant points of $\mathcal{D}$, i.e., that the sets are equal, was proved shortly afterwards in [BGH19b]. Additionally, [BGH19b] identified the Hausdorff dimension of the set of points (y, x), as a subset of $\mathbb{R}^2$, such that there are at least two disjoint geodesics (except for the common start and endpoints) from $(y, 0)$ to $(x, 1)$. This set's dimension was also shown to be one-half. In both these results of [BGH19b] the geodesics are defined in terms of the directed landscape, similar to the earlier mentioned [DSV20].

There are two further recent studies of fractal dimension within KPZ, [CHHM21] and [DG21].

In the first [CHHM21], instead of studying the parabolic Airy sheet or directed landscape directly, the KPZ fixed point is studied. Recall its definition as a process in t from Section 1.1. [CHHM21] identifies as $\frac{2}{3}$ the Hausdorff dimension of the set of exceptional times $t > 0$ where $\mathfrak{h}_t$ has multiple maximizers, for a broad class of initial data $\mathfrak{h}_0$, conditionally on the set of exceptional times being non-empty (an event which is shown to have positive probability, and conjectured to be an almost sure event). In the geodesic picture, these exceptional times are exactly when there is not a unique geodesic with initial condition given by $\mathfrak{h}_0$ and unconstrained ending point.

The second [DG21] investigates the upper and lower laws of iterated logarithm (LIL) in time, at a fixed spatial location, for the solution to the KPZ equation (a canonical stochastic PDE in the KPZ universality class) started from narrow-wedge initial condition. They show that the upper LIL occurs at scale $(\log\log t)^{2/3}$ and the lower at scale $(\log\log t)^{1/3}$. This agrees with the scales of laws of iterated logarithm proved previously in prelimiting LPP models in [Led18, BGHK21]. The result of [DG21] of relevance from the point of view of fractal geometry is their further study of the level sets, parametrized by α, i.e., times t where the solution exceeds $\alpha(\log\log t)^{2/3}$ and their Hausdorff dimensions. They also establish an interesting transition from mono-fractal to multi-fractal behaviour under an appropriate exponential time change.

Finally, in a recent preprint [SS21], the authors announce that in forthcoming work they will prove that, in the model of Brownian LPP, the set of points (m, t) in the semi-discrete plane for which there is a random direction θ such that there exist two semi-infinite geodesics (which will be disjoint) in direction θ starting from (m, t) has Hausdorff dimension $\frac{1}{2}$.

1.4. **Remarks on proof strategy and organization of the paper.** In this section we sketch the basic ideas behind Theorems 2 and 3. Theorem 1 is a straightforward consequence of Theorem 2 and the fact that the support of $\mathcal{L}$ almost surely has Hausdorff dimension one-half. But we start off by giving a brief description of how [BGH19a] proved Theorem 1 to contrast with our approach.

[BGH19a]'s argument worked in Brownian LPP and relied on probability estimates for rare geometric events involving geodesics. To start, let us discuss the geometric significance of Theorem 1 in the continuum LPP model from above.

1.4.1. *Geodesic geometry of non-constant points of* $\mathcal{D}$. The two possible behaviours of $\mathcal{D}$ at x—constant or not—have distinct manifestations in terms of geodesics. For the first, where $\mathcal{D}$ is constant in a neighbourhood of x, the leftmost and rightmost geodesics from y_a and y_b (their existence can be shown by a simple topological argument using planarity) to x have non-trivial overlap near their end; they *coalesce*. See Figure 1. As the endpoint x varies very slightly, the point of coalescence does not change, and the change in the two geodesics is identical; in particular, the change in their weights is identical, so the weight difference $\mathcal{D}$ remains constant.

On the other hand, in the second case, the geodesics from y_a to x and from y_b to x remain disjoint for their entire lifetimes, except the final instant when their endpoint is shared. In this case the geodesics each change in distinct ways as the endpoint shifts. Thus the non-constant points of $\mathcal{D}$

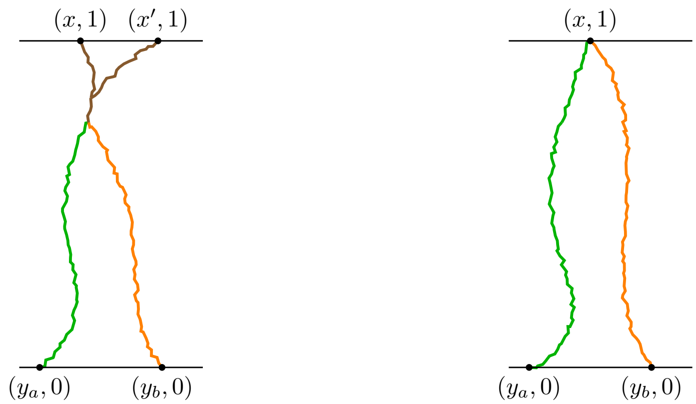

FIGURE 1. A depiction of the two possible behaviours of the geodesics with separate fixed starting points $(y_a, 0)$ and $(y_b, 0)$ and common ending point $(x, 1)$. Left: The two geodesics (green and orange) coalesce, with the common portion shown in brown. When x is varied locally (eg. to x'), the brown portion will be modified, but the green and orange portions will remain fixed; since $\mathcal{D}(z)$ for z close to x is the difference of the orange and green weights, $\mathcal{D}$ is locally constant at x. Right: The contrasting situation when the two geodesics remain disjoint till the last instant. Here moving x locally modifies both geodesics in an unshared way, so $\mathcal{D}$ does not remain constant.

correspond exactly to the endpoint locations admitting disjoint geodesics from the fixed starting points y_a and y_b.

As mentioned above, the containment of the former set in the latter was proved in [BGH19a] and was sufficient for their purposes. But the equality of the two sets is not obvious and was established only later in [BGH19b]. The argument is similar to that of Lemma 1.1 and proceeds by showing that, for any point where $\mathcal{D}$ is locally constant, the leftmost and rightmost geodesics from y_a and y_b respectively must intersect non-trivially.

Thus, on a rough level, [BGH19a]'s arguments used estimates on the probability of disjoint geodesics in Brownian LPP arising from earlier work of Hammond [Ham20] to obtain an upper bound on the Hausdorff dimension. In contrast, the lower bound was obtained by a different argument relying on a quantified form of Brownianity of the parabolic Airy$_2$ process $\mathcal{P}_1$.

We now describe the approach taken in this paper, which, in contrast to [BGH19a], relies on what may be called a continuous Robinson-Schensted-Knuth correspondence which [DOV18] used to define the parabolic Airy sheet $\mathcal{S}$.

1.4.2. *The parabolic Airy sheet via continuous RSK.* [DOV18] defined the parabolic Airy sheet $\mathcal{S}$ in terms of a limiting semi-discrete LPP problem in an environment defined by the parabolic Airy line ensemble $\mathcal{P}$. The latter is an infinite $\mathbb{N}$-indexed collection of random non-intersecting curves whose lowest indexed, but highest in value, curve is $\mathcal{P}_1$, the parabolically shifted Airy$_2$ process; see Figure 2.

The above mentioned last passage problem in $\mathcal{P}$ can be described by fixing a starting coordinate on the k^{th} line (for some $k \geq 1$) and an ending coordinate on the first (i.e., top) line, and considering up-right paths between the two. The weight of a given path is given by the sum (over i) of increments of values of $\mathcal{P}_i$ along the interval that the path spends on the i^{th} line. If the starting point is (y, k) and ending point is $(x, 1)$, we denote this by $\mathcal{P}[(y, k) \to (x, 1)]$.

It was shown in [DOV18] that the set of last passage values obtained this way encodes $\mathcal{S}$. At a high level, the increments of $\mathcal{S}$ with a fixed starting point, i.e., of the form $\mathcal{S}(y, x) - \mathcal{S}(y, z)$, are defined

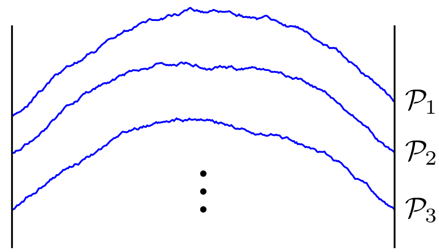

FIGURE 2. A depiction of the parabolic Airy line ensemble.

as the limit of a difference $\mathcal{P}[((y)_k, k) \to (x,1)] - \mathcal{P}[((y)_k, k) \to (z, 1)$ of LPP values in $\mathcal{P}$. Here $\{(y)_k\}_{k\in\mathbb{N}}$ is a sequence of deterministic points, depending on y, which is such that $(y)_k \to -\infty$ in a parabolically curved manner as $k \to \infty$; see Definition 2.6. Similar formulations have recently been used to construct the extended directed landscape in [DZ21].

Note that here the increment of $\mathcal{S}$ is with shared starting point and differing ending points, while our object of interest $\mathcal{D}(x) = \mathcal{S}(y_b, x) - \mathcal{S}(y_a, x)$ has differing starting point but shared ending point. Thus the description of the former type of increment of $\mathcal{S}$ in terms of a difference of LPP values in $\mathcal{P}$ is not directly useful for us; we need a LPP description that holds for $\mathcal{S}(y, x)$ itself. This is rather difficult and in fact an open problem ([DOV18, Conjecture 14.2]).

This issue was encountered by [SV21] as well, who bypassed it by considering an LPP problem in $\mathcal{P}$ with appropriate *boundary data* which encodes $\mathcal{S}(y, x)$. More precisely, [SV21, Lemma 3.10] (and cited here as Lemma 2.8) says that, for given $\lambda \in \mathbb{R}$, there exist random numbers $\{a_i^\lambda\}_{i\in\mathbb{N}}$ and $\{b_j^\lambda\}_{j\in\mathbb{N}}$ such that, for $x \geq 0$,

$$\begin{aligned} \mathcal{S}(y_a, \lambda + x) &= \sup_{i\in\mathbb{N}} \left\{ a_i^\lambda + \mathcal{P}[(\lambda, i) \to (\lambda + x, 1)] \right\}. \\ \mathcal{S}(y_b, \lambda + x) &= \sup_{j\in\mathbb{N}} \left\{ b_j^\lambda + \mathcal{P}[(\lambda, j) \to (\lambda + x, 1)] \right\}. \end{aligned} \tag{2}$$

Thus the RHS is the last passage value in $\mathcal{P}$ from the set $\{\lambda\} \times \mathbb{N}$ to $(\lambda + x, 1)$with boundary values $\{a_i^\lambda\}_{i\in\mathbb{N}}$ and $\{b_j^\lambda\}_{j\in\mathbb{N}}$ respectively. The latter implicitly encode the weight contribution coming from the initial segments of the geodesics between the points $((y)_k, k)$ and $(\lambda + x, 1)$ as $k \to \infty$ when $y = y_a$ and y_b.

Before proceeding further it would be convenient to setup some notation. Given $\mathcal{P}$, for $i < j$ and $\lambda \in \mathbb{R}$, let

$$\mathcal{P}^\lambda_{j\to i}(x) := \mathcal{P}[(\lambda, j) \to (\lambda + x, i)]. \tag{3}$$

For our applications we will often be using the case that $i = 1$. Note also that when $i = j$, $\mathcal{P}^\lambda_{i\to i}(x) = \mathcal{P}_i(\lambda + x) - \mathcal{P}_i(\lambda)$ measures the increment of the i^{th} line $\mathcal{P}_i$ across the interval $[\lambda, \lambda + x]$; for ease of notation, we will often denote $\mathcal{P}^\lambda_{i\to i}$ by simply $\mathcal{P}^\lambda_i$.

Given the above notation and the representation in (2), the starting point of the present work is the observation that to relate $\mathcal{D}$ to Brownian local time we should look at functions of the form

$$\mathcal{P}^\lambda_{j\to 1}(x) - \mathcal{P}^\lambda_{i\to 1}(x), \tag{4}$$

and, indeed, these will essentially be the fabric functions Y_k for different values of λ, i, and j.

Before we outline how to relate such a function to Brownian local time, let us mention the important fact about $\mathcal{P}$ that is the source of all Brownian comparisons we make, namely that it enjoys the *Brownian Gibbs* property [CH14]. This means that conditional on everything apart from the top k curves of $\mathcal{P}$ on an interval $[a, b]$, the distribution of $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ on $[a, b]$ is given by k independent rate two Brownian bridges between the correct endpoints which are conditioned to not intersect each other or the lower curve $\mathcal{P}_{k+1}$. An immediate implication is that the joint law of $\mathcal{P}_i(\cdot) - \mathcal{P}_i(a)$, for $i = 1, \dots, k$, is absolutely continuous with respect to the law of k independent rate two Brownian motions on $[a, b]$. The formal definition of the Brownian Gibbs property is given in Definition 2.3.

1.4.3. *Analyzing LPP values using Pitman transforms.* As we saw, to relate $\mathcal{D}$ to Brownian local times, we need to relate $\mathcal{P}^\lambda_{j\to 1} - \mathcal{P}^\lambda_{i\to 1}$ to the same. To do this, we make fundamental use of a representation of LPP values as the result of a sequence of more basic transformations known as *Pitman transforms*, which will be denoted by PT. Use of the transform and this representation is present most closely in the spirit we utilize them in [DOV18, SV21], but were first introduced in the context of LPP in [OY02]. The formal definition appears in Section 3.

In a general sense, the PT takes as input two functions (f_1, f_2) defined, say, on a compact interval, and outputs two functions (g_1, g_2) defined on the same domain such that $f_1 + f_2 = g_1 + g_2$ as functions.

Proceeding to describe how they naturally appear in our setting, we first consider (4) when $j = 2$ and $i = 1$. It is easy to see that

$$\mathcal{P}^\lambda_{2\to 1}(x) = \mathcal{P}[(\lambda, 2) \to (\lambda + x, 1)] = \mathcal{P}^\lambda_1(x) + \max_{0\le s\le x}\left(\mathcal{P}^\lambda_2(s) - \mathcal{P}^\lambda_1(s)\right). \tag{5}$$

The righthand side appearing at the end of the above equation defines the first component of the Pitman transform of $\mathcal{P}^\lambda_1$ and $\mathcal{P}^\lambda_2$, with the other component being determined by the equality of sums property mentioned above. For us, the important feature of the Pitman transform of stochastic processes X_1 and X_2 is that it is represented via the *maximum* of the difference of X_1 and X_2. Returning to (5), we see that when $j = 2$ and $i = 1$, (4) is equal to

$$\mathcal{P}^\lambda_{2\to 1}(x) - \mathcal{P}^\lambda_{1\to 1}(x) = \max_{0\le s\le x}\left(\mathcal{P}^\lambda_2(s) - \mathcal{P}^\lambda_1(s)\right). \tag{6}$$

By the Brownian Gibbs property, $(\mathcal{P}^\lambda_1, \mathcal{P}^\lambda_2)$ is absolutely continuous to a pair of independent Brownian motions of rate two, and so (6) is absolutely continuous, *on the whole interval* $x \in [0, T]$ for any fixed T, to the running maximum of rate four Brownian motion. By Lévy's famous identity, the latter has exactly the distribution of the local time at the origin of rate four Brownian motion (see Proposition 3.2 to recall this).

For different values of i and j, we will need to use multiple compositions of PT. Indeed, a key step in our argument is to find a particular sequence of transforms which gives tractable representations of $\mathcal{P}^\lambda_{i\to 1}$ and $\mathcal{P}^\lambda_{j\to 1}$ such that the difference in (4) can be analyzed.

However, in this case, we cannot obtain absolute continuity to independent Brownian motions on the entire interval $[0, T]$ for the input functions, unlike the pair $(\mathcal{P}^\lambda_1, \mathcal{P}^\lambda_2)$ for the case $j = 2$ and $i = 1$. Without going into technical details, this is broadly because of the fact that two components of the Pitman transform of independent Brownian motions are not jointly absolutely continuous to independent Brownian motions on an interval containing zero. In fact, the joint law of the output of PT is precisely a 2-Dyson Brownian motion, where an n-Dyson Brownian motion may be defined as n independent Brownian motions conditioned on non-intersection. (The singular conditioning at the origin is achieved via a suitable Doob h-transform, see, for example, [Gra99].) Thus on successively applying PT, the singular output of one round gets fed into the next round, and, hence, one cannot expect the next output to be absolutely continuous to independent Brownian motions on an interval *containing zero*. However, fortunately, zero is the only source of singularity, and the absolute continuity does hold on any interval avoiding the origin. Further, this property is preserved even on successive applications of PT, which suffices for our purposes.

Section 3 contains the arguments making use of the Pitman transform to establish absolute continuity to Brownian local time of the functions which will be the fabric functions Y_k of Definition 1.3.

1.4.4. *Using geodesics to define fabric functions and patches.* The discussion in the previous subsection gave some idea of how to show that functions of the form (4), which will essentially play the role of Y_k, are absolutely continuous to Brownian local time. Next we indicate how the boundaries of the patches $[x_k, x_{k+1}]$ are defined.

We will need to know some monotonicity properties of the $\{a_i^\lambda\}_{i\in\mathbb{N}}$ and $\{b_j^\lambda\}_{j\in\mathbb{N}}$ sequences. Heuristically, a_i^λ can can be thought of as the weight of the infinite geodesic defined as the limit of the finite geodesic from $((y_a)_k, k)$ to (λ, i) as $k \to \infty$. We will refer to this infinite geodesic as the one from y_a to (λ, i).

(Such an infinite geodesic can be defined rigorously, as was done in [SV21]. But it is difficult to make sense of its weight directly through a limit as it will diverge. It is to handle this divergence that the rigorous definition of a_i^λ is different; see Section 2.3.)

This heuristic suggests that $a_i^\lambda \geq a_{i+1}^\lambda$ by considering simple path modifications: any path ending at $(\lambda, i+1)$ can trivially be extended to end at (λ, i) by jumping from line $i+1$ to line i at the last instant, which does not modify its weight. A similar monotonicity holds for $\{b_j^\lambda\}_{j\in\mathbb{N}}$.

An implication of these monotonicity properties is that the minimum indices achieving the two supremums in (2) are equal to 1 when $x = 0$. Let us call the supremum achieving indices in (2) respectively $i^\lambda(x)$ and $j^\lambda(x)$ (always the minimum such in case of non-uniqueness). It will be important to understand their relation to each other and how they behave as x increases. As a first fact we will of course need to know that they are finite for each x, which we cite from the previously mentioned [SV21, Lemma 3.10].

Again the heuristic in terms of an infinite geodesic helps explain the behaviour of i^λ and j^λ. In particular, $i^\lambda(x)$ is the location at λ of the infinite geodesic from y_a to $(\lambda + x, 1)$, and $j^\lambda(x)$ the same for the one from y_b to $(\lambda + x, 1)$. Since $y_b > y_a$, planarity suggests that the geodesic corresponding to y_b should be to the right of that from y_a. This would give that $i^\lambda(x) \leq j^\lambda(x)$. Similarly, if $x' > x$, the geodesic from y_a to x' should be to the right of the one from y_a to x: $x \mapsto i^\lambda(x)$ should be non-decreasing, and the same for j^λ.

Now, from (2), we see that on intervals where i^λ and j^λ are both constant, say with values $i \leq j$, $\mathcal{D}(x)$ is equal to (4) up to a vertical shift given by $b_j^\lambda - a_i^\lambda$. Thus the patch boundaries x_k should be exactly the points where i^λ or j^λ change value, and the vertical shifts y_k should be $b_j^\lambda - a_i^\lambda$ on the interval where i^λ and j^λ are i and j. This gives a patchwork quilt description on an infinite ray, and to cover $\mathbb{R}$, we can also vary λ over a countable sequence going to $-\infty$ (we concretely take λ over $\mathbb{Z}$).

Arguments involving the geometry of geodesics in the LPP problem (4) are handled in Section 2. In Section 4 we combine the geometric information of Section 2 and the absolute continuity information of Section 3 to prove Theorem 2, and then derive Theorem 1 as an easy consequence.

1.4.5. *Understanding geodesic geometry at a random time.* The proof of Theorem 3 requires a more detailed understanding of the geodesic geometry outlined in the previous subsection. A particularly elegant aspect of our proof is that it ultimately reduces to the well-known fact that two-dimensional Brownian motion avoids any given point in the plane forever with probability one.

While the theorem is stated for three different choices of τ, in this proof overview we will mainly discuss the case where $\tau = \tau_\lambda$, which captures the main conceptual points common to all the cases. We will say a few words at the end about the other choices of τ.

First, observe that Theorem 3 concerns behaviour of $\mathcal{D}$ at a *random* time τ_λ, which is almost surely greater than λ. So a first approach to proving Theorem 3 might proceed by arguing as above to conclude that $\mathcal{D}(\tau_\lambda)$ equals $\mathcal{P}_{j\to 1}^\lambda(\tau_\lambda) - \mathcal{P}_{i\to 1}^\lambda(\tau_\lambda)$ with some values of j and i. The problem is that τ_λ, though a stopping time with respect to the right-continuous filtration of $\mathcal{P}$, is not one with respect to the top j curves of $\mathcal{P}$ for any fixed j, and so it unclear how to pin down the behaviour of $\mathcal{P}^{\lambda}_{j\to 1} - \mathcal{P}^{\lambda}_{i\to 1}$ at such a random time; in addition, the random values of j and i at τ_λ introduce complicated correlations.

Instead, we consider the geodesic itself as its endpoint moves from τ_λ slightly forward. More precisely, we look at where the geodesic is at τ_λ as the endpoint moves further. In other words, we need a formula for $\mathcal{S}(y_b, \tau_\lambda + x) - \mathcal{S}(y_a, \tau_\lambda + x)$ in terms of boundary data at τ_λ, analogous to (2) with τ_λ in place of λ. But now the issue is that behaviour at τ_λ of $\mathcal{D}$ corresponds to behaviour of an expression of the form (4) (again with τ_λ in place of λ) *at the origin*. But as we saw, unless $j = 2$ and $i = 1$, expressions like (4) are not absolutely continuous to Brownian local time around the origin!

Conversely, if $j = 2$ and $i = 1$, it is an easy argument, using the scale invariance and independence across scales of Brownian motion, that the local limit at the origin of an expression like $\max_{0\le s\le x}\left(\mathcal{P}^{\lambda}_2(s) - \mathcal{P}^{\lambda}_1(s)\right)$ is Brownian local time; see Lemma 5.3 and Corollary 5.4. We can handle the fact that we are looking at $\mathcal{P}^{\lambda}_2$ and $\mathcal{P}^{\lambda}_1$ at a stopping time of $\mathcal{P}$ by making use of a stronger version of the Brownian Gibbs property which works for spatial analogues of stopping times for Markov processes; τ_λ is such a random quantity. So $\mathcal{P}^{\lambda}_2$ and $\mathcal{P}^{\lambda}_1$ look Brownian even at τ_λ.

So our proof consists of showing that in the immediate right-neighbourhood of τ_λ, we do in fact have $j = 2$ and $i = 1$.

Ignoring certain technical details, such as setting up the appropriate right-continuous filtration, we review the the basic argument here. Note that the infinite geodesics (one for each starting point y_b and y_a) ending at $(\tau_\lambda, 1)$ will certainly both be at the top line at τ_λ. Now, because τ_λ is a point of increase of $\mathcal{D}$, at least one of the geodesic must switch lines immediately after τ_λ; if not, $j = i = 1$ in that neighbourhood and $\mathcal{D}$ would be a constant. Because $y_b > y_a$, the ordering of the geodesic says that the one corresponding to y_b must certainly jump.

To say that $j = 2$ and $i = 1$ in the right-neighbourhood of τ_λ, we need to say that that the y_b-geodesic jumps to line 2 and not lower (which will also imply the y_a-geodesic stays on line 1).

The above argument applies not only to τ_λ, but to any stopping time which is almost surely a point of increase of $\mathcal{D}$ and is greater than λ. Dropping the condition of being greater than λ is a simple technical task, which then takes care of the case where $\tau = \rho^h$ (hitting time of $h \in \mathbb{R}$) in Theorem 3. The final case of $\xi_{[c,d]}$ (an independent uniform sample from $\mathcal{D}$ on $[c, d]$) is slightly more complicated as it is not a stopping time. For this, we rely on a representation of it as the hitting time ρ^U_c, i.e., the hitting time of $\mathcal{D}(c) + U$, with U a uniform random variable on $[0, \mathcal{D}(d) - \mathcal{D}(c)]$, which can then be decomposed into a mixture of stopping times ρ^h_c.

Now we turn to explaining why the y_b-geodesic must jump to line 2 and not lower.

1.4.6. *Boundary data at random locations.* To ensure this we analyze the boundary data $\{b^{\tau_\lambda}_j\}_{j\in\mathbb{N}}$ at the random location τ_λ (which needs some care to define precisely, as we will shortly discuss). Recall first the stated monotonicity ($b^{\tau_\lambda}_1 \ge b^{\tau_\lambda}_2 \ge \ldots$). Observe that under the strict inequality $b^{\tau_\lambda}_2 > b^{\tau_\lambda}_3$, the continuity of $\mathcal{P}$ will imply from (2) that

$$b^{\tau_\lambda}_2 + \mathcal{P}[(\tau_\lambda, 2) \to (\tau_\lambda + \varepsilon x, 1)] > b^{\tau_\lambda}_3 + \mathcal{P}[(\tau_\lambda, 3) \to (\tau_\lambda + \varepsilon x, 1)]$$

for all small ε and x in a compact set, and so the y_b-geodesic must be at line two for all such small ε and bounded x.

We also know that $b^{\tau_\lambda}_1 = b^{\tau_\lambda}_2$, for otherwise the y_b-geodesic could not jump to line 2 at τ_λ. So our proof has essentially reduced to showing that $b^{\tau_\lambda}_1 > b^{\tau_\lambda}_3$ almost surely.

To do this, as mentioned above, we first have to find an expression for the boundary data as a process (since τ_λ is random) that agrees with the definition of $\{b^{\lambda}_j\}_{j\in\mathbb{N}}$ adopted earlier from [SV21], which only works for fixed values of λ. To emphasize this distinction, we call the boundary data process $Z^{\lambda,b}_j$, where $Z^{\lambda,b}_j(x)$ should be thought of as $b^{\lambda+x}_j$. So it suffices to show that, almost surely for all $x \ge 0$, $Z^{\lambda,b}_1(x) > Z^{\lambda,b}_3(x)$.

To define $Z_i^{\lambda,b}(x)$, we consider the LPP problem from the vertical line at λ, with boundary data $\{b_j^\lambda\}_{j\in\mathbb{N}}$, to $(\lambda+x,i)$. In other words, $Z_i^{\lambda,b}$ the righthand side of the second equality of (2) with $(\lambda+x,i)$ in place of $(\lambda+x,1)$.

Now, by arguments similar to before, it is plausible that $Z_3^{\lambda,b}$ is absolutely continuous to Brownian motion away from the origin. We can now simply consider the three line ensemble given by $(\mathcal{P}_1^\lambda,\mathcal{P}_2^\lambda,Z_3^{\lambda,b})$ and use Pitman transforms to compute $Z_1^{\lambda,b}$. In fact for any $i\in\mathbb{N}$, $Z_i^{\lambda,b}$ can essentially be expressed as $\mathcal{P}_i^\lambda(\cdot)$ reflected off of $Z_{i+1}^{\lambda,b}$, in the sense of Skorohod reflection; see [War07] for work within KPZ on Brownian motions reflecting of Dyson's Brownian motion.

With this, and forms of absolute continuity with respect to independent Brownian motions, the event that $Z^{\lambda,1}(x)=Z^{\lambda,3}(x)$ for some x can be related to the event that two dimensional Brownian motion hits the origin at some strictly positive time. But of course, it is a classical fact that this event has probability zero, which, by absolute continuity, implies what we need.

These arguments are developed in Section 5.

1.5. **Extensions.** It is an interesting question whether a stronger statement of absolute continuity of $\mathcal{D}$ to Brownian local time can be made. We note that some kind of random shift to $\mathcal{D}$ will be required for any such comparison: for example, $\mathcal{D}$ itself can adopt negative values, which is clearly not the case for Brownian local time. We believe that a statement avoiding random intervals is more subtle and challenging to obtain, for reasons similar to why we need to avoid the origin when making our absolute continuity comparisons as, for example, in Definition 1.3(i); we expand on these aspects slightly in Remarks 3.11 and 4.2. But, in brief, it appears that to remove the patches one will need a good understanding of geodesic behaviour deep in the parabolic Airy line ensemble, which is currently not very well understood; indeed, tools such as the Brownian Gibbs property which work well at the top of $\mathcal{P}$ are not useful in this regime of its depths.

In the case of the KPZ fixed point, the decomposition into random patches was avoided in a sense in [SV21], where the KPZ fixed point was proven to be absolutely continuous to Brownian motion on any fixed interval; though finiteness of higher moments of the Radon-Nikodym derivative is not asserted. As mentioned before, our proof makes use of a representation involving suitable boundary data introduced in [SV21]. Further, there is an overall similarity of theme with the latter, in that they too study absolute continuity of the Pitman transform of Brownian-like processes in the context of an LPP problem through $\mathcal{P}$, and we make use of two of their results: see (11) and Lemma 2.8. The difficulty [SV21] faced in obtaining Radon-Nikodym derivatives is similarly obstructive in that respect for this work.

Nonetheless, despite some thematic similarities with past works, including [BGH19a] and [SV21] in particular, the results of this paper require a considerable number of novel arguments, as may be seen by the proof overview in Section 1.4. We highlight a few of those points here. Firstly, Theorem 3 requires a careful study of local geodesic behaviour at certain random locations, which we believe has not appeared in the literature before. A primary device to undertake this study is the analysis of a *process* version of the boundary data, introduced in [SV21] for fixed locations, which leads to comparisons with planar Brownian motion. Finally, on a more technical level, Theorem 2 requires new absolute continuity statements of Pitman transforms to Brownian motion.

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2. Parabolic Airy line ensemble, LPP, and geometry of geodesics

This section develops formally the central objects that will be in play throughout the paper. We then derive some useful geometric lemmas. We start by introducing the precise definition of the parabolic Airy sheet $\mathcal{S}$. Because this is defined in terms of a last passage problem in a random environment defined by the parabolic Airy line ensemble $\mathcal{P}$, we first define the former concept and the latter object in Section 2.1.

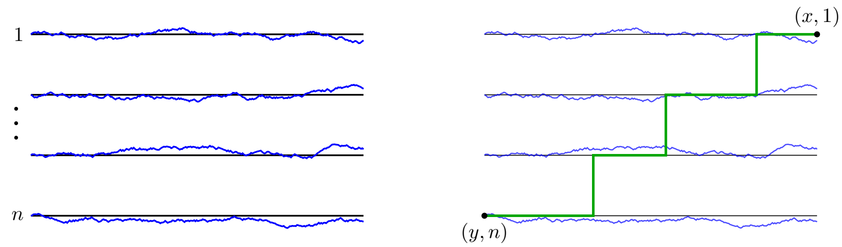

Figure 3. Left: An illustration of (a subset of) the environment defined by f. The functions f_i corresponding to each line are shown in blue on the corresponding black line for visual clarity; the function values themselves are not necessarily ordered. Right: An up-right path γ from (y,n) to $(x,1)$ is shown in green; note that in the formal definition the depicted vertical portions are not part of γ. The path's weight is the sum of the increments of f_i along the portion of the i^{th} line γ spends on it.

Notation. We denote the integer interval $\{i,\ldots,j\}$ by $[\![i,j]\!]$, and the set $\{1,2,\ldots,\}$ by $\mathbb{N}$. For an interval $I \subseteq \mathbb{R}$, $\mathcal{C}(I)$ will be the set of continuous functions $f : I \to \mathbb{R}$. For a finite set A, $\#A$ will denote its cardinality. Finally, given a function f defined on a set A, we will denote its restriction to a subset $B \subseteq A$ by $f|_B$.

2.1. Last passage percolation and the parabolic Airy line ensemble.

Definition 2.1 (Last passage percolation)**.** Let $f = (f_1, f_2, \ldots) : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a given sequence of continuous functions. We will think of the curves $f_1, f_2, \ldots,$ as lying on horizontal lines which are ordered vertically, indexed such that f_1 is the top curve, f_2 is below it, and so on; though the function values themselves are not assumed to be ordered. See Figure 3. We will refer to these curves as the *environment* given by f, or defined by f.

Let $y \le x$ and $n \ge m$. An *up-right* path from (y,n) to (x,m) is a path which begins at (y,n) and moves rightward, jumping from one line to the next at various times, till it reaches (x,m); see Figure 3. Up-right paths are parametrized by their *jump times* $\{t_i\}_{i=m+1}^{n}$ at which they jump from the i^{th} line to the $(i-1)^{\text{th}}$ line. The *weight* of an up-right path γ from (y,n) to (x,m) through f is given by

$$f[\gamma] = \sum_{i=m}^{n} \big(f_i(t_i) - f_i(t_{i+1})\big),$$

where $t_{n+1} = y$ and $t_m = x$. In other words, $f[\gamma]$ is the sum over i of the increments of f_i over the interval that γ spends on the i^{th} line. The *last passage value* from (y,n) to (x,m) through f is given by

$$f[(y,n) \to (x,m)] = \sup_{\gamma:(y,n)\to(x,m)} f[\gamma], \tag{7}$$

where the supremum is over all up-right paths from (y,n) to (x,m). A path which achieves the supremum is called a *geodesic* from (y,n) to (x,m).

Note that the definition (7) immediately adapts to the case that f has domain $I \times \mathbb{R}$ or $I \times [0, \infty)$ for some finite integer interval $I \subseteq \mathbb{N}$, so long as $n, m \in I$, and, in the latter case, $x, y \geq 0$.

While there are broad similarities, the heuristic LPP model described in Section 1 should not be confused for the precise LPP model defined above. We emphasize that the directedness constraint in Definition 2.1 imposes the inequality $y \leq x$ and that the paths are not well-defined functions of their height, i.e., the line index.

Having defined the general LPP model, we now move on to formally introducing another central object in this paper, the parabolic Airy line ensemble.

Definition 2.2 (Parabolic Airy line ensemble)**.** The parabolic Airy line ensemble $\mathcal{P} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is an $\mathbb{N}$-indexed family of random non-intersecting continuous curves, such that the ensemble $\mathcal{A}$ given by $\mathcal{A}_i(x) = \mathcal{P}_i(x) + x^2$ has finite dimensional distributions determined as follows: for every $m \in \mathbb{N}$ and real $t_1 < \ldots < t_m$, the point process $\{(\mathcal{A}_i(t_j), t_j) : i \in \mathbb{N}, j \in [\![1, m]\!]\}$ is determinantal with correlation kernel given by the extended Airy kernel $K_{\mathrm{Ai}}^{\mathrm{ext}}$, where

$$K_{\mathrm{Ai}}^{\mathrm{ext}}\big((x,t);(y,s)\big) = \begin{cases} \int_0^\infty e^{-\lambda(t-s)} \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda & t \geq s \\ -\int_{-\infty}^0 e^{-\lambda(t-s)} \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda & t < s; \end{cases}$$

here Ai is the classical Airy function. The reader is referred to [HKPV09] for background on determinantal point processes.

The parabolic Airy line ensemble was first constructed in [CH14]. That paper also proved $\mathcal{P}$ enjoys the *Brownian Gibbs property*, a central invariance property which we define next.

Definition 2.3 (Brownian Gibbs property)**.** Let $X : \mathbb{N} \to \mathbb{R}$ be a collection of random continuous non-intersecting curves, and fix $k \in \mathbb{N}$ and $[\ell, r] \subset \mathbb{R}$. Let $\mathcal{F}_{\mathrm{ext}}(k, \ell, r)$ be the σ-algebra generated by $\{X_i(x) : (i, x) \notin [\![1, k]\!] \times [\ell, r]\}$, i.e., everything external to $[\ell, r]$ on the top k curves. X is said to possess the *Brownian Gibbs* property if, conditionally on $\mathcal{F}_{\mathrm{ext}}(k, \ell, r)$, the distribution of X on $[\![1, k]\!] \times [\ell, r]$ is that of k independent Brownian bridges $(B_1, \ldots, B_k)$ of rate *two*, with $B_i(x) = X_i(x)$ for $x \in \{\ell, r\}$ and $i \in [\![1, k]\!]$, conditioned on not intersecting each other or $X_{k+1}(\cdot)$ on $[\ell, r]$.

Theorem 2.4 (Theorem 3.1 of [CH14])**.** *The parabolic Airy line ensemble $\mathcal{P}$ has the Brownian Gibbs property.*

We will be making use of the Brownian Gibbs property of $\mathcal{P}$ quite often; indeed, it is the ultimate source of all the statements we make regarding absolute continuity to Brownian objects, the first of which is the following.

Corollary 2.5. *Let $[\ell, r] \subset \mathbb{R}$ and $k \in \mathbb{N}$. Then $\{\mathcal{P}_i(\cdot) - \mathcal{P}_i(\ell) : i \in [\![1, k]\!]\}$, as a process on $[\ell, r]$, is absolutely continuous to the law of k independent Brownian motions of rate two, all started at $(\ell, 0)$, on $[\ell, r]$.*

This proof is the same as the one given for [CH14, Proposition 4.1], where the $k = 1$ case of Corollary 2.5 is proved, and we reproduce it here.

Proof of Corollary 2.5. Applying the Brownian Gibbs property on $[\![1, k]\!] \times [\ell, r]$ tells us that, conditionally on $\mathcal{F}_{\mathrm{ext}}(\ell, r, k)$, $\{\mathcal{P}_i(\cdot) - \mathcal{P}_i(\ell) : i \in [\![1, k]\!]\}$ is absolutely continuous to the law of k independent Brownian bridges of rate two. To move from here to Brownian motions, we must know that $(\mathcal{P}_i(r) - \mathcal{P}_i(\ell))_{i=1}^k$ has law which is absolutely continuous to product Lebesgue measure. This is implied by applying the Brownian Gibbs property on $[\![1, k]\!] \times [\ell, 2r]$. ☐

We will often make use of this corollary without explicitly referring to it by name.

2.2. **The parabolic Airy sheet.** The parabolic Airy sheet was proved to exist in [DOV18]. Its definition, as given in [DOV18, Definition 8.1], is in terms of a last passage problem in the parabolic Airy line ensemble. First, for $k \in \mathbb{N}$ and $y > 0$, let

$$(y)_k = \Big(-(k/2y)^{1/2}, k\Big).$$

Definition 2.6 (Parabolic Airy sheet)**.** The parabolic Airy sheet $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$ is a continuous process with the following two properties.

(i) $\mathcal{S}(\cdot + z, \cdot + z) \stackrel{d}{=} \mathcal{S}(\cdot, \cdot)$ for each $z \in \mathbb{R}$, where the equality is in distribution of processes.
(ii) $\mathcal{S}$ can be coupled with the parabolic Airy line ensemble $\mathcal{P}$ so that $\mathcal{S}(0, \cdot) = \mathcal{P}_1(\cdot)$ and, almost surely for all $x, y, z \in \mathbb{Q}$ with $y > 0$, there exists a random constant $K_{x,y,z}$ such that, for all $k \geq K_{x,y,z}$,

$$\mathcal{S}(y, z) - \mathcal{S}(y, x) = \mathcal{P}\big[(y)_k \to (z, 1)\big] - \mathcal{P}\big[(y)_k \to (x, 1)\big].$$

The existence of the parabolic Airy sheet was proven in [DOV18] via Brownian last passage percolation. To state this, let $B : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be a $\mathbb{N}$-indexed family of independent two-sided Brownian motions of rate one.

Theorem 2.7 (Theorem 1.3 of [DOV18])**.** *The Airy sheet exists and is unique in law. Further,*

$$\mathcal{S}(y, x) = \lim_{n\to\infty} n^{-1/3} \left(B[(2yn^{2/3}, n) \to (n + 2xn^{2/3}, n)] - 2n - 2(x - y)n^{2/3}\right),$$

where the limit is in distribution, in the topology of uniform convergence on compact sets.

Note that the stationarity asserted in (i) of Definition 2.6 above allows us to assume without loss of generality that $y_a, y_b > 0$ in the definition (1) of $\mathcal{D}$, and we do so for the rest of the paper. This is useful as it gives us access to the formula in (ii) of Definition 2.6, which requires the initial point y to be positive.

2.3. **LPP boundary values in $\mathcal{P}$.** We now set up the framework of LPP models with *boundary data* which, as indicated in Section 1.4, will play a vital role in many of our arguments. This was introduced in [SV21]. However, before formally defining things, we first discuss the various issues one encounters in making certain definitions and how they are addressed.

Observe that item (ii) in Definition 2.6 is suggestive of the possibility that $\mathcal{S}(y, x)$ itself can be written as a limiting LPP value in the parabolic Airy line ensemble, without having to resort to an expression which only involves a difference. This is not directly true, as $\mathcal{P}[(y)_k \to (x, 1)]$ will diverge to ∞ as $k \to \infty$. Instead, it is conjectured (see [DOV18, Conjecture 14.2]) that there exists a deterministic function $a : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}$ such that, for every fixed $y > 0$, almost surely

$$\mathcal{S}(y, 0) = \lim_{k\to\infty} \big(\mathcal{P}[(y)_k \to (0, 1)] - a(y, k)\big). \tag{8}$$

However, something of this sort appears quite difficult to prove. The main utility for us of such an expression is that it would allow the use of reasoning about geodesics in analyzing $x \mapsto \mathcal{D}(x) = \mathcal{S}(y_b, x) - \mathcal{S}(y_a, x)$, for which the LPP description of Definition 2.6 is not available as it is not a difference of the parabolic Airy sheet at different ending values with common starting point.

Imagine for a moment that we could express $\mathcal{S}(y, x)$ as the weight of an infinite geodesic through the parabolic Airy line ensemble ending at $(x, 1)$ with a formula like (8). Then, by considering the index i of the line this geodesic is at at a location $\lambda \leq x$, we see that it would be possible to write $\mathcal{S}(y, x)$ as

$$\mathcal{S}(y, x) = \sup_{i\in\mathbb{N}} \big(\mathcal{P}[y \to (\lambda, i)] + \mathcal{P}[(\lambda, i) \to (x, 1)]\big), \tag{9}$$

where $\mathcal{P}[y \to (\lambda, i)]$ represents a renormalized form of the weight of the infinite geodesic ending at (λ, i), akin to the righthand side of (8) with i replacing 1. Note that, while the first term in the supremum is currently difficult to define directly, the second is a standard finite LPP problem. The collection $\{\mathcal{P}[y \to (\lambda, i)] : i \in \mathbb{N}\}$ can be thought of as *boundary data*, with respect to which the finite LPP problem $\mathcal{P}[(\lambda, i) \to (x, 1)]$ is considered.

The crucial observation, however, is that even in the absence of (8), equation (9) is also useful for analyzing $\mathcal{D}$, for the expressions for both $\mathcal{S}(y_a, x)$ and $\mathcal{S}(y_b, x)$ involve the *same* finite LPP problems (though with *differing* boundary data).

Thus it remains to look for a well-defined quantity that can take the place of $\mathcal{P}[y \to (\lambda, i)]$ in the above formula and play the role of boundary data. The problem with defining $\mathcal{P}[y \to (\lambda, i)]$ directly was that $\mathcal{P}[(y)_k \to (\lambda, i)]$ diverges to infinity and we do not currently have a statement of the form (8) that would have allowed us normalize the latter by a quantity like $a(y, k)$ to obtain a well defined limit.

To get around this, with a device similar in spirit to how differences were considered in Definition 2.6, [SV21] considered the quantity

$$\mathcal{P}[(y)_k \to (\lambda, i)] - \mathcal{P}[(y)_k \to (\lambda, 1)]. \tag{10}$$

Owing to the fact that the starting points are the same, this quantity's limit can be shown to exist more easily, and it is finite. This is essentially because the finite geodesics corresponding to the two terms share the same path from their starting point to a certain height, i.e., *coalesce*, in a uniform way, only to separate at a unit order distance away from the destination points $(\lambda, 1)$ and (λ, i). This makes the difference of these geodesics' weights be of unit order.

Thus the difference in (10) is essentially an LPP problem from a random point, say the point of separation whose depth, as mentioned, is uniformly bounded in k.

In spite of the issues with defining its weight directly, the notion of an infinite geodesic through $\mathcal{P}$ can in fact be made precise, and this is the approach [SV21] adopts. They do this by carefully considering the limit of the finite geodesics mentioned in the previous paragraph. We shall not require that and choose simply to work with finite LPP problems with boundary data, as it avoids some of the technical issues in defining the infinite geodesic. However, we do make use of a few results of [SV21] which are proved via arguments involving infinite geodesics in order to ensure the existence of the boundary data and that it satisfies a relation involving $\mathcal{S}$ as in (9) which will be discussed next.

We now introduce the precise objects. We fix $\lambda \in \mathbb{R}$ for the rest of this section and, for $i \in \mathbb{N}$, define

$$\begin{aligned} a_i^\lambda &= \lim_{k\to\infty} \left(\mathcal{P}\big[(y_a)_k \to (\lambda, i)\big] - \mathcal{P}\big[(y_a)_k \to (\lambda, 1)\big] + \mathcal{S}(y_a, \lambda)\right) \\ b_i^\lambda &= \lim_{k\to\infty} \left(\mathcal{P}\big[(y_b)_k \to (\lambda, i)\big] - \mathcal{P}\big[(y_b)_k \to (\lambda, 1)\big] + \mathcal{S}(y_b, \lambda)\right). \end{aligned} \tag{11}$$

(The term $\mathcal{S}(y, \lambda)$ allows a formula like (9), see Lemma 2.8 below.) It is proved in [SV21, Theorem 3.7] that, for $\lambda = 0$ and every $i \in \mathbb{N}$, these limits exist and are finite almost surely; in fact, there exists a random integer K_i (which corresponds to the depth at which the geodesics from $(y)_k$ to (λ, i) and $(\lambda, 1)$ coalesce) such that the limiting values in (11) are achieved for all $k \geq K_i$.

While [SV21] states the existence of the limits (11) only for $\lambda = 0$, their argument applies verbatim for any fixed $\lambda \in \mathbb{R}$.

Next, we define some notation for the finite LPP value term in (9). Recall from (3) the functions $\mathcal{P}^\lambda_{i\to 1} : [0, \infty) \to \mathbb{R}$ given by

$$\mathcal{P}^\lambda_{i\to 1}(x) = \mathcal{P}[(\lambda, i) \to (\lambda + x, 1)]. \tag{12}$$

The following lemma is the rigorous analogue of (9) with the boundary data $\{a_i^\lambda\}_{i\in\mathbb{N}}$ and $\{b_j^\lambda\}_{j\in\mathbb{N}}$. It was proved in [SV21], again in the $\lambda = 0$ case, but the argument applies to $\lambda \in \mathbb{R}$.

Lemma 2.8 (Lemma 3.10 of [SV21])**.** *Fix $\lambda \in \mathbb{R}$. For all $x \geq 0$,*

$$\begin{aligned}\mathcal{S}(y_a, \lambda + x) &= \sup_{i\in\mathbb{N}} \left\{a_i^\lambda + \mathcal{P}_{i\to 1}^\lambda(x)\right\}\\ \mathcal{S}(y_b, \lambda + x) &= \sup_{j\in\mathbb{N}} \left\{b_j^\lambda + \mathcal{P}_{j\to 1}^\lambda(x)\right\}.\end{aligned} \tag{13}$$

Further, the supremums are almost surely achieved at finite values of i and j.

That the supremums in (13) are attained will be important for us, and we will need to understand properties of the supremum-achieving indices. This will require arguments involving the geometry of the geodesics for the finite LPP problems encoded by $\mathcal{P}_{i\to 1}^\lambda$, and the next section is devoted to developing these arguments.

2.4. **Geometry of geodesics.** For $\lambda \in \mathbb{R}$ fixed and $x \geq 0$, let $i^\lambda(x)$ be the minimum i which achieves the supremum in the first equation of (13), and $j^\lambda(x)$ be defined similarly for the second equation. We will need the following properties.

Lemma 2.9. *The following hold almost surely: (i) $i^\lambda(x) \leq j^\lambda(x)$ for all $x \geq 0$, (ii) i^λ and j^λ are left-continuous non-decreasing functions, and (iii) there exists (random) $\varepsilon > 0$ such that $i^\lambda(x) = j^\lambda(x) = 1$ for all $x \in [0, \varepsilon]$.*

The intuition behind Lemma 2.9 is that $i^\lambda(x)$ and $j^\lambda(x)$ should be thought of as the line index at λ of the infinite geodesic from y_a to $(\lambda + x, 1)$ and y_b to $(\lambda + x, 1)$ respectively. Since $y_a < y_b$, by planarity, the first infinite geodesic should be to the left of the second, which is encoded by (i) above. Similarly, as x increases, the geodesic corresponding to it should move to the right, which is encoded by (ii). The reason behind (iii) is an almost surely strict monotonicity property enjoyed by the sequences $\{a_i^\lambda\}_{i\in\mathbb{N}}$ and $\{b_j^\lambda\}_{j\in\mathbb{N}}$.

The weaker version, namely that the boundary data satisfy $a_{i+1}^\lambda \leq a_i^\lambda$ and $b_{i+1}^\lambda \leq b_i^\lambda$, is in fact immediate from the inequality $\mathcal{P}[(y_a)_k \to (\lambda, i+1)] \leq \mathcal{P}[(y_a)_k \to (\lambda, i)]$ for all k. The latter follows by considering the geodesic corresponding to the lefthand side and extending it to jump to line i at the last instant. The extended path is in the set of paths that $\mathcal{P}[(y_a)_k \to (\lambda, i)]$ is maximizing over, which gives the inequality. To show that it is almost surely strict we will make use of the Brownian Gibbs property, the first of many times in this paper.

Lemma 2.10. *Almost surely, $a_i^\lambda > a_{i+1}^\lambda$ and $b_j^\lambda > b_{j+1}^\lambda$ for every $i, j \in \mathbb{N}$.*

Proof. We give the proof for a_i^λ as that for b_i^λ is identical. Let K_i be the random constant introduced after the definition (11) of a_i^λ and b_i^λ. Then writing out their definition, we see that the inequality $a_i^\lambda > a_{i+1}^\lambda$ is implied if we show that, almost surely, for all $k \geq K_i$,

$$\mathcal{P}[(y_a)_k \to (\lambda, i)] > \mathcal{P}[(y_a)_k \to (\lambda, i+1)]. \tag{14}$$

Consider the probability one event Ω that, for each $i \in \mathbb{N}$, the function

$$x \mapsto \big(\mathcal{P}_i(\lambda - x) - \mathcal{P}_i(\lambda)\big) - \big(\mathcal{P}_{i+1}(\lambda - x) - \mathcal{P}_{i+1}(\lambda)\big), \tag{15}$$

defined for $x \in [0, 1]$, is strictly positive on a sequence of points $\{x_m^i\}_{m\in\mathbb{N}}$ such that $x_m^i < \lambda$ for all $i, m \in \mathbb{N}$ and $\lim_{m\to\infty} x_m^i = \lambda$ for all $i \in \mathbb{N}$. That this event, to be denoted by Ω, has probability one follows by using the absolute continuity of the displayed function to Brownian motion of rate four by the same argument as in Corollary 2.5; Brownian motion, of course, has the required property. We will show that (14) holds on Ω for all $i, k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. Let $\pi_{k\to i}$ be the geodesic from $(y_a)_k$ to (λ, i). We *claim* that, on Ω, it holds that $\pi_{k\to i}(\lambda^-) = i$ for each $i \in \mathbb{N}$, i.e., the geodesic does not jump to its destination line at the last instant. For suppose that $\pi_{k\to i}(\lambda^-) = j > i$. Note that j is finite since we are considering the geodesic from

$(y_a)_k$ and hence $j \le k$. The argument now is to construct, on this event, a path which has weight greater than $\pi_{k\to i}$. Let $(\lambda - \varepsilon, \lambda)$ be an interval on which $\pi_{k\to i} = j$, and let $x_0 \in (\lambda - \varepsilon, \lambda)$ be such that (15) with $i = j - 1$ is positive at x_0; such a point exists by the definition of Ω. Consider the modified path π' which is $\pi_{k\to i}(x)$ for $x < x_0$, equals $j - 1$ for $x \in (x_0, \lambda)$, and is i at $x = \lambda$. By our choice of x_0, the weight of π' exceeds that of $\pi_{k\to i}$, which is a contradiction.

Thus we know that for any given i, the geodesic $\pi_{k\to i+1}$ lies on $i + 1$ for a short interval $(\lambda - \varepsilon, \lambda]$. Now consider a point $x_0 < \lambda$ close to λ such that $\pi_{k\to i+1}(x_0) = i + 1$ and such that (15) is positive at x_0. We can do a similar path construction as described in the previous paragraph to find a path which agrees with $\pi_{k\to i+1}$ till x_0, and then equals i on $[x_0, \lambda]$; this is a path from $(y_a)_k$ to (λ, i) whose weight is greater than $\mathcal{P}[(y_a)_k \to (\lambda, i + 1)]$, and so we get (14). This completes the proof of Lemma 2.10. ☐

However, we will not need the monotonicity of $\{a_i^\lambda\}_{i\in\mathbb{N}}$ or $\{b_i^\lambda\}_{i\in\mathbb{N}}$ to prove parts (i) and (ii) of Lemma 2.9. Instead, we will require the monotonicity of the *difference*, i.e., that we have $a_{i+1}^\lambda - b_{i+1}^\lambda \le a_i^\lambda - b_i^\lambda$, which is a consequence of a simple planarity argument that we record next.

Lemma 2.11 (Crossing lemma)**.** *Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be any sequence of functions. Let $n_1 \ge m_1$, $n_2 \ge m_2$, $n_1 \le n_2$, and $m_1 \le m_2$. Also let $y_1 \le x_1$, $y_2 \le x_2$, $x_1 \le x_2$, and $y_1 \le y_2$. Then we have*

$$f[(y_1, n_1) \to (x_1, m_1)] + f[(y_2, n_2) \to (x_2, m_2)] \ge f[(y_1, n_1) \to (x_2, m_2)] + f[(y_2, n_2) \to (x_1, m_1)].$$

This lemma has appeared many times in the LPP literature, see for instance [DOV18, BBS20, CLP19]. See also Figure 4 for a visual aid for the proof.

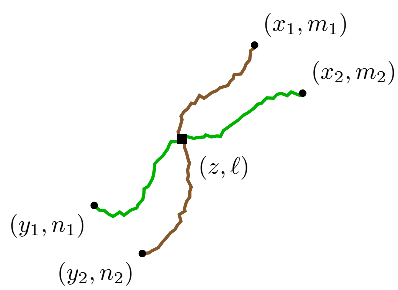

Figure 4. An illustration of a choice of starting and ending points which ensures, by planarity, that the geodesics between those points must intersect. The intersection, depicted by a square, is at (z, ℓ). By following the brown geodesic from (y_2, n_2) to (z, ℓ), and then following the green geodesic from (z, ℓ) to (x_2, m_2), we obtain a path from (y_2, n_2) to (x_2, m_2) whose weight can be at most $f[(y_2, n_2) \to (x_2, m_2)]$, and similarly for from (y_1, n_1) to (x_1, m_1).

Proof of Lemma 2.11. By the positioning of the points assumed and planarity, we have that any geodesic from (y_1, n_1) to (x_2, m_2) must cross any geodesic from (y_2, n_2) to (x_1, m_1) at some point (z, ℓ), with $z \in [y_2, x_1]$ and $m_2 \le \ell \le n_1$. This implies

$$\begin{aligned} f[(y_1, n_1) \to (x_2, m_2)] + f[(y_2, n_2) \to (x_1, m_1)] &= f[(y_1, n_1) \to (z, \ell)] + f[(z, \ell) \to (x_2, m_2)] \\ &\quad + f[(y_2, n_2) \to (z, \ell)] + f[(z, \ell) \to (x_1, m_1)]. \end{aligned}$$

But we have that $f[(y_1, n_1) \to (z, \ell)] + f[(z, \ell) \to (x_1, m_1)] \le f[(y_1, n_1) \to (x_1, m_1)]$ as the lefthand side is the weight of a particular up-right path from (y_1, n_1) to (x_1, m_1). We have a similar inequality with y_2, n_2, x_2, m_2. Applying these inequalities to the last display completes the proof of Lemma 2.11. ☐

Corollary 2.12. *Suppose $j \geq i$. Then we have $b_j^\lambda - a_j^\lambda \geq b_i^\lambda - a_i^\lambda$.*

Proof. Recalling the definitions (11) of a_i^λ and b_j^λ, it is enough to show that, for all $k \in \mathbb{N}$,

$$\mathcal{P}[(y_b)_k \to (\lambda, j)] - \mathcal{P}[(y_a)_k \to (\lambda, j)] \geq \mathcal{P}[(y_b)_k \to (\lambda, i)] - \mathcal{P}[(y_a)_k \to (\lambda, i)]$$

and then take the limit as $k \to \infty$. This last inequality is an immediate consequence of Lemma 2.11 since $y_a \leq y_b$ implies $(y_a)_k \leq (y_b)_k$. ☐

Remark 2.13. It is not the case that $b_j^\lambda - a_j^\lambda > b_i^\lambda - a_i^\lambda$ almost surely, i.e., strict inequality does not always hold, though it may hold with positive probability. Indeed, observe from the proof of Lemma 2.11 that equality holds in that lemma when the geodesic from (y_1, n_1) to (x_1, m_1) coalesces with the geodesic from (y_2, n_2) to (x_2, m_2), i.e, when those two geodesics are not disjoint. This does occur for finite geodesics from $(y_a)_k$ to $(y_b)_k$ in $\mathcal{P}$ with positive probability, and the coalescence asserted via the existence of K_i in the definition (11) of a_i^λ and b_i^λ implies that equality can hold in the limit as $k \to \infty$ as well with positive probability.

The outstanding proof of Lemma 1.1 is also an immediate consequence of Lemma 2.11:

Proof of Lemma 1.1. We have to show that $\mathcal{S}(y_b, x_1) - \mathcal{S}(y_a, x_1) \leq \mathcal{S}(y_b, x_2) - \mathcal{S}(y_a, x_2)$ whenever $x_1 \leq x_2$, i.e.,

$$\mathcal{S}(y_b, x_2) + \mathcal{S}(y_a, x_1) \geq \mathcal{S}(y_b, x_1) + \mathcal{S}(y_a, x_2).$$

The analogous inequality for Brownian LPP follows from Lemma 2.11, and the convergence of Brownian LPP to the parabolic Airy sheet (Theorem 2.7) implies the displayed inequality. ☐

Now we turn to the proof of Lemma 2.9 on the ordering and monotonicity properties of i^λ and j^λ. Based on the intuition in terms of planarity of infinite geodesics explained after Lemma 2.9, the proof idea for each property is to show that assuming the contrary leads to a contradiction between the definitions of i and j as maximizers and the relations imposed by planarity via Corollary 2.12.

Proof of Lemma 2.9. We start with (i). Suppose to the contrary that there is an $x^* \geq 0$ such that $i^\lambda(x^*) > j^\lambda(x^*)$. Let us for shorthand call these values i and j. Then,

$$a_i^\lambda + \mathcal{P}_{i\to 1}^\lambda(x^*) > a_j^\lambda + \mathcal{P}_{j\to 1}^\lambda(x^*)$$
$$b_j^\lambda + \mathcal{P}_{j\to 1}^\lambda(x^*) \geq b_i^\lambda + \mathcal{P}_{i\to 1}^\lambda(x^*);$$

the first is a strict inequality since $i = i^\lambda(x^*)$ is the minimum index which achieves the supremum in (13) and we have assumed $i > j$. The above pair of inequalities implies that $a_i^\lambda - b_i^\lambda > a_j^\lambda - b_j^\lambda$, which contradicts Corollary 2.12.

Next we turn to (ii). The proof is the same for both i^λ and j^λ and we show here that i^λ is left-continuous and non-decreasing. We start with showing that i^λ is non-decreasing.

Again to the contrary, suppose we have $x_1 < x_2$ with $i^\lambda(x_1) > i^\lambda(x_2)$. Let $i_1 = i^\lambda(x_1)$ and $i_2 = i^\lambda(x_2)$ so that $i_1 > i_2$. Then,

$$a_{i_1}^\lambda + \mathcal{P}_{i_1\to 1}^\lambda(x_1) > a_{i_2}^\lambda + \mathcal{P}_{i_2\to 1}^\lambda(x_1)$$
$$a_{i_2}^\lambda + \mathcal{P}_{i_2\to 1}^\lambda(x_2) \geq a_{i_1}^\lambda + \mathcal{P}_{i_1\to 1}^\lambda(x_2);$$

again the first inequality is strict since i_1 is the minimum index which achieves the supremum in (13). These two inequalities combined imply that

$$\mathcal{P}_{i_1\to 1}^\lambda(x_1) + \mathcal{P}_{i_2\to 1}^\lambda(x_2) > \mathcal{P}_{i_1\to 1}^\lambda(x_2) + \mathcal{P}_{i_2\to 1}^\lambda(x_1),$$

which is equivalent to

$$\mathcal{P}[(\lambda, i_1) \to (\lambda + x_1, 1)] + \mathcal{P}[(\lambda, i_2) \to (\lambda + x, 1)]$$

$$> \mathcal{P}[(\lambda, i_1) \to (\lambda + x_2, 1)] + \mathcal{P}[(\lambda, i_2) \to (\lambda + x_1, 1)];$$

but since $i_1 > i_2$ and $x_1 < x_2$, this contradicts Lemma 2.11.

Now we turn to showing i^λ is left-continuous. Left-continuity is a consequence of the fact that $a_i^\lambda + \mathcal{P}_{i\to 1}^\lambda(x)$ is a continuous function of x for each $i \in \mathbb{N}$ and $i^\lambda(x)$ is defined as the minimum index which achieves the supremum in (13). Indeed, suppose that $\lim_{x\uparrow x^*} i^\lambda(x) = i^*$. This implies that

$$a_{i^*}^\lambda + \mathcal{P}_{i^*\to 1}^\lambda(x) \geq a_i^\lambda + \mathcal{P}_{i\to 1}^\lambda(x)$$

for all $i \neq i^*$ and $x \in [x^* - \varepsilon, x^*)$ with $\varepsilon > 0$ sufficiently small, since, for x in this interval, $i^\lambda(x) = i^*$. Taking the limit of $x \uparrow x^*$ shows that the displayed inequality also holds for $x = x^*$. Since $i^\lambda(x^*)$ is the minimum index achieving the supremum in (13), we have that $i^\lambda(x^*) \leq i^*$. But the monotonicity of i^λ implies $i^\lambda(x^*) \geq i^*$, which completes the proof of part (ii) of Lemma 2.9.

For part (iii), observe from Lemma 2.10 that $a_i^\lambda > a_{i+1}^\lambda$ for each $i \in \mathbb{N}$ almost surely. For $\delta > 0$, let M_δ be such that $i^\lambda(1) \leq M_\delta$ with probability at least $1 - \delta$. On this event, we know $i^\lambda(x) \leq M_\delta$ for all $x \in [0, 1]$ by the monotonicity of i^λ. We pick $\varepsilon > 0$ such that $a_1 + \mathcal{P}_{1\to 1}^\lambda(x) \geq a_i + \mathcal{P}_{i\to 1}^\lambda(x)$ for all $x \in [0, \varepsilon]$ and $i \in \mathbb{N}$, which is possible by the strict inequalities of a_i^λ, since $\mathcal{P}_{i\to 1}^\lambda(0) = 0$, and the uniform continuity of $\{\mathcal{P}_{i\to 1}^\lambda : i \in [\![1, M_\delta]\!]\}$ on $[0, 1]$. This implies that $i^\lambda(x) = 1$ for all $x \in [0, \varepsilon]$. Here ε was defined on an event with probability at least $1 - \delta$. Taking $\delta \to 0$ along a countable sequence, and defining $\varepsilon > 0$ on each resulting event, completes the proof of item (iii) of Lemma 2.9. □

3. Absolute continuity with respect to Brownian local time

In this section we identify the processes which will play the role of the patch functions $\{Y_k\}_{k\in\mathbb{Z}}$ in Definition 1.3 and prove that they are absolutely continuous on compact intervals away from zero to Brownian local time; see Proposition 3.3. We start by formally defining Brownian local time (at zero), which, heuristically, measures the amount of time Brownian motion spends at the origin.

Definition 3.1 (Brownian local time)**.** Let $B : [0, \infty) \to \mathbb{R}$ be a one-dimensional Brownian motion of rate σ^2 started at the origin. The associated Brownian local time at zero $\mathcal{L} : [0, \infty) \to [0, \infty)$ of rate σ^2 is defined as

$$\mathcal{L}(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{-\varepsilon < B(s) < \varepsilon} \, \mathrm{d}s.$$

In our arguments, we will generally relate last passage values to the running maximum process of various processes which are absolutely continuous to Brownian motion. We move from these running maximum processes to the local time process of Brownian motion via the famous identity of Lévy; see for example [MP10, Theorems 6.16 and 7.38].

Proposition 3.2 (Lévy's identity)**.** *Let $B : [0, \infty) \to \mathbb{R}$ be Brownian motion of rate σ^2 started at the origin. Let $\mathcal{L} : [0, \infty) \to [0, \infty)$ be its associated local time at zero and $M : [0, \infty) \to [0, \infty)$ be its running maximum process, i.e., $M(t) = \sup_{s\leq t} B(s)$. Then $(\mathcal{L}, |B|)$ and $(M, M - B)$ are equal in law as processes on $[0, \infty)$.*

We will be making many absolute continuity statements in this section, and we introduce some convenient notation and terminology to streamline this. For two probability measures μ_1 and μ_2 on a measure space, $\mu_1 \ll \mu_2$ will denote that μ_1 is absolutely continuous with respect to μ_2. We will often abuse notation and, for random variables X_1 and X_2, say that $X_1 \ll X_2$ to mean that the distributions of X_1 and X_2 satisfy the corresponding relation; obviously, the joint distribution of X_1 and X_2 has no relevance to the statement. When for two processes X and Y defined on $[0, \infty)$ we have that $X|_{[\varepsilon,T]} \ll Y|_{[\varepsilon,T]}$ for every $0 < \varepsilon < T$, we will sometimes say that X is *locally absolutely continuous away from zero* to Y.

In this section too we fix $\lambda \in \mathbb{R}$, and we recall the definition of $\mathcal{P}_i^\lambda$ from (12). Finally, here and in the rest of this section, a continuous function $f : [0, \infty) \to \mathbb{R}$ is said to have a *point of increase* at the origin if $f(x) > f(0)$ for all $x > 0$.

The goal of this section is to prove the following proposition.

Proposition 3.3. *Let $i < j$. There is a process $X : [0, \infty) \to \mathbb{R}$ with $X|_{[\varepsilon,T]} \ll B|_{[\varepsilon,T]}$ for all $0 < \varepsilon < T$ with B a Brownian motion of rate four, such that $(\mathcal{P}_{j\to 1}^\lambda - \mathcal{P}_{i\to 1}^\lambda)(t) = \max_{0\le s\le t} X(s)$. Further, $(\mathcal{P}_{j\to 1}^\lambda - \mathcal{P}_{i\to 1}^\lambda)|_{[\varepsilon,T]} \ll \mathcal{L}|_{[\varepsilon,T]}$, where $\mathcal{L}$ is Brownian local time of rate four. Finally, $\mathcal{P}_{j\to 1}^\lambda - \mathcal{P}_{i\to 1}^\lambda$ almost surely has a point of increase at the origin.*

Here we consider a restriction to $[\varepsilon, T]$ as we expect the processes to exhibit singular behaviour "at" the origin; we will say more about this shortly in Remark 3.11.

The following is an overview of the proof of Proposition 3.3. We start with the environment given by $\{\mathcal{P}_\ell^\lambda(\cdot) : \ell \in [\![1, j]\!]\}$ where the latter was defined in (3).

As outlined in Section 1.4.3, we make multiple use *Pitman transforms* which we formally define next. For continuous functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ with $f_1(0) = f_2(0) = 0$, define the Pitman transform $\mathsf{PT}(f_1, f_2) : [0, \infty) \to \mathbb{R}^2$ by

$$\begin{aligned} \big(\mathsf{PT}(f_1, f_2)\big)_1(x) &= f[(0, 2) \to (x, 1)] \\ &= f_1(x) + \max_{0\le s\le x} (f_2(s) - f_1(s))\,; \\ \big(\mathsf{PT}(f_1, f_2)\big)_2(x) &= f_1(x) + f_2(x) - f[(0, 2) \to (x, 1)] \\ &= f_2(x) - \max_{0\le s\le x} (f_2(s) - f_1(s))\,. \end{aligned} \tag{16}$$

Observe that the first component is nothing but the solution of the LPP problem over the two lines (f_1, f_2); this gives a hint of an important fact that we rely on, namely that composing Pitman transforms in particular ways on an environment of k curves gives the LPP value across the k curves.

In fact, and more precisely, there are many sequences of Pitman transforms which, when applied to adjacent curves within the environment of k curves $\mathcal{P}_1, \ldots, \mathcal{P}_k$, take the original environment into a new sequence of k curves, such that the new top curve is the last passage value $\mathcal{P}_{k\to 1}^\lambda$. An important idea of Proposition 3.3 is to find a sequence of Pitman transforms which have the property that, when applied to the environment, they yield an ensemble of lines with the top line being $\mathcal{P}_{i\to 1}^\lambda$, and, on applying a *final* Pitman transform to the top two lines, turns the top line into $\mathcal{P}_{j\to 1}^\lambda$.

This last line along with (16) gives a representation of $\mathcal{P}_{j\to 1}^\lambda - \mathcal{P}_{i\to 1}^\lambda$ as the running maximum of a complicated third process X. The form of (16) suggests that X will be the difference of two processes which, by the Brownian Gibbs property, are plausibly jointly absolutely continuous to independent rate two Brownian motions. This will allow us to compare X to rate four Brownian local time.

However, we will in fact only make claims about local absolute continuity away from zero, i.e, that $X|_{[\varepsilon,T]} \ll B|_{[\varepsilon,T]}$ for all $0 < \varepsilon < T$ (with B a rate four Brownian motion), and, similarly, that this implies that $(\mathcal{P}_{j\to 1}^\lambda - \mathcal{P}_{i\to 1}^\lambda)|_{[\varepsilon,T]} \ll \mathcal{L}|_{[\varepsilon,T]}$ for all $0 < \varepsilon < T$.

Thus we will need to know that applying Pitman transforms to a pair of processes which are locally absolutely continuous away from zero to a pair of independent Brownian motions preserves this property. This is Lemma 3.5. As a preliminary step, we will need to show that the same occurs when the transform is applied to independent Brownian motions; this is Lemma 3.4. That the running maximum of a process which is locally absolutely continuous away from zero to Brownian motion is itself locally absolutely continuous away from zero to Brownian local time is Lemma 3.6. These three statements and their proofs comprise Section 3.1. Then in Section 3.2 we prove Proposition 3.3

according to the above outline, after recalling the details of which sequences of Pitman transforms yield the top curve as the LPP function.

3.1. **Absolute continuity properties of Pitman transforms.** To prove our claims about absolute continuity to Brownian local time or to Brownian motion on intervals only away from zero, since the Pitman transform involves the full function from zero onwards in the form of the maximizations in (16), it will be convenient to have a form of PT which excludes behaviour of the input functions near the origin.

To this end, for $\delta \geq 0$, we define a truncated Pitman transform $\mathsf{PT}^\delta(f_1, f_2) : [\delta, \infty) \to \mathbb{R}^2$ by replacing the condition $0 \leq s \leq x$ over which the maximizations are done in (16) by the condition $\delta \leq s \leq x$, i.e.,

$$\begin{aligned}\big(\mathsf{PT}^\delta(f_1, f_2)\big)_1(x) &= f_1(x) + \max_{\delta \leq s \leq x} (f_2(s) - f_1(s))\\ \big(\mathsf{PT}^\delta(f_1, f_2)\big)_2(x) &= f_2(x) - \max_{\delta \leq s \leq x} (f_2(s) - f_1(s)).\end{aligned}$$

The $\delta = 0$ case gives back the original Pitman transform.

Our first lemma says that the truncated Pitman transform of independent Brownian motions is locally absolutely continuous away from zero to independent Brownian motions.

Lemma 3.4. *Let $B_1, B_2 : [0, \infty) \to \mathbb{R}$ be independent rate two Brownian motions, and $0 \leq \delta < \varepsilon < T$. Then $\mathsf{PT}^\delta(B_1, B_2)|_{[\varepsilon,T]} \ll (B_1, B_2)|_{[\varepsilon,T]}$.*

A special case of this statement where $\delta = 0$ and $(\mathsf{PT}(B_1, B_2))_1$ is compared to a single Brownian motion is proven in [SV21, Lemma 4.1], and our proof is similar.

Proof of Lemma 3.4. Observe that we can write $\mathsf{PT}^\delta(B_1, B_2)$ as follows:

$$\begin{aligned}\big(\mathsf{PT}^\delta(B_1, B_2)\big)_1(x) &= B_1(x) - B_1(\delta) + \max_{\delta \leq s \leq x} \big(\big[B_2(s) - B_2(\delta)\big] - \big[B_1(s) - B_1(\delta)\big]\big) + B_2(\delta)\\ \big(\mathsf{PT}^\delta(B_1, B_2)\big)_2(x) &= B_2(x) - B_2(\delta) - \max_{\delta \leq s \leq x} \big(\big[B_2(s) - B_2(\delta)\big] - \big[B_1(s) - B_1(\delta)\big]\big) + B_1(\delta).\end{aligned}$$

This implies, by the Markov property of Brownian motion, that

$$\mathsf{PT}^\delta(B_1, B_2)(\cdot) \stackrel{d}{=} \mathsf{PT}(B_1, B_2)(\cdot - \delta) + N,$$

where the equality in distribution is as processes on $[\delta, \infty)$, and $N = (B_2(\delta), B_1(\delta))$ is a random element of $\mathbb{R}^2$ whose components are independent normal random variables of mean zero and variance 2δ which are also independent of $B_1(\cdot) - B_1(\delta)$ and $B_2(\cdot) - B_2(\delta)$ as processes on $[\delta, \infty)$. Now, it is sufficient to show that $\mathsf{PT}(B_1, B_2)(\cdot - \delta)|_{[\varepsilon,T]} \ll (B_1, B_2)|_{[\varepsilon,T]}$ for all $\delta < \varepsilon < T$, because of the independence of N and since $(B_1(\varepsilon), B_2(\varepsilon))$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$.

So we now show that, for any $\eta > 0$, $\mathsf{PT}(B_1, B_2)|_{[\eta,T]} \ll (B_1, B_2)|_{[\eta,T]}$. Note that we have adopted an implicit relabeling of T in the domain of $\mathsf{PT}(B_1, B_2)$, and translated the domain of (B_1, B_2) so that the domains of both objects on which we do the comparison are the same. The change in domain of (B_1, B_2) is inconsequential as it is easy to see that $(B_1, B_2)|_{[\eta_1,T]}$ is mutually absolutely continuous with $(B_1, B_2)|_{[\eta_1+\eta_2,T+\eta_2]}$ for any $\eta_1, \eta_2 > 0$.

That $\mathsf{PT}(B_1, B_2)|_{[\eta,T]} \ll (B_1, B_2)|_{[\eta,T]}$ follows by recalling that $\mathsf{PT}(B_1, B_2)$ is 2-level Dyson Brownian motion (of rate two), which has the distribution of two independent Brownian motions of rate two conditioned to not intersect (with the singular conditioning at the origin made precise via a Doob h-transform [Gra99]). The explicit entrance law of Dyson Brownian motion implies that $\mathsf{PT}(B_1, B_2)(\eta)$ is absolutely continuous to Lebesgue measure on the quadrant $\{(x, y) \in \mathbb{R}^2 : x \geq y\}$, and the transition probability formula shows that, conditionally on $\mathsf{PT}(B_1, B_2)(\eta)$, $\mathsf{PT}(B_1, B_2)|_{[\eta,T]}$

has the law of two independent Brownian motions of rate two with appropriate starting points conditioned not to intersect. The latter is absolutely continuous to the law of two independent Brownian motions of rate two on $[\delta, T]$ with the same starting points since they are almost surely distinct. Together, this implies that $\mathsf{PT}(B_1, B_2)|_{[\eta,T]} \ll (B_1, B_2)|_{[\eta,T]}$ and completes the proof of Lemma 3.4. ☐

The next lemma uses the previous one to say that if the input functions to the Pitman transform are only known to be absolutely continuous away from zero to independent Brownian motions, the same will be true of the output functions. The form of this lemma allows it to be applied iteratively in an induction argument, as we will do in the proof of Proposition 3.3.

Lemma 3.5. *Let $B_1, B_2 : [0,\infty) \to \mathbb{R}$ be independent rate two Brownian motions. Suppose $X_1, X_2 : [0,\infty) \to \mathbb{R}$ satisfy $X_1(0) = X_2(0) = 0$ and are such that, almost surely, $X_2 - X_1$ has positive maximum over every neighbourhood of the origin. Further assume that, for every $0 < \varepsilon < T$, $(X_1, X_2)|_{[\varepsilon,T]} \ll (B_1, B_2)|_{[\varepsilon,T]}$. Then, for every $0 < \varepsilon < T$,*

$$\mathsf{PT}(X_1, X_2)|_{[\varepsilon,T]} \ll (B_1, B_2)|_{[\varepsilon,T]}.$$

The basic idea of the proof is to use the hypothesis that the origin is not the maximizer of $X_2 - X_1$ in any neighbourhood to equate $\mathsf{PT}(X_1, X_2)$ and $\mathsf{PT}^\delta(X_1, X_2)$ for a small random $\delta > 0$. Then $\mathsf{PT}^\delta(X_1, X_2) \ll \mathsf{PT}^\delta(B_1, B_2)$ (which need not be true when $\delta = 0$) and we can make use of Lemma 3.4.

Proof of Lemma 3.5. Fix $0 < \varepsilon < T$. For notational convenience, let $Y^\varepsilon : [\varepsilon, T] \to \mathbb{R}^2$ be $\mathsf{PT}(X_1, X_2)|_{[\varepsilon,T]}$. Next, let $A \subseteq \mathcal{C}([\varepsilon, T])$ (recall that the latter denotes the set of all continuous functions on $[\varepsilon, T]$) be a measurable event such that

$$\mathbb{P}\Big((B_1, B_2)\big|_{[\varepsilon,T]} \in A\Big) = 0. \tag{17}$$

We must show that

$$\mathbb{P}\big(Y^\varepsilon \in A\big) = 0. \tag{18}$$

Let E_n be defined by (where $\max \varnothing = -\infty$)

$$E_n = \left\{ \max_{0 \le s \le x} (X_2(s) - X_1(s)) = \max_{n^{-1} \le s \le x} (X_2(s) - X_1(s)) \text{ for all } x \in [\varepsilon, T] \right\}.$$

Let $E = \cup_{n=2\lceil \varepsilon^{-1} \rceil}^{\infty} E_n$. We have that

$$\mathbb{P}\big(Y^\varepsilon \in A\big) \le \sum_{n=2\lceil \varepsilon^{-1} \rceil}^{\infty} \mathbb{P}\big(Y^\varepsilon \in A, E_n\big) + \mathbb{P}\big(Y^\varepsilon \in A, E^c\big). \tag{19}$$

We will show that each summand is zero. First we deal with the term containing E^c. On E^c, the unique maximizer of $X_2(s) - X_1(s)$ on $s \in [0, \varepsilon]$ is attained at $s = 0$, where $X_2 - X_1$ equals zero. Therefore by our hypothesis that the maximum on $[0, \varepsilon]$ is positive, $\mathbb{P}(E^c) = 0$.

Now we handle the terms in the summation in (19). It holds on E_n that $Y^\varepsilon = \mathsf{PT}^{1/n}(X_1, X_2)$ on $[\varepsilon, T]$. Since $\mathsf{PT}^{1/n}(X_1, X_2)$ is a function of $(X_1, X_2)|_{[n^{-1},T]}$, by our hypothesis it is absolutely continuous with respect to $\mathsf{PT}^{1/n}(B_1, B_2)$. Now since $\varepsilon > n^{-1}$, Lemma 3.4 with $\delta = n^{-1}$ asserts that $\mathsf{PT}^{1/n}(B_1, B_2)|_{[\varepsilon,T]} \ll (B_1, B_2)|_{[\varepsilon,T]}$. This, along with (17), implies that the summand of (19) indexed by n equals zero, completing the proof of Lemma 3.5. ☐

Finally, while the previous two lemmas were regarding absolute continuity to Brownian motion, we will need to make a comparison to Brownian local time. This is done in the form of comparing the running maximum of a process which is locally absolutely continuous away from zero to Brownian motion to the running maximum of Brownian motion.

Lemma 3.6. *Let B be a Brownian motion of rate σ^2. Suppose $X : [0,\infty) \to \mathbb{R}$ is continuous and such that $X|_{[\varepsilon,T]} \ll B|_{[\varepsilon,T]}$ for every $0 < \varepsilon < T$, $X(0) = 0$, and X almost surely has positive maximum over every neighbourhood of zero. Let $M^B(t) = \max_{0\le s\le t} B(s)$ and $M^X(t)$ be defined similarly. Then, for every $0 < \varepsilon < T$, $M^X|_{[\varepsilon,T]} \ll M^B|_{[\varepsilon,T]}$.*

Because here we compare with running maximum of Brownian motion instead of Brownian motion itself, the earlier trick of isolating the origin and considering the maximization away from it, as used in Lemma 3.5, does not work; essentially because $\max_{\varepsilon\le s\le t} B(s)$ is not absolutely continuous to $\max_{0\le s\le t} B(s)$ (both as processes of $t \in [\varepsilon, T]$). For this reason, the proof of Lemma 3.6 is technically slightly more subtle. Instead of isolating the origin, we first perform a "surgery" near the origin to get a process which is absolutely continuous to B on the entire interval $[0, T]$.

Proof. Let $\eta > 0$. Define $X^\eta : [0,\infty) \to \mathbb{R}$ by

$$X^\eta(x) = \begin{cases} B^{\mathrm{br},\eta}(x) & 0 \le x \le \eta \\ X(x) & x \ge \eta, \end{cases}$$

where, conditionally on $X|_{[\eta,\infty)}$, $B^{\mathrm{br},\eta}$ is an independent Brownian bridge from $(0, 0)$ to $(\eta, X(\eta))$. We may couple X^η such that $B^{\mathrm{br},\eta}$ is constructed for all $\eta > 0$ by scaling and affine shifting of a single independent Brownian bridge from $(0, 0)$ to $(1, 0)$.

We claim that $X^\eta \ll B$ as processes on $[0, T]$. Indeed, this can be seen by decomposing the Brownian motion path on $[0, T]$ in a similar way as the definition of X^η. First, sample a Brownian motion $\tilde{B}$ on $[0, T]$ and retain the portion on $[\eta, T]$; by hypothesis, $X|_{[\eta,T]}$ is absolutely continuous to the law of this portion, i.e., the law of $\tilde{B}|_{[\eta,T]}$. Then, conditionally on $\tilde{B}|_{[\eta,T]}$, sample an independent Brownian bridge $B^{\mathrm{br},\eta}$ with endpoints $(0, 0)$ and $(\eta, \tilde{B}(\eta))$. The resulting process on $[0, T]$ has the law of Brownian motion. Since we have performed the same sequence of operations to obtain X^η or B, depending on whether we started with $X|_{[\eta,T]}$ or $B|_{[\eta,T]}$, our claim is proved.

Now, let $\varepsilon > 0$ and $A \subseteq \mathcal{C}([\varepsilon, T])$ be an event such that $\mathbb{P}(M^B|_{[\varepsilon,T]} \in A) = 0$. We must show that

$$\mathbb{P}\left(M^X|_{[\varepsilon,T]} \in A\right) = 0. \tag{20}$$

Let $\delta > 0$ be given. We claim that there exists $0 < \eta < \varepsilon$ small enough such that, with probability at least $1 - \delta$, and for all $t \in [\varepsilon, T]$,

$$\max_{0\le s\le t} X(s) = \max_{0\le s\le t} X^\eta(s). \tag{21}$$

This is essentially a consequence of continuity of X. To prove this, we first find a simpler event which implies (21). Observe that for any $\eta \ge 0$ (noting that $X^\eta = X$ when $\eta = 0$), since $X^\eta|_{[\eta,T]} = X|_{[\eta,T]}$,

$$\max_{0\le s\le t} X^\eta(s) = \max\left\{\max_{0\le s\le \eta} X^\eta(s), \max_{\eta\le s\le t} X(s)\right\}. \tag{22}$$

Since $\varepsilon > \eta$, (22) implies that (21) is equivalent to the $t = \varepsilon$ case, i.e., to $\max_{0\le s\le\varepsilon} X(s) = \max_{0\le s\le\varepsilon} X^\eta(s)$. Now taking $t = \varepsilon$ in (22), and since $B^{\mathrm{br},\eta}|_{[0,\eta]} = X^\eta|_{[0,\eta]}$ for all $\eta > 0$, we see that (21) is implied by the event that

$$\max\left\{\max_{0\le s\le\eta} X(s), \max_{0\le s\le\eta} B^{\mathrm{br},\eta}(s)\right\} \le \max_{\eta\le s\le\varepsilon} X(s). \tag{23}$$

We will show that this event holds with probability at least $1 - \delta$ for $\eta > 0$ small enough.

Since $X(0) = 0$ and is continuous almost surely, and by our coupling of X^η across $\eta > 0$, the lefthand side of (23) decreases to 0 almost surely as $\eta \to 0$. Since the maximum of X over every neighbourhood of zero is strictly positive, the righthand side of (23) has a strictly positive limit as $\eta \to 0$. Thus there exists $\eta > 0$ such that (21) holds with probability at least $1 - \delta$.

Now we see that, for such an η,

$$\mathbb{P}\left(M^X|_{[\varepsilon,T]} \in A\right) \le \mathbb{P}\left(M^{X^\eta}|_{[\varepsilon,T]} \in A\right) + \delta = \delta,$$

the last equality since $X^\eta|_{[0,T]} \ll B|_{[0,T]}$ and $\mathbb{P}(M^B|_{[\varepsilon,T]} \in A) = 0$. Since δ was arbitrary, we have verified (20) and proved Lemma 3.6. ☐

3.2. **Proving Proposition 3.3.** Recall that we mentioned earlier that certain sequences of Pitman transforms, applied to an environment of k curves, results in the top most curve of the transformed environment being the last passage value function across the original k curves. We start this section by specifying this more precisely.

Given an environment $f : [\![1, n]\!] \times [0, \infty) \to \mathbb{R}$ of n curves, we will apply the Pitman transform (PT) to adjacent pairs, say the k^{th} and $(k+1)^{\text{st}}$ curves. We associate this application of PT with the adjacent transposition $(k\ k+1)$. Indeed, denote the obtained transformed environment by $\sigma_{(k\ k+1)}f$, which is defined by $(\sigma_{(k\ k+1)}f)_\ell(\cdot) = f_\ell(\cdot)$ for $\ell \neq k, k+1$ and

$$(\sigma_{(k\ k+1)}f)_k(\cdot) = \big(\mathsf{PT}(f_k, f_{k+1})\big)_1(\cdot),$$
$$(\sigma_{(k\ k+1)}f)_{k+1}(\cdot) = \big(\mathsf{PT}(f_k, f_{k+1})\big)_2(\cdot).$$

A sufficient condition that a sequence of Pitman transforms results in the top curve being the LPP value across the environment, i.e., equals $x \mapsto f[(0, n) \to (x, 1)]$ can be expressed in terms of the compositions of the associated adjacent transpositions. Indeed, suppose $N = \binom{n}{2}$ adjacent transpositions $\sigma_1, \ldots, \sigma_N$ are such that $\sigma_1 \circ \cdots \circ \sigma_N$ is the reverse permutation rev_n on $[\![1, n]\!]$. (N is the minimum number such that this is possible.) Then,

$$\big((\sigma_1 \circ \cdots \circ \sigma_N)f\big)_1(x) = f[(0, n) \to (x, 1)],$$

where $(\sigma_1 \circ \cdots \circ \sigma_N)f = \sigma_1(\sigma_2(\cdots(\sigma_N f))\cdots)$ is the environment obtained by applying the associated Pitman transforms iteratively right to left (see [DOV18, Section 4] and the references therein, or [BBO05]).

(Setting $\tau = \sigma_1 \circ \cdots \sigma_k$, we will at times adopt the slight abuse of notation that τf is the environment obtained by iteratively applying the Pitman transforms associated to $\sigma_1, \ldots, \sigma_k$ from right-to-left, keeping in mind that in general τf is not well-defined without its expression as a particular product of adjacent transpositions specified.)

Our previous displayed equation implies that if, for some $k \le N$, $\sigma_1, \ldots, \sigma_k$ are adjacent transpositions such that there exist $N - k$ adjacent transpositions $\rho_1, \ldots, \rho_{N-k}$ *that act only on* $\{2, \ldots, n\}$ such that $\rho_1 \circ \cdots \circ \rho_{N-k} \circ \sigma_1 \circ \cdots \sigma_k = \mathrm{rev}_n$, then also

$$\big((\sigma_1 \circ \cdots \circ \sigma_k)f\big)_1(x) = f[(0, n) \to (x, 1)], \tag{24}$$

for the Pitman transforms corresponding to $\rho_1, \ldots, \rho_{N-k}$ do not modify the top line.

There is a simple condition that ensures, for given $\sigma_1, \ldots, \sigma_k$, that there exist $N - k$ more adjacent transpositions whose left-composition with the original k is rev_n. Let us recall the inversion number $\mathrm{Inv}(\sigma)$ of a permutation σ on $[\![1, n]\!]$, defined as

$$\mathrm{Inv}(\sigma) = \#\big\{(i, j) \in [\![1, n]\!]^2 : i < j \text{ and } \sigma(i) > \sigma(j)\big\}.$$

Then the mentioned condition is the following lemma.

Lemma 3.7. *Let σ be a permutation on $[\![1, n]\!]$, $k = \mathrm{Inv}(\sigma)$, and $N = \binom{n}{2}$. Then there exists a sequence of $N - k$ adjacent transpositions $\rho_1, \ldots, \rho_{N-k}$ such that $\rho_1 \circ \cdots \circ \rho_{N-k} \circ \sigma = \mathrm{rev}_n$.*

Proof. We may assume $N - k \ge 1$. It is sufficient to find a single adjacent transposition ρ such that $\mathrm{Inv}(\rho \circ \sigma) = k + 1$ and then iterate. Now, since $\sigma \neq \mathrm{rev}_n$, there must exist an i such that $\sigma^{-1}(i) < \sigma^{-1}(i + 1)$. Then it is easy to check that $\mathrm{Inv}(\rho \circ \sigma) = k + 1$ when $\rho = (i\ i + 1)$. ☐

Using this, we can obtain a simple criterion that ensures that (24) holds which will be used several times.

Lemma 3.8. *Let* $n \in \mathbb{N}$, $\ell \in [\![1, n]\!]$, $\sigma_1, \ldots, \sigma_k$ *be adjacent transpositions on* $[\![1, \ell]\!]$, *and* $\tau = \sigma_1 \circ \cdots \circ \sigma_k$. *If* $\mathrm{Inv}(\tau) = k$ *and* $\tau(\ell) = 1$, *then*

$$(\tau f)_1(x) = f[(0, \ell) \to (x, 1)].$$

Proof. Let $L = \binom{\ell}{2}$. By (24), we only need to verify that there exist $L - k$ adjacent transpositions $\rho_1, \ldots, \rho_{L-k}$ on $[\![2, \ell]\!]$ such that $\rho_1 \circ \cdots \circ \rho_{L-k} \circ \tau = \mathrm{rev}_\ell$. But since $\tau^{-1}(1) = \ell$, which is the maximum possible value, the sequence of $L - k$ adjacent transpositions defined by the proof of Lemma 3.7 do not include (1 2). □

The main idea of the proof of Proposition 3.3 is to find a sequence of adjacent transpositions $\sigma_1, \ldots, \sigma_k$ satisfying the hypotheses of Lemma 3.8 such that $(\sigma_1 \circ \cdots \sigma_k)\mathcal{P}$ has top line equal to $\mathcal{P}^\lambda_{i\to 1}$ and second line, labeled B', such that $(\mathsf{PT}(\mathcal{P}^\lambda_{i\to 1}, B'))_1 = \mathcal{P}^\lambda_{j\to 1}$. The latter equality, with the definition of PT from (16), implies that $\mathcal{P}^\lambda_{j\to 1}(x) - \mathcal{P}^\lambda_{i\to 1}(x) = \max_{0\le s\le x}(B'(s) - \mathcal{P}^\lambda_{i\to 1}(s))$. If $\mathcal{P}^\lambda_{i\to 1}$ and B' are jointly locally absolutely continuous away from zero to independent Brownian motions, then, by Lemma 3.6 and Lévy's identity (Proposition 3.2), $\mathcal{P}^\lambda_{j\to 1} - \mathcal{P}^\lambda_{i\to 1}$ is absolutely continuous away from zero to Brownian local time and the proof is complete.

This joint absolute continuity of $\mathcal{P}^\lambda_{i\to 1}$ and B' to Brownian motion is essentially a result of induction and Lemma 3.5 on the absolute continuity of the Pitman transform to independent Brownian motions. To make certain technical aspects of the argument easier, however, we will actually do the above after replacing the environment given by the top j lines of $\mathcal{P}$ with j independent Brownian motions using the absolute continuity of the former to the latter ensured by the Brownian Gibbs property (Corollary 2.5).

Proof of Proposition 3.3. Let $T > 0$. By the Brownian Gibbs property, $\{\mathcal{P}^\lambda_\ell(\cdot) = \mathcal{P}_\ell(\lambda + \cdot) - \mathcal{P}_\ell(\lambda) : \ell \in [\![1, j]\!]\}_{[0,T]}$ is absolutely continuous with respect to j independent Brownian motions of rate two started at the origin. Note that LPP values in $\mathcal{P}$ (with argument translated by λ) and in the environment $\{\mathcal{P}^\lambda_i(\cdot) : i \in \mathbb{N}\}$ are identical, as constant vertical shifts for each function are not seen by the increments. Thus to prove that $\mathcal{P}^\lambda_{j\to 1}(\cdot) - \mathcal{P}^\lambda_{i\to 1}(\cdot)$ is locally absolutely continuous away from zero with respect to rate four Brownian local time, and further that it almost surely has a point of increase at the origin, it suffices to prove instead the same for $B^\lambda_{j\to 1}(\cdot) - B^\lambda_{i\to 1}(\cdot)$, where $B^\lambda_{\ell\to 1} : [0, T] \to \mathbb{R}$ is defined, for $\ell \in [\![1, j]\!]$, by

$$B^\lambda_{\ell\to 1}(x) = B[(\lambda, \ell) \to (\lambda + x, 1)]$$

with $B = (B_1, \ldots, B_j)$ a collection of j independent Brownian motions of rate two. That $\mathcal{P}^\lambda_{j\to 1} - \mathcal{P}^\lambda_{i\to 1}$ can be written as the running maximum of a process which is locally absolutely continuous away from zero to Brownian motion does not logically follow from this fact holding for $B^\lambda_{j\to 1} - B^\lambda_{i\to 1}$, and we will indicate this aspect of the proof separately.

We *claim* that there exists a function $B' : [0, T] \to \mathbb{R}$ such that

$$B^\lambda_{j\to 1}(\cdot) = \big(\mathsf{PT}(B^\lambda_{i\to 1}, B')\big)_1(\cdot) \tag{25}$$

with B' such that, for every $\delta > 0$, $(B^\lambda_{i\to 1}, B')|_{[\delta,T]} \ll (B_1, B_2)|_{[\delta,T]}$, $B'(0) = 0$, and $B' - B^\lambda_{i\to 1}$ almost surely has a positive maximum over every neighbourhood of the origin. We show how to finish the proof of local absolute continuity away from zero to Brownian local time given the claim. Using (25), it would follow from (16) that, for $x \in [0, T]$,

$$B^\lambda_{j\to 1}(x) - B^\lambda_{i\to 1}(x) = \max_{0\le s\le x}\Big(B'(s) - B^\lambda_{i\to 1}(s)\Big). \tag{26}$$

Since $(B^{\lambda}_{i\to 1}, B')|_{[\delta,T]} \ll (B_1, B_2)|_{[\delta,T]}$ for every $\delta > 0$, it follows that, for every $\varepsilon > 0$, $(B' - B^{\lambda}_{i\to 1})|_{[\varepsilon,T]} \ll B|_{[\varepsilon,T]}$, where B is a Brownian motion of rate *four*. The same holds for the analogue of $B' - B^{\lambda}_{i\to 1}$ in the original environment defined by $\mathcal{P}$, and thus we may take X in the statement of Proposition 3.3 to be this analogue of $B' - B^{\lambda}_{i\to 1}$.

By Lemma 3.6, (26) is locally absolutely continuous away from zero to the running maximum of B. By Lévy's identity (Proposition 3.2), the latter process has the same distribution as Brownian local time of rate four. So $B^{\lambda}_{j\to 1} - B^{\lambda}_{i\to 1}$ is locally absolutely continuous away from zero to Brownian local time of rate four. Since we claim $B' - B^{\lambda}_{i\to 1}$ almost surely has a positive maximum over every neighbourhood of the origin, $B^{\lambda}_{j\to 1} - B^{\lambda}_{i\to 1}$ has a point of increase at the origin, by (26), as required.

Thus we only need to prove the claim (25). This requires identifying a sequence of adjacent transpositions satisfying the hypotheses of Lemma 3.8. Consider the compositions of adjacent transpositions given by

$$\begin{aligned} \tau_1 &= (1\ 2) \circ (2\ 3) \circ \cdots \circ (i-2\ i-1) \circ (i-1\ i) \\ \tau_2 &= (2\ 3) \circ (3\ 4) \circ \cdots \circ (j-2\ j-1) \circ (j-1\ j); \end{aligned} \tag{27}$$

this encodes the transpositions depicted in Figure 5.

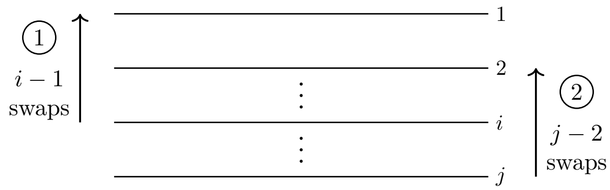

FIGURE 5. The sequence of adjacent transpositions $\tau_2 \circ \tau_1$ is depicted. First, for τ_1, lines from the i^{th} upwards are iteratively swapped (i.e., the Pitman transform is applied) till the top two are swapped. Then, for τ_2, the same is done from the j^{th} line upwards till the second and third lines are swapped.

Observe that $\mathrm{Inv}(\tau_1) = i - 1$ and τ_1 maps i to 1. Then by Lemma 3.8 (taking $k = i - 1$ and $\ell = i$), it follows that

$$(\tau_1 B)_1(\lambda + x) = B^{\lambda}_{i\to 1}(x).$$

We define B' by $B'(x) = \big((\tau_2 \circ \tau_1)B\big)_2(\lambda + x)$ and claim that this satisfies (25), the independence condition following that equation, and the condition of having a positive maximum (for $B' - B^{\lambda}_{i\to 1}$) over every neighbourhood of the origin.

First note that $\big((\tau_2 \circ \tau_1)B\big)_1(\lambda + \bullet) = (\tau_1 B)_1(\lambda + \bullet) = B^{\lambda}_{i\to 1}$. Since the righthand side of (25) is the result of applying the Pitman transform to the top two lines of $(\tau_2 \circ \tau_1)B$, we now verify that $(1\ 2) \circ \tau_2 \circ \tau_1$ satisfies the hypotheses of Lemma 3.8 with $\ell = j$.

We see that $(1\ 2) \circ \tau_2 \circ \tau_1$ in the given representation consists of $1 + (j-2) + (i-1) = j + i - 2$ transpositions. Now, it is easy to check that $\mathrm{Inv}((1\ 2) \circ \tau_2 \circ \tau_1) = j + i - 2$, since each transposition increases the number of inversions by one. Finally, it is also immediate that $(1\ 2) \circ \tau_2 \circ \tau_1$ maps j to 1. Then applying Lemma 3.8 (with $k = j + i - 2$ and $\ell = j$) verifies (25).

We next need to verify the independence condition following (25) and that $B' - B^{\lambda}_{i\to 1}$ almost surely has a positive maximum over every neighbourhood of the origin. For the first aim, we will iteratively apply Lemma 3.5, which asserts the local absolute continuity away from zero to independent Brownian motions of $\mathsf{PT}(X_1, X_2)$ if X_1 and X_2 jointly satisfy the same absolute continuity condition. Lemma 3.5 has a hypothesis that $X_2 - X_1$ almost surely has a positive maximum over every neighbourhood of the origin, and we need to verify that this holds at every step of applying PT corresponding to the adjacent transpositions encoded by $\tau_2 \circ \tau_1$. This task is performed by the following Lemma 3.9; Lemma 3.9 also implies that $B' - B^{\lambda}_{i\to 1}$ has positive maximum over every neighbourhood of the origin.

This completes the proof of the claim preceding (25) and so also the proof of Proposition 3.3, save for the proof of Lemma 3.9. □

Lemma 3.9. *Let $i < j$ and τ_1 be as in (27). Let $\sigma = (k\ k+1) \circ \cdots \circ (i-2\ i-1) \circ (i-1\ i)$ (with $k \in [\![1, i-1]\!]$) or $\sigma = (k\ k+1) \circ \cdots \circ (j-2\ j-1) \circ (j-1\ j) \circ \tau_1$ (with $k \in [\![2, j-1]\!]$). Let $B = (B_1, \ldots, B_j)$ be the environment of independent Brownian motions, $X_1 = (\sigma B)_{k-1}$, and $X_2 = (\sigma B)_k$. Then $X_2 - X_1$ almost surely has positive maximum over every neighbourhood of the origin.*

The two cases defining different forms of σ (i.e., with τ_1 not present or present) correspond to the Pitman transforms associated to the adjacent transpositions in $\tau_2 \circ \tau_1$. In other words, the two cases correspond to verifying that the process has a positive maximum over every neighbourhood of zero at every intermediate step of the swaps on the left and right side of Figure 5 respectively.

To prove Lemma 3.9, we will need a simple scaling property of the Pitman transform; it is to have this scaling property that we transformed to the Brownian environment instead of working in the original parabolic Airy environment in the proof of Proposition 3.3.

Lemma 3.10. *Suppose $T > 0$ and $X_1, X_2 : [0, T] \to \mathbb{R}$ are random processes with $X_1(0) = X_2(0) = 0$ and which satisfy $(X_1, X_2)(t) \overset{d}{=} \alpha^{-1/2}(X_1, X_2)(\alpha t)$, where $\alpha \in (0, 1]$ and the distributional equality is as processes in $t \in [0, T]$. Then for $\alpha \in (0, 1]$, $\mathsf{PT}(X_1, X_2)(t) \overset{d}{=} \alpha^{-1/2}\mathsf{PT}(X_1, X_2)(\alpha t)$ again as processes in $t \in [0, T]$.*

Proof. We will show the scaling property for the first component of $\mathsf{PT}(X_1, X_2)$ as the argument for the other component is quite similar. Recall that $(\mathsf{PT}(X_1, X_2))_1(t)$ is given by

$$\big(\mathsf{PT}(X_1, X_2)\big)_1(t) = X_1(t) + \max_{0 \le s \le t} \left(X_2(s) - X_1(s)\right)$$

For fixed $\alpha > 0$, the scaling property for (X_1, X_2) implies that

$$\begin{aligned}\big(\mathsf{PT}(X_1, X_2)\big)_1(t) &\overset{d}{=} \alpha^{-1/2}X_1(\alpha t) + \alpha^{-1/2} \max_{0 \le s \le t} \left(X_2(\alpha s) - X_1(\alpha s)\right) \\ &= \alpha^{-1/2}X_1(\alpha t) + \alpha^{-1/2} \max_{0 \le s \le \alpha t} \left(X_2(s) - X_1(s)\right),\end{aligned}$$

which is clearly $\alpha^{-1/2}\mathsf{PT}(X_1, X_2)(\alpha t)$. That the distributional equality holds at the level of processes is immediate. □

Proof of Lemma 3.9. Let $\rho = \inf\{t > 0 : X_2(t) - X_1(t) > 0\}$. We must show that $\rho = 0$ almost surely. First, note that X_1 and X_2 are functions of B by a sequence of Pitman transforms, and that $\{\rho = 0\}$ is in the germ σ-algebra of these Brownian motions. Hence by Blumenthal's zero-one law, $\mathbb{P}(\rho = 0) \in \{0, 1\}$. It is therefore sufficient to show that $\mathbb{P}(\rho = 0) > 0$.

We note that

$$\mathbb{P}(\rho = 0) = \lim_{t \to 0} \mathbb{P}(\rho \le t) \ge \lim_{t \to 0} \mathbb{P}\big(X_2(t) > X_1(t)\big) = \mathbb{P}\big(X_2(T) > X_1(T)\big),$$

the last equality by the scaling invariance of X_1 and X_2 guaranteed by Lemma 3.10 and the same property of Brownian motion. Thus it is sufficient to show that

$$\mathbb{P}\big(X_2(T) > X_1(T)\big) > 0. \tag{28}$$

We break into cases depending on the two cases of σ in the statement, i.e., based on whether σ contains a composition with τ_1 or not.

Case 1: τ_1 not present. In this case, $X_1 = (\sigma B)_{k-1}$ is a rate two Brownian motion, namely the $(k-1)^{\text{st}}$ line of B, as that has not been involved in the Pitman transforms associated to σ that have been performed; further, X_1 is independent of X_2 since X_2 is a function of $\{B_\ell : \ell \in [\![k, i]\!]\}$. This immediately implies (28), as $X_2(T)$ is almost surely finite and $X_1(T)$ is an independent Gaussian random variable of variance $2T$.

Case 2: τ_1 is present. For this case we don't have explicit independence of X_1 and X_2. Instead, we will find a Brownian motion in the environment which is at most X_2 and independent of X_1, which suffices.

Let $Y = \tau_1 B$ and $\sigma^k = (k\ k+1) \circ \cdots \circ (j-1\ j)$, where recall $k \in [\![2, j-1]\!]$. Then $X_1 = (\sigma^k Y)_{k-1}$ and $X_2 = (\sigma^k Y)_k$. Now, $\mathrm{Inv}(\sigma^k) = j - k$, the same number of adjacent transpositions making up σ^k. Further, $\sigma^k(j) = k$. Thus, viewing the environment formed by the k^{th} to j^{th} lines of Y separately and applying Lemma 3.8, we see that

$$X_2(T) = Y[(0, j) \to (T, k)].$$

This implies that $X_2(T) \geq Y_j(T)$ as $Y_j(0) = 0$. Next note that $Y_\ell = B_\ell$ for $\ell \in [\![i+1, j]\!]$. Therefore $X_2(T) \geq B_j(T)$, so to show (28) it suffices to show $B_j(T) > X_1(T)$ with positive probability. But B_j is independent of $X_1 = (\sigma B)_{k-1} = (\tau_1 B)_{k-1}$ since $(\tau_1 B)_{k-1}$ is a function of only $\{B_\ell : \ell \in [\![1, i]\!]\}$ and $i < j$. Thus since $X_1(T)$ is almost surely finite and $B_j(T)$ is an independent Gaussian random variable of variance $2T$, we get (28).

This completes the proof of Lemma 3.9 and so also of Proposition 3.3. □

Remark 3.11. Observe that, in Proposition 3.3, we do not claim absolute continuity with respect to Brownian local time on $[0, T]$. This is because we believe that absolute continuity in fact does not hold on the entire interval in general with a singularity at 0.

As a first step, one might try to show that $B^\lambda_{j\to 1} - B^\lambda_{i\to 1}$, the analogue of $\mathcal{P}^\lambda_{j\to 1} - \mathcal{P}^\lambda_{i\to 1}$ in a Brownian environment used in the proof, is not absolutely continuous to $\mathcal{L}$, Brownian local time of rate four, on $[0, T]$. Using the invariance under Brownian scaling of the process to express $\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{i\to 1})(t)]$ as an almost sure limit at zero of an empirical mean of the process, the lack of absolute continuity would follow from showing that $\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{i\to 1})(1)] \neq \mathbb{E}[\mathcal{L}(1)]$ (where we may take $t = 1$ by Brownian scaling).

When $j = 2$ and $i = 1$, $B^\lambda_{j\to 1} - B^\lambda_{i\to 1}$ actually has the distribution of $\mathcal{L}$. However, we believe that $\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{i\to 1})(1)] \neq \mathbb{E}[\mathcal{L}(1)]$ whenever $(i, j) \neq (1, 2)$. This follows in the case that $i = 1, j > 2$ by monotonicity, as well as by a different form of monotonicity in the case $B^\lambda_{j\to 1} - B^\lambda_{j-1\to 1}$ for $j > 2$. The latter can be seen by observing $B^\lambda_{j\to 1} = \mathsf{PT}(B^\lambda_1, B^\lambda_{j\to 2})$ and $B^\lambda_{j\to 2} \stackrel{d}{=} B^\lambda_{j-1\to 1}$ and noting

$$\begin{aligned}
(B^\lambda_{j\to 1} - B^\lambda_{j\to 2})(1) &= B^\lambda_1(1) + \max_{0\leq t\leq 1}\big(B^\lambda_{j\to 2}(t) - B^\lambda_1(t)\big) - B^\lambda_{j\to 2}(1) \\
&= B^\lambda_1(1) - B^\lambda_{j\to 2}(1) + \max_{0\leq t\leq 1}\left\{B^\lambda_2(t) + \max_{0\leq s\leq t}\big(B^\lambda_{j\to 3}(s) - B^\lambda_2(s)\big) - B^\lambda_1(t)\right\} \\
&= B^\lambda_1(1) - \Big(B^\lambda_2(1) + \max_{0\leq t\leq 1}\big(B^\lambda_{j\to 3}(t) - B^\lambda_2(t)\big)\Big) \\
&\quad + \max_{0\leq t\leq 1}\Big\{B^\lambda_2(t) + \max_{0\leq s\leq t}\big(B^\lambda_{j\to 3}(s) - B^\lambda_2(s)\big) - B^\lambda_1(t)\Big\} \\
&\leq B^\lambda_1(1) - B^\lambda_2(1) - \max_{0\leq t\leq 1}\big(B^\lambda_{j\to 3}(t) - B^\lambda_2(t)\big) + \max_{0\leq t\leq 1}\big(B^\lambda_2(t) - B^\lambda_1(t)\big) \\
&\quad + \max_{0\leq s\leq 1}\big(B^\lambda_{j\to 3}(s) - B^\lambda_2(s)\big) \\
&= B^\lambda_1(1) - B^\lambda_2(1) + \max_{0\leq t\leq 1}\big(B^\lambda_2(t) - B^\lambda_1(t)\big) \\
&= \big(\mathsf{PT}(B^\lambda_1, B^\lambda_2)\big)(1) - B^\lambda_2(1)
\end{aligned} \tag{29}$$

where in the second and third equalities we use that $B^\lambda_{j\to 2} = \mathsf{PT}(B^\lambda_2, B^\lambda_{j\to 3})$. Finally, note that

$$\mathsf{PT}(B^\lambda_1, B^\lambda_2) = B^\lambda_{2\to 1} \quad \text{and} \quad B^\lambda_2 \stackrel{d}{=} B^\lambda_1,$$

which, together with the above, implies that $\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{j-1\to 1})(1)] < \mathbb{E}[\mathcal{L}(1)]$; this inequality is strict since the inequality in (29) can easily be seen to be strict with positive probability, using the independence of the Brownian motions and the fact that $\max_t(f(t)+g(t)) < \max_t(f(t))+\max_t(g(t))$ holds whenever the set of maximizers of f and that of g are disjoint. Note also that the inequality in (29) is *not* almost surely strict, which is essentially because $t \mapsto \max_{0\le s\le t}(B^\lambda_{j\to 3}(s) - B^\lambda_2(s))$ is non-decreasing and so can have its set of maximizers on $[0,1]$ be $[\varepsilon, 1]$ with positive probability for any $\varepsilon > 0$.

It is not so easy to show the inequality $\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{i\to 1})(1)] \neq \mathbb{E}[\mathcal{L}(1)]$ for the remaining cases, and the issue can be glimpsed by the fact that

$$\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{j-1\to 1})(1)] < \mathbb{E}[(B^\lambda_{2\to 1} - B^\lambda_{1\to 1})(1)] < \mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{1\to 1})(1)];$$

i.e., a single inequality does not hold between $\mathbb{E}[(B^\lambda_{j\to 1} - B^\lambda_{i\to 1})(1)]$ and $\mathbb{E}[\mathcal{L}(1)]$ for all $(i,j) \neq (1,2)$. Thus monotonicity arguments alone are unlikely to yield the desired conclusion.

One intriguing possible direction is to use the distributional identity between $B^\lambda_{k\to 1}(1)$ and the top eigenvalue of the size k GUE for each fixed k; the coupling of $B^\lambda_{k\to 1}(1)$ across k given by the GUE minor process is distinct from that given by the LPP coupling here, and so may furnish a different argument. We have not explored this avenue in detail.

4. Proofs of the patchwork quilt and Hausdorff dimension results

4.1. **$\mathcal{D}$ is a Brownian local time patchwork quilt: proving Theorem 2.** In this section we prove a technically more precise theorem, from which Theorem 2 immediately follows. Recall the objects involved in the definition of the Brownian local time patchwork quilt (Definition 1.3): Y_k are fabric functions, and $[x_k, x_{k+1}]$ are endpoints of the k^{th} patch at which the fabric functions are stitched together after a vertical shift y_k (to ensure continuity) and a horizontal shift μ_k. The horizontal shift μ_k is needed as the weight difference profile $\mathcal{D}$, the function being written as a patchwork quilt, is defined on $\mathbb{R}$, while each Y_k is defined on $[0,\infty)$.

Finally, we recall the canonical filtration of the process $\mathcal{P}$. In the notation of Definition 2.3, this can be written as the family of σ-algebras $\mathcal{F}_{\text{ext}}(k=\infty, t, \infty)$, i.e., the data of all the curves in $\mathcal{P}$ before location t. We will actually need the standard right-continuous augmentation of the filtration, which, recall, gives an "infinitesimal peak" into the future. This is defined in a more general setting that we will need later in Remark 5.8.

W may now state our more detailed theorem.

Theorem 4.1. *$\mathcal{D}$ is a Brownian local time patchwork quilt of rate four. In addition, adopting the notation of Definition 1.3, we have the following properties for each $k \in \mathbb{Z}$:*

- *(i) $\mu_k \in \mathbb{Z}$ almost surely.*
- *(ii) either Y_k is identically zero or there exists a process $B'_k : [0,\infty) \to [0,\infty)$ locally absolutely continuous away from zero to Brownian motion of rate four, such that $Y_k(t) = \max_{0\le s\le t} B'_k(s)$. In the latter case, Y_k almost surely increases at the origin.*
- *(iii) x_k is a stopping time with respect to the right-continuous filtration of $\mathcal{P}$.*

In Proposition 3.3 we have identified a family of functions, $\mathcal{P}^\lambda_{j\to 1} - \mathcal{P}^\lambda_{i\to 1}$ with $i < j$ and $\lambda \in \mathbb{R}$, which are locally absolutely continuous away from zero to Brownian local time of rate four. These will essentially play the role of the Y_k after some reindexing and translations (the latter will define μ_k and y_k). The other task of defining $\{x_k\}_{k\in\mathbb{Z}}$ will be done using the functions i^λ and j^λ defined after

Lemma 2.8; it is because these functions are only left-continuous (Lemma 2.9) that we specified that the filtration be right-continuous in Theorem 4.1.

Proof of Theorem 4.1. We will define the functions Y_k, points x_k, and variables μ_k and y_k with a more convenient indexing, and later show that an indexing as in Definition 1.3 is possible. We will use an extra index λ which in this proof will take values in $\mathbb{Z}$.

So for $\lambda \in \mathbb{Z}$, let N^λ be the number of times either i^λ or j^λ changes its values in $[0,1]$, i.e.,

$$N^\lambda = \#\left\{x \in [\lambda, \lambda+1) : i^\lambda(x^+ - \lambda) \neq i^\lambda(x-\lambda) \ \text{ or } \ j^\lambda(x^+ - \lambda) \neq j^\lambda(x-\lambda)\right\}. \tag{30}$$

We compare the values of i^λ and j^λ at x^+ to that at x since the functions are left-continuous from Lemma 2.9(ii); the shift by λ is to ensure that the points in the set lie in disjoint sets for different values of λ.

Note that since i^λ and j^λ take values in $\mathbb{N}$ and are non-decreasing, $N^\lambda < \infty$ almost surely. Let the elements of the set in (30) be $x_1^\lambda < x_2^\lambda < \ldots < x_{N^\lambda}^\lambda$ (if $N^\lambda \neq 0$), which all lie in $[\lambda, \lambda+1)$. Regardless of whether $N^\lambda = 0$ or not, define $x_0^\lambda = \lambda$ and $x_{N^\lambda+1}^\lambda = \lambda + 1$.

Observe that $x_0^\lambda < x_1^\lambda$ by part (iii) of Lemma 2.9, which says that both i^λ and j^λ equal one in a small right-neighbourhood of λ. Also, we have by definition $x_0^{\lambda+1} = x_{N^\lambda+1}^\lambda$. While Definition 1.3 requires that $x_k < x_{k+1}$ for each $k \in \mathbb{Z}$, we have allowed ourselves this violation for convenience of indexing. Such coincidences of values of x_k^λ can trivially be removed by a change in indexing while preserving the other properties of Definition 1.3.

That each x_ℓ^λ is a stopping time with respect to the right-continuous filtration of $\{\mathcal{P}_k : k \in \mathbb{N}\}$ is immediate from the fact that i^λ and j^λ are left continuous. (Note that $i^\lambda(x-\lambda)$ and $j^\lambda(x-\lambda)$ are determined by the right-continuous filtration of $\mathcal{P}$ at x, not $x - \lambda$.)

Now for $k \in [\![0, N^\lambda]\!]$, we have by definition that i^λ and j^λ are constant on $(x_k^\lambda - \lambda, x_{k+1}^\lambda - \lambda]$ (including the right endpoint since i^λ and j^λ are left-continuous from Lemma 2.9(ii)). Fixing $k \in [\![0, N^\lambda]\!]$, let these values be labeled i and j. We define, for $z \geq 0$,

$$Y_k^\lambda(z) = \mathcal{P}_{j\to 1}^\lambda(z) - \mathcal{P}_{i\to 1}^\lambda(z), \tag{31}$$

where we recall the definition of $\mathcal{P}_{k\to 1}^\lambda$ from (12).

By Proposition 3.3, Y_k^λ satisfies condition (i) of Definition 1.3, on the absolute continuity to Brownian local time of rate four whenever $i^\lambda(x) < j^\lambda(x)$. In the other case that $i^\lambda(x) = j^\lambda(x)$ (recall that $i^\lambda(x) \leq j^\lambda(x)$ by Lemma 2.9(i)), $Y_k^\lambda \equiv 0$, which is indeed locally absolutely continuous away from zero to Brownian local time of any rate.

We next set

$$\mu_0^\lambda = \lambda - 1 \quad \text{and} \quad \mu_k^\lambda = \lambda \tag{32}$$

for $k \in [\![1, N^\lambda]\!]$. The definition of μ_0^λ is different merely to ensure that $x_0^\lambda > \mu_0^\lambda$ as required by Definition 1.3, since $x_0^\lambda = \lambda$ by definition. Note that, by (30) and since $x_1^\lambda > x_0^\lambda$, we have $x_k^\lambda > \mu_k^\lambda$ for $\lambda \in \mathbb{Z}$ and $k \in [\![1, N^\lambda]\!]$ already. Now define

$$y_k^\lambda = b_j^\lambda - a_i^\lambda \tag{33}$$

for $k \in [\![0, N^\lambda]\!]$, with i and j as above (31).

Next, we verify that Y_k^λ satisfies $\mathcal{D}(z) = Y_k^\lambda(z - \mu_k^\lambda) + y_k^\lambda$ for $z \in [x_k^\lambda, x_{k+1}^\lambda]$, thus establishing (iii) of Definition 1.3.

From the definitions of i and j as the constant values of i^λ and j^λ on $(x_k^\lambda - \lambda, x_{k+1}^\lambda - \lambda]$ we observe that, for all $k \in [\![0, N^\lambda]\!]$ and $z \in (x_k^\lambda, x_{k+1}^\lambda]$,

$$\mathcal{D}(z) = \mathcal{S}(y_b, z) - \mathcal{S}(y_a, z) = b_j^\lambda + \mathcal{P}_{j\to 1}^\lambda(z - \lambda) - a_i^\lambda - \mathcal{P}_{i\to 1}^\lambda(z - \lambda). \tag{34}$$

By the continuity of both sides of the last equality, the equality also holds for $z = x_k^\lambda$. Comparing with the definitions (31), (32), and (33) we see that $\mathcal{D}(z) = Y_k^\lambda(z - \mu_k^\lambda) + y_k^\lambda$ for all $k \in [\![1, N^\lambda]\!]$ and $z \in [x_k^\lambda, x_{k+1}^\lambda]$. So item (iii) of Definition 1.3 has been verified when $k \in [\![1, N^\lambda]\!]$.

Next we consider the $k = 0$ case. We know from Lemma 2.9(iii) there exists a small random interval $[0, \varepsilon]$ such that $i^\lambda(x) = j^\lambda(x) = 1$ for all $x \in [0, \varepsilon]$ and all $\lambda \in \mathbb{Z}$. Thus, from (31), Y_0^λ must be identically zero. Thus in this case we may replace $z - \lambda$ by $z - (\lambda - 1)$ in both of the instances on the righthand side of (34) and the equality still holds for $z \in [x_0^\lambda, x_1^\lambda]$. This verifies that $\mathcal{D}(z) = Y_0^\lambda(z - \mu_0^\lambda) + y_0^\lambda$ when $z \in [x_0^\lambda, x_1^\lambda]$.

Now we let λ vary over elements of $\mathbb{Z}$ and consider the collection of random functions $\{Y_k^\lambda : \lambda \in \mathbb{Z}, k \in [\![0, N^\lambda]\!]\}$ and random points $\{x_k^\lambda : \lambda \in \mathbb{Z}, k \in [\![0, N^\lambda]\!]\}$. Note that, since $N^\lambda < \infty$ for all $\lambda \in \mathbb{Z}$ almost surely, it holds that only finitely many x_k^λ lie in any compact set $\mathcal{K} \subseteq \mathbb{R}$. This verifies item (ii) of Definition 1.3.

Finally we reindex these collections by $\mathbb{Z}$ instead of a subset of $\mathbb{Z}^2$ (namely, $\bigcup_{\lambda \in \mathbb{Z}} \{\lambda\} \times [\![0, N^\lambda]\!]$). Item (i) of Theorem 4.1 is true by the definition (32) of μ_k^λ and (ii) follows from Proposition 3.3 and (31). This completes the proof of Theorem 2. □

Remark 4.2. Here we briefly discuss the difficulties in formulating an absolute continuity result without patches. One might expect that it would be possible to "stitch" together adjacent patches using some degree of independence structure in their endpoints (for example, the endpoints are defined by the locations of discontinuities of i^λ and j^λ, which are stopping times with respect to $\mathcal{P}$).

In our argument, we have implicitly been conditioning on data to the left of λ and proving absolute continuity of fabric functions to Brownian local time under this conditioning, which is preserved on averaging over the conditioned data. However, observe that, under this conditioning, there are patches corresponding to $j = i$, where $\mathcal{D}$ is flat with deterministic value $b_j^\lambda - a_j^\lambda$. On the other hand, note that $\mathcal{L}$ has a flat portion at a given deterministic height with probability zero. Thus the absolute continuity statement cannot hold under the conditioning.

Of course, it may still hold after averaging and we indeed expect this to be true. However for this, we would at least need to know that $b_j^\lambda - a_j^\lambda$ are continuous random variables. It turns out that this can be shown with further arguments which bear similarity to the ones already introduced in this paper—i.e., making use of geodesic geometry and the underlying Brownian nature of the environment and its last passage values. However one needs further ingredients to obtain a result without patches. Such results and related ones involving refined understanding of geodesic behavior will be pursued in future work.

In summary, the fact that the absolute continuity to $\mathcal{L}$ with a single patch does *not* hold conditionally on data to the left of the interval in question, but may (and does) on averaging, indicates the subtleties in obtaining such patch-less absolute continuity statements.

Given Theorem 4.1, the proof of Theorem 1 is quick, and we proceed to it next.

4.2. **The Hausdorff dimension of E: proving Theorem 1.** We start by recalling the definition of Hausdorff dimension.

Definition 4.3 (Hausdorff dimension)**.** The α-Hausdorff measure of a set $A \subseteq \mathbb{R}$, denoted $H^\alpha(A)$, is defined as $H^\alpha(A) = \lim_{\delta \downarrow 0} H_\delta^\alpha(A)$, where, for $\delta > 0$,

$$H_\delta^\alpha(A) = \inf\left\{\sum_{i=1}^\infty \operatorname{diam}(U_i)^\alpha : A \subseteq \bigcup_{i=1}^\infty U_i \text{ and } 0 < \operatorname{diam}(U_i) < \delta\right\}.$$

The Hausdorff dimension of A, denoted $\dim(A)$, is defined as

$$\dim(A) = \inf\big\{\alpha > 0 : H^\alpha(A) < \infty\big\}.$$

The Hausdorff dimension enjoys a *countable stability* property; see for example [Mat99, below Definition 4.8]. It states that

$$\dim\left(\bigcup_{i\in I} S_i\right) = \sup_{i\in I} \dim(S_i),$$

where I is any countable set.

We next state a lemma on the Hausdorff dimension of the set of non-constant points of process locally absolutely continuous away from zero to Brownian local time.

Lemma 4.4. *Let $Y : [0,\infty) \to \mathbb{R}$ be locally absolutely continuous away from zero to Brownian local time of any given rate, and $\mathsf{NC}(Y)$ be its set of non-constant points. Then the following holds with probability one. For any $0 \le z_1 < z_2 \le \infty$ and $j > i$, $\dim(\mathsf{NC}(Y) \cap (z_1, z_2))$ is equal to zero if $\mathsf{NC}(Y) \cap (z_1, z_2) = \varnothing$ and one-half if $\mathsf{NC}(Y) \cap (z_1, z_2) \neq \varnothing$. Also, if Y almost surely increases at zero, then $\dim(\mathsf{NC}(Y) \cap [0, z)) = \frac{1}{2}$ for all $z \in \mathbb{R}$.*

Note that the statement holds for all the sets $\mathsf{NC}(Y) \cap (z_1, z_2)$ on the same probability one event. Note also that in the first statement we consider an intersection with an *open* interval; the statement is not true with a closed interval. For example, consider a sufficiently small closed interval whose right endpoint is the first non-constant point after zero, for which the intersection with E^Y will be a singleton.

Proof of Lemma 4.4. By the countable stability property of Hausdorff dimension, it is sufficient to prove the statement for all $\mathsf{NC}(Y) \cap [q_1 \cap q_2]$ with $q_1, q_2 \in \mathbb{Q}$ and $q_1 < q_2$ by writing the interval (z_1, z_2) as a union of intervals with rational endpoints. Since the rationals are countable, it is sufficient to prove the almost sure statement for a fixed pair of rationals $q_1 < q_2$. We may also assume $q_1, q_2 > 0$ by a similar approximation argument for intervals with left endpoint equal to zero.

We next recall the well-known fact [MP10, Theorem 4.24] that the set of non-constant points of Brownian local time of any given rate almost surely has Hausdorff dimension one-half on the event that it is non-empty. Now, the statement we are proving follows immediately from our hypothesis on the absolute continuity of Y on $[q_1, q_2]$ to the restriction to $[q_1, q_2]$ of Brownian local time.

For the case of $\mathsf{NC}(Y) \cap [0, q]$ with $q \in \mathbb{Q}$, we note that 0 is almost surely a point of increase of Y by hypothesis, and so both $\mathsf{NC}(Y) \cap [0, q]$ and $\mathsf{NC}(Y) \cap (0, q)$ are almost surely not empty. Thus both have the same Hausdorff dimension. The case of the latter set has been dealt with above. □

We may now prove Theorem 1 using Theorem 2 and Lemma 4.4. Recall that $\mathsf{NC} = \mathsf{NC}(\mathcal{D})$ is the set of non-constant points of $\mathcal{D}$. We first start by showing that NC is almost surely non-empty, in a slightly stronger form that we will need later. This was essentially proven in [BGH19a, Proposition 3.10] in a pre-limiting setting and we reproduce the argument for completeness.

Lemma 4.5. *With probability one, $\lim_{M\to\infty} \mathcal{D}(M) = \infty$.*

Proof. Let $\mathcal{S}^{\cup}$ be the parabolic Airy sheet with the parabola compensated for, i.e., $\mathcal{S}^{\cup}(y, x) = \mathcal{S}(y, x) + (y - x)^2$. The distribution of $\mathcal{S}^{\cup}(y, x)$ is the same for all $x, y \in \mathbb{R}$, which follows from the stationarity of $\mathcal{S}$, that $\mathcal{S}(0, \cdot) = \mathcal{P}_1(\cdot)$ from Definition 2.6(ii), and that the Airy$_2$ process (given by $x \mapsto \mathcal{P}_1(x) + x^2$) is stationary. (In fact, this implies that the common distribution is the GUE Tracy-Widom distribution [TW94], though we will not make use of this fact.)

It is enough to show that $\limsup_{M\to\infty} \mathcal{D}(M) = \infty$ almost surely. We observe that

$$\mathbb{P}\left(\limsup_{M\to\infty} \mathcal{D}(M) = \infty\right) \ge \limsup_{M\to\infty} \mathbb{P}\big(\mathcal{D}(M) > 2M^{1/2}\big). \tag{35}$$

Writing out $\mathcal{D}(M)$ we see

$$\mathcal{D}(M) = \mathcal{S}(y_b, M) - \mathcal{S}(y_a, M) = \mathcal{S}^{\cup}(y_b, M) - \mathcal{S}^{\cup}(y_a, M) - (y_b - M)^2 + (y_a - M)^2$$

$$= \mathcal{S}^{\cup}(y_b, M) - \mathcal{S}^{\cup}(y_a, M) - (y_b^2 - y_a^2) + 2M(y_b - y_a).$$

Note that $\mathcal{S}^{\cup}(y_b, \pm M)$, $\mathcal{S}^{\cup}(y_a, \pm M)$ all have the same distribution and are almost surely finite. For notational simplicity, let X be a random variable with the same distribution. Now, by a union bound,

$$\mathbb{P}\big(\mathcal{D}(M) \le 2M^{1/2}\big) \le \mathbb{P}\big(X \le M^{1/2} + y_a^2 - (y_b - y_a)M\big) + \mathbb{P}\big(X \ge -M^{1/2} - y_b^2 + (y_b - y_a)M\big).$$

Since X is almost surely finite and $y_b > y_a$, this implies that

$$\limsup_{M\to\infty} \mathbb{P}\big(\mathcal{D}(M) - \mathcal{D}(-M) > 2M^{1/2}\big) = 1,$$

which, with (35), completes the proof of Lemma 4.5. □

Proof of Theorem 1. By Lemma 4.5, we have that E is almost surely uncountable. Let E_k be the set of non-constant points of Y_k, which are as defined in Definition 1.3. Then note that, by Theorem 2,

$$E = \bigcup_{k\in\mathbb{Z}} E \cap [x_k, x_{k+1}] = \bigcup_{k\in\mathbb{Z}} E_k \cap [x_k, x_{k+1}].$$

The second equality follows by noting that the set of non-constant points of function is unchanged under constant vertical shifts of the function. By the countable stability property, the Hausdorff dimension of the right-hand side is the same as if $[x_k, x_{k+1}]$ is replaced by (x_k, x_{k+1}) for each k. Now, since E is uncountable, there must exist at least one k such that $E_k \cap (z_k, z_{k+1})$ is non-empty. Then by Lemma 4.4 and the countable stability property, we obtain that $\dim(E) = \frac{1}{2}$ almost surely. □

Remark 4.6. It is worth mentioning that besides Hausdorff dimension, there are other natural notions of dimensions that one might also consider, such as the Minkowski dimension and the packing dimension. While they do not agree in general, it is known that, for Brownian local time, all these notions of dimension agree and are equal to 1/2. This can then be used to obtain versions of Lemma 4.4 for all the different notions, leading to analogues of Theorem 1.

5. Proof that local limit of $\mathcal{D}$ is Brownian local time

5.1. **Outline and setup for the proof.** We first recall and expand on the proof ideas of Theorem 3 as discussed in Section 1.4. As in the latter section, this overview will only discuss the case where $\tau = \tau_\lambda$, as the important conceptual points of the proof are captured here. Recall for $\lambda \in \mathbb{R}$ that τ_λ is the first time after λ that $\mathcal{D}$ is non-constant.

Consider the geodesic π_a^t from the vertical line at λ, with the boundary data $\{a_i^\lambda\}_{i\in\mathbb{N}}$ as in Lemma 2.8, to $(t, 1)$, and similarly the geodesic π_b^t with boundary data $\{b_i^\lambda\}_{i\in\mathbb{N}}$. When $t = \tau_\lambda$, both geodesics are at the top line at τ_λ since they end at $(\tau_\lambda, 1)$, although it may intersect the line $\{\tau_\lambda\} \times \mathbb{N}$ at other levels as well; in fact, as we will show, it will also intersect $(\tau_\lambda, 2)$. As t increases, if both the geodesics only use the top line in $[\tau_\lambda, t]$, then $\mathcal{D}$ will be constant in that interval by an analogue of Lemma 2.8 expressing $\mathcal{S}(y, \tau_\lambda + x)$ in terms of boundary data at τ_λ; see Lemma 5.1 ahead.

Thus by definition of τ_λ (it being a point of increase), at least one of the geodesics must jump to a different line (i.e., its line index at τ_λ increases) as soon as t increases past τ_λ. Now planarity implies π_b^t must jump.

If we know that it jumps to the *second* line, then it follows that π_a^t must not jump (since otherwise $\mathcal{D}$ continues to be constant past τ_λ) and $\mathcal{D}$ in the right-neighbourhood of τ_λ will essentially be the running maximum of $\mathcal{P}_2 - \mathcal{P}_1$; this is because the weight of the geodesic which jumps to line two is $(\mathsf{PT}(\mathcal{P}_1, \mathcal{P}_2))_1(t)$, while that of the one which remains on line one is $\mathcal{P}_1(t)$, and the difference of the two is the running maximum of $\mathcal{P}_2 - \mathcal{P}_1$ from the definition of PT (16). Since $\mathcal{P}_2 - \mathcal{P}_1$ is absolutely continuous to rate four Brownian motion, even including the origin, this running maximum is absolutely continuous to Brownian local time, again without avoiding the origin. The local limit at zero of the increment of such a process can easily be shown to be Brownian local time by scale invariant considerations.

Thus there are three aspects to the proof. The first is a notion of boundary data at the random time τ_λ, for we are concerned with the behaviour of geodesics immediately following this point; in particular, we need an analogue of Lemma 2.8 involving the boundary data at this random point (Lemma 5.1). Second, we need to know that at τ_λ, the geodesic which jumps does so to the second line and not a lower one (Lemma 5.9, using Proposition 5.2). Finally, we need a statement on local limits of processes absolutely continuous to Brownian local time, where we will make use of the added information that the process can be written as a running maximum of a Brownian-like process (Corollary 5.4).

For the first step, because the limits (11) defining $\{a_i^\lambda\}_{i\in\mathbb{N}}$ and $\{b_i^\lambda\}_{i\in\mathbb{N}}$ are only known to exist almost surely for fixed λ, and it is not a priori clear from their definition that they should be continuous in λ, we need a different specification of the boundary data as a process in the location λ. This is the role of the next definition, which is based on the idea that we can get boundary data at $\lambda + x$ by considering the boundary data at λ and an LPP problem on $[\lambda, \lambda + x]$.

For $\lambda \in \mathbb{R}$ and $i \in \mathbb{N}$, define $Z_i^{\lambda,a} : [0,\infty) \to \mathbb{R}$ by

$$Z_i^{\lambda,a}(x) = \sup_{j\geq i}\left\{a_j^\lambda + \mathcal{P}_{j\to i}^\lambda(x)\right\} = \sup_{j\geq i}\left\{a_j^\lambda + \mathcal{P}[(\lambda, j)\to(\lambda+x, i)]\right\}; \tag{36}$$

similarly define $Z_i^{\lambda,b}$ with the boundary data $\{b_i^\lambda\}_{i\in\mathbb{N}}$ in place of $\{a_i^\lambda\}_{i\in\mathbb{N}}$. In this section we will state results in terms of $Z_i^{\lambda,a}$, $\{a_i^\lambda\}_{i\in\mathbb{N}}$, and y_a. The analogous results with $Z_i^{\lambda,b}$, $\{b_i^\lambda\}_{i\in\mathbb{N}}$, and y_b also hold, though we will not mention this explicitly.

Note that, almost surely, for all $i \in \mathbb{N}$ and $x \geq 0$, $Z_i^{\lambda,a}(x) \geq Z_{i+1}^{\lambda,a}(x)$. We also have that $\mathcal{S}(y_a, \lambda + x) = Z_1^{\lambda,a}(x)$ for all $x \geq 0$ by Lemma 2.8. However, we can get another representation of $\mathcal{S}(y_a, \lambda + x)$ in terms of all the $Z_i^{\lambda,a}$ analogous to Lemma 2.8, which is the sense in which $Z_i^{\lambda,a}(x)$ is the boundary data from y_a at x and can be thought of as the process $x \mapsto a_i^{\lambda+x}$.

Lemma 5.1. *Let $\lambda \in \mathbb{R}$ be fixed. Then almost surely, for all $x \geq y \geq 0$,*

$$\mathcal{S}(y_a, \lambda + x) = \sup_{i\in\mathbb{N}}\left\{Z_i^{\lambda,a}(y) + \mathcal{P}[(\lambda + y, i)\to(\lambda + x, 1)]\right\}.$$

Moreover, the supremum is attained at a finite index.

Proof. We have from Lemma 2.8 and the definition (36) of $Z_1^{\lambda,a}$ that

$$\mathcal{S}(y_a, \lambda + x) = Z_1^{\lambda,a}(x).$$

Expanding the definition of the righthand side, and decomposing $\mathcal{P}[(\lambda, j)\to(\lambda + x, 1)]$ based on the location of the corresponding geodesic at $\lambda + y$, we see that

$$\begin{aligned}
Z_1^{\lambda,a}(x) &= \sup_{j\in\mathbb{N}}\left\{a_j^\lambda + \mathcal{P}[(\lambda, j)\to(\lambda + x, 1)]\right\}\\
&= \sup_{j\in\mathbb{N}}\sup_{i\leq j}\left\{a_j^\lambda + \mathcal{P}[(\lambda, j)\to(\lambda + y, i)] + \mathcal{P}[(\lambda + y, i)\to(\lambda + x, 1)]\right\}\\
&= \sup_{i\in\mathbb{N}}\left\{Z_i^{\lambda,a}(y) + \mathcal{P}[(\lambda + y, i)\to(\lambda + x, 1)]\right\};
\end{aligned}$$

for the final equality, we use that $\sup_{j\in\mathbb{N}}\sup_{i\leq j} a_{ij} = \sup_{i\in\mathbb{N}}\sup_{j\geq i} a_{ij}$ for any real collection $\{a_{ij}\}$, and the definition (36) of $Z_i^{\lambda,a}$. Observe from this manipulation that if the supremum in the first equality is attained at a finite index, then the same is true of the supremum in the final equality. Since we know from Lemma 2.8 that the first supremum is indeed attained at a finite index, the proof of Lemma 5.1 is complete. ∎

The second aspect of the proof outline of Theorem 3 above was that the geodesic, on jumping after τ, would jump to the second line and not a lower one. This is implied if the boundary data at τ at the third line is strictly smaller than at the first line, i.e., if $Z_1^{\lambda,b}(\tau-\lambda) > Z_3^{\lambda,b}(\tau-\lambda)$ by Lemma 5.1 and the continuity of $x \mapsto \mathcal{P}[(\tau,i)\to(x,1)]$, and is written out in Lemma 5.9. That $a_1^\lambda > a_3^\lambda$ and $b_1^\lambda > b_3^\lambda$ for any fixed λ is true almost surely, as we proved in Lemma 2.10, but we require such a statement at the random time $\tau-\lambda$. This is the content of the next proposition.

Proposition 5.2. *Fix $\lambda \in \mathbb{R}$. Almost surely, for all $\tau > 0$, it holds that $Z_1^\lambda(\tau) > Z_3^\lambda(\tau)$.*

The proof of Proposition 5.2, which we give in Section 5.5, has several technical details to be handled, but finally relies on the well-known fact that two-dimensional Brownian motion, started at any given point in the plane, almost surely never subsequently hits the origin.

The final aspect of the proof outline was that the local limit of the running maximum of a process absolutely continuous to Brownian motion is Brownian local time. This follows from an analogous statement that the local limit of a process absolutely continuous to Brownian motion is itself Brownian motion. We cite this from [DSV20].

Lemma 5.3 (Lemma 4.3 of [DSV20])**.** *Let $B' : [0,T] \to \mathbb{R}$ be a process such that $B'|_{[0,T]} \ll B|_{[0,T]}$ for all $T > 0$, where B is Brownian motion of rate σ^2. Then,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} B'(\varepsilon t) = B(t),$$

where the limit is in distribution in the topology of uniform convergence on compact sets.

The argument for Lemma 5.3 given in [DSV20] proceeds by contradiction, but is essentially to identify, for a given event A in a convenient measure-determining class, a functional of $t \mapsto \varepsilon^{-1/2}B(\varepsilon t)$ which converges almost surely to $\mathbb{P}(B \in A)$ as $\varepsilon \to 0$. The functional applied to $t \mapsto B'_\varepsilon(t) := \varepsilon^{-1/2}B'(\varepsilon t)$ converges to $\mathbb{P}(\lim_{\varepsilon\to 0} B'_\varepsilon \in A)$, which, by absolute continuity, must be equal to $\mathbb{P}(B \in A)$.

Corollary 5.4. *Let $B' : [0,\infty) \to \mathbb{R}$ be a stochastic process such that $B|_{[0,T]} \ll B|_{[0,T]}$ for any $T > 0$, where B is Brownian motion B of rate σ^2. Let $Y : [0,\infty) \to [0,\infty)$ be defined by $Y(t) = \max_{0\le s\le t} B'(s)$. Let $\mathcal{L} : [0,\infty)$ be the local time at the origin associated to B. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} Y(\varepsilon t) = \mathcal{L}(t),$$

where the convergence is in distribution in the topology of uniform convergence on compact sets.

Proof. We see that $Y(\varepsilon t) = \max_{0\le s\le \varepsilon t} B'(s) = \max_{0\le s\le t} B'(\varepsilon s)$. It is an easy exercise that the map taking a function defined on $[0,\infty)$ to its running maximum on $[0,\infty)$ is continuous in the topology of uniform convergence on compact sets. Then by Lemma 5.3 and the continuous mapping theorem,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2} Y(\varepsilon t) = \max_{0\le s\le t} B(s),$$

where the limit is in distribution in the same topology. Lévy's identity (Proposition 3.2) completes the proof of the corollary. □

5.2. **Brownian Gibbs at nice random times.** In the proof of Theorem 3, we will need to understand the parabolic Airy line ensemble $\mathcal{P}$ at the random time τ_λ. For this we recall from [CH14] the concept of a stopping domain and the associated strong Brownian Gibbs property of $\mathcal{P}$.

Definition 5.5 (Stopping domain and strong Brownian Gibbs)**.** Let $X : \mathbb{N} \to \mathbb{R}$ be an $\mathbb{N}$-indexed collection of continuous curves. Recall the σ-algebra $\mathcal{F}_{\mathrm{ext}}(k,\ell,r)$ from Definition 2.3 generated by the data external to the top k curves of X on $[\ell,r]$. A pair of random variables $\mathfrak{l},\mathfrak{r}$ is a *stopping domain* for $X_1,\ldots,X_k$ if, for all $\ell < r$,

$$\{\mathfrak{l} \le \ell, \mathfrak{r} \ge r\} \in \mathcal{F}_{\mathrm{ext}}(k,\ell,r).$$

Define the σ-algebra $\mathcal{F}_{\text{ext}}(k, \mathfrak{l}, \mathfrak{r})$ to be the one generated by events A such that $A \cap \{\mathfrak{l} \leq \ell, \mathfrak{r} \geq r\} \in \mathcal{F}_{\text{ext}}(k, \ell, r)$ for all $\ell < r$.

An infinite collection of random continuous curves X has the *strong Brownian Gibbs* property if, for any $k \in \mathbb{N}$ and conditionally on $\mathcal{F}_{\text{ext}}(k, \mathfrak{l}, \mathfrak{r})$, the distribution of X on $[\![1, k]\!] \times [\mathfrak{l}, \mathfrak{r}]$ is that of k independent Brownian bridges $(B_1, \ldots, B_k)$ of rate *two*, with $B_i(x) = X_i(x)$ for $x \in \{\mathfrak{l}, \mathfrak{r}\}$ and $i \in [\![1, k]\!]$, conditioned on not intersecting each other or $X_{k+1}(\cdot)$ on $[\mathfrak{l}, \mathfrak{r}]$.

Proposition 5.6 (Lemma 2.5 of [CH14])**.** *The parabolic Airy line ensemble $\mathcal{P}$ has the strong Brownian Gibbs property.*

Remark 5.7. The definition of a stopping domain and the strong Brownian Gibbs property in [CH14] considers ensembles with N curves instead of infinitely many. We require the infinite case as our random time τ_λ is defined via all the curves of $\mathcal{P}$. However, the proof given in [CH14] applies verbatim for the infinite case.

Alternatively, the $N = \infty$ case can be derived from the finite N one with a certain augmentation to handle boundary data. Given a stopping domain $[\mathfrak{l}, \mathfrak{r}]$ defined in terms of the entire line ensemble, one can apply, for any $\delta > 0$, the Brownian Gibbs property on the top k curves on an interval $[-M, M]$ with M chosen such that $[\mathfrak{l}, \mathfrak{r}] \subseteq [-M, M]$ with probability at least $1 - \delta$. Conditionally on $\mathcal{F}_{\text{ext}}(k, -M, M)$, $[\mathfrak{l}, \mathfrak{r}]$ is determined by the conditioned data and $\mathcal{P}|_{[-M,M]}, \ldots, \mathcal{P}_k|_{[-M,M]}$. Thus we may consider the $\mathcal{F}_{\text{ext}}(k, -M, M)$-conditional ensemble $\mathcal{P}|_{[-M,M]}, \ldots, \mathcal{P}_k|_{[-M,M]}$. Note that this ensemble has the Brownian Gibbs property with the boundary condition $\mathcal{P}_{k+1}$, i.e., the extra condition that $\mathcal{P}_k|_{[-M,M]}$ remains above $\mathcal{P}_{k+1}$.

Thus the finite N case (with $N = k$) with a boundary condition given by the curve $\mathcal{P}_{k+1}$, for which the argument in [CH14] goes through verbatim, can be applied to this ensemble defined on $[\![1, k]\!] \times [-M, M]$ to obtain the conclusion that $\mathcal{P}_1|_{[\mathfrak{l},\mathfrak{r}]}, \ldots, \mathcal{P}_k|_{[\mathfrak{l},\mathfrak{r}]}$ is distributed as non-intersecting Brownian bridges with appropriate endpoints, on the high-probability event that $[\mathfrak{l}, \mathfrak{r}] \subseteq [-M, M]$. Taking $\delta \to 0$ and $M \to \infty$ appropriately gives the conclusion.

Remark 5.8. In the proof of Theorem 3 we will want to apply the strong Brownian Gibbs property to an interval like $[\tau_\lambda, \tau_\lambda + 1]$. But observe that this interval is actually not a stopping domain, for the same reason that τ_λ is not a stopping time with respect to the canonical filtration $\sigma(\mathcal{P}_i(s) : s \leq t, i \in \mathbb{N})$ of $\mathcal{P}$: determining the occurrence of τ_λ requires an "infinitesimal peak" into the future.

To address this, we will actually need the strong Brownian Gibbs property to hold with respect to the analogue of right-continuous filtrations in this spatial setting. Indeed, for $\ell < r$ and $k \in \mathbb{N}$, consider the σ-algebra $\mathcal{F}_{\text{ext}}(k, \ell^+, r^-)$ defined as

$$\mathcal{F}_{\text{ext}}(k, \ell^+, r^-) = \bigcap_{n=1}^{\infty} \mathcal{F}_{\text{ext}}(k, \ell + n^{-1}, r - n^{-1}).$$

To know that an ensemble X has the strong Brownian Gibbs property with respect to the above family of augmented σ-algebras, it is sufficient to know that X has the Brownian Gibbs property with respect to the same family. Indeed, the proof of Proposition 5.6 as given in [CH14] goes through verbatim with $\mathcal{F}_{\text{ext}}(k, \ell^+, r^-)$ in place of $\mathcal{F}_{\text{ext}}(k, \ell, r)$ if it is known that X has the Brownian Gibbs property with respect to the former family of σ-algebras.

To prove that X having the Brownian Gibbs property with respect to $\mathcal{F}_{\text{ext}}(k, \ell, r)$ implies the same with respect to $\mathcal{F}_{\text{ext}}(k, \ell^+, r^-)$, it is sufficient to show a form of Blumenthal's zero-one law for the family of augmented σ-algebras, i.e., that almost surely for any $A \in \mathcal{F}_{\text{ext}}(k, \ell^+, r^-)$, it holds that

$$\mathbb{P}\left(A \mid \mathcal{F}_{\text{ext}}(k, \ell, r)\right) \in \{0, 1\}. \tag{37}$$

(The argument to show X has the Brownian Gibbs property with respect to $\mathcal{F}_{\mathrm{ext}}(k, \ell^+, r^-)$ via (37) is analogous to the one showing that a process which is Markov with respect to its canonical filtration is also so with respect to the right-continuous filtration; see eg. [RY13, Proposition 2.14].)

That (37) is true is easy to see. We may assume that A is a function of only $\mathcal{P}_1|_{[\ell,r]}, \ldots, \mathcal{P}_k|_{[\ell,r]}$, as everything else will be conditioned on. Conditionally on $\mathcal{F}_{\mathrm{ext}}(k, \ell, r)$, the distribution of $\mathcal{P}_1, \ldots, \mathcal{P}_k$ on $[\ell, r]$ is that of k independent Brownian bridges with boundary values determined by $\mathcal{F}_{\mathrm{ext}}(k, \ell^+, r^-)$, conditioned not to intersect one another or $\mathcal{P}_{k+1}$. Since the boundary values are almost surely separated, the Brownian bridges are conditioned on a positive probability event. Further, A lies in the σ-algebra generated by the germ σ-algebras of the Brownian bridges as well as their time reversals (so as to capture behaviour near r). We may assume that it lies in exactly one of these σ-algebras. Then by Blumenthal's zero-one law applied to these Brownian bridges (or their time reversals), (37) follows.

5.3. **The local limit argument: proving Theorem 3.** We begin by proving a representation for $\mathcal{D}(\tau + \varepsilon t) - \mathcal{D}(\tau)$ at a point of increase τ, using Proposition 5.2 which states that the boundary data $Z_3^{\lambda,b}$ of the third curve never equals that of the first curve, $Z_1^{\lambda,b}$.

Lemma 5.9. *Let τ be a random variable which is almost surely a point of increase of $\mathcal{D}$, and let $\mathcal{K} \subseteq [0, \infty)$ be a compact set. There exists random $\varepsilon_0 = \varepsilon_0(\tau, \mathcal{K}) > 0$ such that, almost surely, for $0 < \varepsilon < \varepsilon_0$ and $t \in \mathcal{K}$,*

$$\mathcal{D}(\tau + \varepsilon t) - \mathcal{D}(\tau) = \max_{0 \le s \le t} \Big[\big(\mathcal{P}_2(\tau + \varepsilon s) - \mathcal{P}_2(\tau)\big) - \big(\mathcal{P}_1(\tau + \varepsilon s) - \mathcal{P}_1(\tau)\big) \Big]. \tag{38}$$

Proof. Fix $\lambda \in \mathbb{R}$. We will show the representation (38) on the event that $\tau > \lambda$, which will suffice as we may take $\lambda \to -\infty$ at the end.

Let $t \in \mathcal{K}$. Using Lemma 5.1 with $y = \tau$ and $x = \tau + \varepsilon t$, we see

$$\begin{aligned}\mathcal{D}(\tau + \varepsilon t) = \sup_{i \in \mathbb{N}} &\Big\{ Z_i^{\lambda,b}(\tau - \lambda) + \mathcal{P}[(\tau, i) \to (\tau + \varepsilon t, 1)] \Big\} \\ &- \sup_{i \in \mathbb{N}} \Big\{ Z_i^{\lambda,a}(\tau - \lambda) + \mathcal{P}[(\tau, i) \to (\tau + \varepsilon t, 1)] \Big\}.\end{aligned}$$

Note that, by Proposition 5.2, $Z_1^{\lambda,a}(\tau - \lambda) > Z_3^{\lambda,a}(\tau - \lambda)$, and similarly $Z_1^{\lambda,b}(\tau - \lambda) > Z_3^{\lambda,b}(\tau - \lambda)$. Since by Lemma 5.1 the supremums in the last displayed equation are attained at finite indices, the continuity of $t \mapsto \mathcal{P}[(\tau, i) \to (\tau + \varepsilon t, 1)]$ for each i implies that there exists random $\varepsilon_0 > 0$ (depending on the supremum-achieving indices and on the compact set $\mathcal{K}$) such that, for all $0 < \varepsilon < \varepsilon_0$ and $t \in \mathcal{K}$,

$$\begin{aligned}\mathcal{D}(\tau + \varepsilon t) = \max_{i=1,2} &\Big\{ Z_i^{\lambda,b}(\tau - \lambda) + \mathcal{P}[(\tau, i) \to (\tau + \varepsilon t, 1)] \Big\} \\ &- \max_{i=1,2} \Big\{ Z_i^{\lambda,a}(\tau - \lambda) + \mathcal{P}[(\tau, i) \to (\tau + \varepsilon t, 1)] \Big\}.\end{aligned}$$

Now by definition $\mathcal{D}(\tau) = Z_1^{\lambda,b}(\tau - \lambda) - Z_1^{\lambda,a}(\tau - \lambda)$. We also know, since τ is a point of increase of $\mathcal{D}$ and by the monotonicity (in t) of the maximizing index as proven in Lemma 2.9, that the first maximum in the previous display is attained at $i = 2$ and the second at $i = 1$ for all $0 < \varepsilon < \varepsilon_0$ and $t \in \mathcal{K}$; this implies by continuity that $Z_1^{\lambda,b}(\tau - \lambda) = Z_2^{\lambda,b}(\tau - \lambda)$. It also yields Lemma 5.9 since

$$\begin{aligned}\mathcal{D}(\tau + \varepsilon t) - \mathcal{D}(\tau) &= \mathcal{P}[(\tau, 2) \to (\tau + \varepsilon t, 1)] - \mathcal{P}[(\tau, 1) \to (\tau + \varepsilon t, 1)] \\ &= \max_{0 \le s \le t} \Big(\big(\mathcal{P}_2(\tau + \varepsilon s) - \mathcal{P}_2(\tau)\big) - \big(\mathcal{P}_1(\tau + \varepsilon s) - \mathcal{P}_1(\tau)\big) \Big).\end{aligned}$$

∎

Proof of Theorem 3. We must prove that $\varepsilon^{-1/2}(\mathcal{D}(\tau+\varepsilon t)-\mathcal{D}(\tau))$ converges weakly, as a process on any compact set, to $\mathcal{L}$, for τ equal to τ_λ, ρ^h, or $\xi_{[c,d]}$ (the last conditionally on the event that $\mathsf{NC}(\mathcal{D})\cap[c,d]\neq\varnothing$. We start by fixing a compact set $\mathcal{K}$.

We will prove in Lemma 5.12 ahead that all of the types of random variables listed are almost surely points of increase of $\mathcal{D}$. Taking this statement for granted, by Lemma 5.9 we have, for each of the three cases of the definition of τ,

$$\varepsilon^{-1/2}\left(\mathcal{D}(\tau+\varepsilon t)-\mathcal{D}(\tau)\right)=\varepsilon^{-1/2}\max_{0\le s\le t}\Big[\big(\mathcal{P}_2(\tau+\varepsilon s)-\mathcal{P}_2(\tau)\big)-\big(\mathcal{P}_1(\tau+\varepsilon s)-\mathcal{P}_1(\tau)\big)\Big]. \tag{39}$$

We *claim* that the expression being maximized on the righthand side is absolutely continuous, as a process in s, to rate four Brownian motion, for each case of τ. Given the claim, Theorem 3 follows immediately by applying Corollary 5.4.

It remains to prove the claim for each case of τ.

Now, τ_λ and ρ^h are each stopping times with respect to the augmented filtration generated by $(\mathcal{P}_i)_{i\in\mathbb{N}}$ as defined in Remark 5.8. Thus it follows that $[\tau,\tau+2]$ is a stopping domain in these two cases. Then, by the strong Brownian Gibbs property and Remark 5.8, an argument as in Corollary 2.5 implies that $\mathcal{P}_i(\tau+\varepsilon s)-\mathcal{P}_i(\tau)$ for $i=1$ and 2 are jointly absolutely continuous to two independent rate two Brownian motions. This implies the claim in these two cases.

So it only remains to prove the claim for $\xi_{[c,d]}$, on the event that $\mathsf{NC}(\mathcal{D})\cap[c,d]\neq\varnothing$. We start by observing a convenient representation for $\xi_{[c,d]}$. Namely, let U be an uniform random variable on the interval $[0,\mathcal{D}(d)-\mathcal{D}(c)]$, which is conditionally independent of $\mathcal{D}$ given $\mathcal{D}(d)-\mathcal{D}(c)$, and let

$$\rho_c^U=\inf\big\{t>c:\mathcal{D}(t)=\mathcal{D}(c)+U\big\};$$

the infimum is over a non-empty set on the event that $\mathcal{D}(d)>\mathcal{D}(c)$, i.e., $\mathsf{NC}(\mathcal{D})\cap[c,d]\neq\varnothing$. It is easy to see that ρ_c^U has the same distribution as $\xi_{[c,d]}$ by the definition of the distribution function of the latter. Also let ρ_c^h be defined analogously to ρ_c^U with h in place of U; observe that ρ_c^h is a stopping time with respect to $\mathcal{P}$.

Let $\mathcal{P}'_U(s)=(\mathcal{P}_1(\rho_c^U+s)-\mathcal{P}_1(\rho_c^U),\mathcal{P}_2(\rho_c^U+s)-\mathcal{P}_2(\rho_c^U))$ for notational convenience, and $\mathcal{P}'_h$ be defined with ρ_c^h in place of ρ_c^U. We must show that $\mathcal{P}'_U\ll(B_1,B_2)$, with B_1 and B_2 independent rate two Brownian motions. So let $A\subseteq\mathcal{C}([0,1])$ be an event such that $\mathbb{P}((B_1,B_2)\in A)=0$. Now,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{P}'_U\in A,\mathcal{D}(d)-\mathcal{D}(c)>0\right)&=\mathbb{E}\left[\mathbb{P}\left(\mathcal{P}'_U\in A\mid U,\mathcal{D}(d)-\mathcal{D}(c)\right)\mathbb{1}_{\mathcal{D}(d)-\mathcal{D}(c)>0}\right]\\
&=\mathbb{E}\left[\int_0^{\mathcal{D}(d)-\mathcal{D}(c)}\frac{\mathbb{P}\left(\mathcal{P}'_h\in A\mid\mathcal{D}(d)-\mathcal{D}(c)\right)}{\mathcal{D}(d)-\mathcal{D}(c)}\mathbb{1}_{\mathcal{D}(d)-\mathcal{D}(c)>0}\,dh\right]\\
&\le\mathbb{E}\left[\int_0^\infty\sum_{k=1}^\infty\frac{\mathbb{P}\left(\mathcal{P}'_h\in A\mid\mathcal{D}(d)-\mathcal{D}(c)\right)}{\mathcal{D}(d)-\mathcal{D}(c)}\mathbb{1}_{\mathcal{D}(d)-\mathcal{D}(c)\in[(k+1)^{-1},k^{-1}]}\,\mathrm{d}h\right]\\
&\le\int_0^\infty\sum_{k=1}^\infty(k+1)\mathbb{P}\left(\mathcal{P}'_h\in A\right)\,dh=0.
\end{aligned}$$

In the final line, we used Fubini's theorem and, in the final equality, that $\mathcal{P}'_h\ll(B_1,B_2)$ (and so $\mathbb{P}(\mathcal{P}'_h\in A)=0$) as above since ρ_c^h is a stopping time. This along with the fact that $\mathcal{D}(d)-\mathcal{D}(c)>0$ with positive probability which is proved next, completes the proof of the claim in the case that $\tau=\xi_{[c,d]}$ and the proof of Theorem 3. □

Remark 5.10. The above proof in the case of $\xi_{[c,d]}$ can be easily adapted to prove that the local limit of Brownian local time $\mathcal{L}$ at a point sampled according to $\mathcal{L}$ on a compact interval is also $\mathcal{L}$, a statement for which we were unable to find a reference in the literature.

Lemma 5.11 (Positive probability of $\mathsf{NC}(\mathcal{D}) \cap [c,d] \neq \varnothing$)**.** *Fix $d > c$. Then $\mathcal{D}(d) - \mathcal{D}(c) > 0$ with positive probability.*

Proof. As in the proof of Lemma 4.5, we can write, where $\mathcal{S}^{\cup}(y,x) = \mathcal{S}(y,x) + (y-x)^2$,

$$\begin{aligned}\mathcal{D}(d) - \mathcal{D}(c) = \mathcal{S}^{\cup}(y_b, d) - \mathcal{S}^{\cup}(y_a, d) - (\mathcal{S}^{\cup}(y_b, c) - \mathcal{S}^{\cup}(y_a, c))\\ - (y_b - d)^2 + (y_b - c)^2 + (y_a - d)^2 - (y_a - c)^2.\end{aligned}$$

Since $\mathcal{S}^{\cup}(y,x)$ has the same distribution for every fixed $y, x \in \mathbb{R}$, taking expectations shows that $\mathbb{E}[\mathcal{D}(d) - \mathcal{D}(c)] > 0$. That $\mathcal{D}(d) - \mathcal{D}(c) \geq 0$ almost surely completes the proof. ☐

We next prove the small missing step of the proof of Theorem 3, that the random locations in question are almost surely points of increase of $\mathcal{D}$, before turning to the larger task of proving Proposition 5.2.

Lemma 5.12. *Each of τ_λ, ρ^h, and $\xi_{[c,d]}$ is almost surely a point of increase (the final conditionally on $\mathsf{NC}(\mathcal{D}) \cap [c,d] \neq \varnothing$).*

Proof. First, τ_λ is a point of increase by definition.

Next we consider ρ^h. Observe that, on the event that ρ^h is not a point of increase, there must be a non-trivial interval with left endpoint ρ^h where $\mathcal{D}$ is flat, i.e., equals h. In particular, the mentioned event implies the existence of a $q \in \mathbb{Q}$ such that $\mathcal{D}(q) = h$. It is thus sufficient, by a union bound over rationals, to show that $\mathcal{D}(x)$ is a continuous random variable for any fixed $x \in \mathbb{R}$.

We will use the symmetry property of the parabolic Airy sheet $\mathcal{S}$ that $\mathcal{S}(x,y) \stackrel{d}{=} \mathcal{S}(-y,-x)$ as processes on $\mathbb{R}^2$ (see [DOV18, Lemma 9.1]; this follows by a similar relation that holds in Brownian LPP) and the stationarity of $\mathcal{S}$ from item (i) of Definition 2.6. Now,

$$\begin{aligned}\mathcal{D}(x) = \mathcal{S}(y_b, x) - \mathcal{S}(y_a, x) &\stackrel{d}{=} \mathcal{S}(-x, -y_b) - \mathcal{S}(-x, -y_a)\\ &\stackrel{d}{=} \mathcal{S}(0, x - y_b) - \mathcal{S}(0, x - y_a).\end{aligned}$$

By the coupling of $\mathcal{S}$ with the parabolic Airy$_2$ process $\mathcal{P}_1$ from item (ii) of Definition 2.6, we see that

$$\mathcal{D}(x) \stackrel{d}{=} \mathcal{P}_1(x - y_b) - \mathcal{P}_1(x - y_a).$$

That the righthand side is a continuous random variable is an immediate consequence of Corollary 2.5 on the absolute continuity of increments of $\mathcal{P}_1$ to Brownian motion.

Finally, we consider $\xi_{[c,d]}$. By definition, $\xi_{[c,d]}$ is a non-constant point. Now observe that there are only countably many non-constant points of $\mathcal{D}$ that are not points of increase, again by considering a small interval to the right of any such point and invoking the denseness of rationals. Since the measure with distribution function $\mathcal{D}(\cdot) - \mathcal{D}(c)/(\mathcal{D}(d) - \mathcal{D}(c))$ has no atoms, $\xi_{[c,d]}$ almost surely does not lies in this countable subset of $\mathsf{NC}(\mathcal{D})$. This completes the proof of Lemma 5.12. ☐

Remark 5.13. It is natural to try to prove a local limit theorem like Theorem 3 with $\tau = \rho_\lambda^h$ as introduced in the proof of Theorem 3 above and $h > 0$. In fact, much of that proof remains intact when $\tau = \rho_\lambda^h$ as it is a stopping time. Unfortunately, a difficulty analogous to the one indicated in Remark 4.2 arises in showing that ρ_λ^h is almost surely a point of increase of $\mathcal{D}$.

To see this, let us condition on the data of $\mathcal{P}$ to the left of λ. Then $\mathcal{D}(\lambda) = b_1^\lambda - a_1^\lambda$ is deterministic, as are $b_j^\lambda - a_j^\lambda$ for all $j \in \mathbb{N}$, and ρ_λ^h is the hitting time of the deterministic $h + \mathcal{D}(\lambda)$. Now, $\mathcal{D}$ has flat portions which equal $b_j^\lambda - a_j^\lambda$ for various values of j. If $h = b_j^\lambda - a_j^\lambda - \mathcal{D}(\lambda)$, then ρ_λ^h will not be a point of increase.

Thus, conditionally on the data to the left of λ, there are countably many values of h such that ρ_λ^h is not a point of increase. Again as in Remark 4.2, it is possible that, on averaging over the conditioned data, ρ_λ^h will be a point of increase for every $h > 0$ almost surely—but this will require further information such as quantities like $b_j^\lambda - a_j^\lambda - \mathcal{D}(\lambda)$ are continuous random variables, which we aim to establish in future work.

It remains to prove Proposition 5.2, which is the task of the next two sections.

5.4. **Absolute continuity to Brownian motion of $Z_i^{\lambda,a}$ and $Z_i^{\lambda,b}$.** To prove Proposition 5.2, we will need that $Z_i^{\lambda,a}$ and $Z_i^{\lambda,b}$ are locally absolutely continuous away from zero to Brownian motion. Using our earlier analysis of the same property for functions of the form $x \mapsto \mathcal{P}[(\lambda, j) \to (\lambda + x, i)]$, this is straightforward, provided we know that the supremum in (36) is achieved at a finite index with probability one.

The latter statement is obtained by a somewhat delicate geometric argument, using the $i = 1$ case proved in [SV21] and recorded in this article as Lemma 2.8. The majority of this section will be devoted to proving the following proposition.

Proposition 5.14. *Almost surely, for all $x \geq 0$ and $j \in \mathbb{N}$, the supremum in the definition (36) of $Z_j^{\lambda,a}(x)$ is attained at a finite index. The index is uniformly bounded for x in any given compact set.*

While the proof strategy as adopted in [SV21] involving infinite geodesics and the fact that they follow roughly parabolic trajectories provides an approach to proving Proposition 5.14, here we choose to adopt a different strategy, in keeping with our reliance on arguments involving boundary data instead of infinite geodesics.

In the coming pages, we will refer several times to the geodesic implicit in the definition (36) of $Z_1^{\lambda,a}(z)$. This means, if the minimum index at which the supremum in that definition is attained is i_0 (which is known to be finite by Lemma 2.8), to consider the geodesic from (λ, i_0) to $(\lambda + z, 1)$ in the environment defined by $\mathcal{P}$. In the case that this geodesic is not unique, we will consider the left-most one; it is a standard consequence of planarity that this is a well-defined notion, and we refer the reader, for example, to [DOV18, Lemma 3.5].

The basic idea of Proposition 5.14 is geometric and uses the ordering or monotonicity properties of geodesics. Namely, consider the geodesic π^z implicitly defined by $Z_1^{\lambda,a}(z)$. If for some $z \geq x$ this geodesic leaves line j after $\lambda + x$, then the maximizing index for $Z_j^{\lambda,a}(x)$ must be at most the starting line number of π^z, by geodesic ordering, which we know is finite from Lemma 2.8 (Lemma 5.15). See Figure 6. The proof of Proposition 5.14 comes down to showing that the location at which π^z leaves line j goes to ∞ as $z \to \infty$, thus covering all values of x for $Z_j^{\lambda,a}(x)$. This is shown by arguing that, otherwise, the geodesic from vertical line $\lambda + x$ (with boundary data $\{a_i^{\lambda+x}\}_{i\in\mathbb{N}}$) to $(\lambda + z, 1)$ would have uniformly bounded starting line number for all z (Lemma 5.16); that this is not possible is Lemma 5.17.

Lemma 5.15. *Fix $\lambda \in \mathbb{R}$ and $0 \leq x \leq z$, and let π^z be the geodesic implicit in the definition (36) of $Z_1^{\lambda,a}(z)$. Fix $j \in \mathbb{N}$ and suppose that π^z exits line j after $\lambda + x$. Then the maximizing index (the minimum such in the case of non-uniqueness) in the definition of $Z_j^{\lambda,a}(x)$ is at most $i^\lambda(z)$.*

Proof. Let ℓ be the minimum index achieving the supremum in the definition (36) of $Z_j^{\lambda,a}(x)$. Since z is fixed, let $\pi = \pi^z$. Suppose to the contrary that π leaves line j after $\lambda + x$, and $\ell > i^\lambda(z)$.

Consider the geodesic from (λ, ℓ) to $(\lambda + x, j)$ and call it ρ. Note that our hypothesis that $\ell > i^\lambda(z)$ and planarity imply that ρ and π must have non-empty intersection, though it may possibly be just $(\lambda + x, j)$. Let $y \in \mathbb{R} \times \mathbb{N}$ be the point of intersection with minimum first coordinate. Consider the restriction of π from its starting point to y, which we label π', and we define ρ' from ρ similarly. Recall from Definition 2.1 that the weight of a path γ in the environment given by $\mathcal{P}$ is denoted by $\mathcal{P}[\gamma]$. Note that

$$\mathcal{P}[\pi'] + a_{i^\lambda(z)}^\lambda \geq \mathcal{P}[\rho'] + a_\ell^\lambda, \tag{40}$$

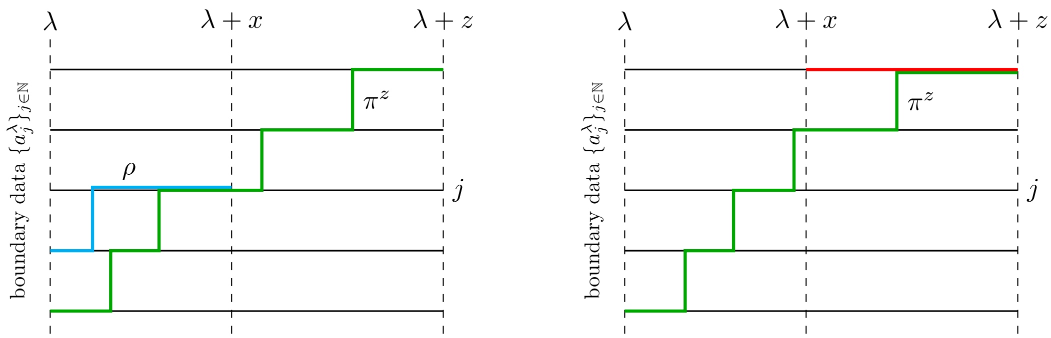

FIGURE 6. On the left panel is depicted the situation where the geodesic π^z (in green) from the vertical line λ with boundary data $\{a_i^\lambda\}_{i\in\mathbb{N}}$ to $(\lambda+z,1)$ leaves line j after $\lambda+x$. In this case, planarity and the weight maximization property of geodesics implies that the geodesic ρ (in light blue) from the vertical line λ, again with boundary data $\{a_i^\lambda\}_{i\in\mathbb{N}}$, to $(\lambda+x,j)$ is to the left of π^z; in particular, ρ starts at a line whose index is lower than that of π^z. (In fact in the depiction, since π^z is at line j at $\lambda+x$, the portion of π^z till $\lambda+x$ must coincide with ρ. We have not shown this as it need not hold in general when π^z is at an index greater than j at $\lambda+x$.) In the right panel, π^z leaves line j before $\lambda+x$; this implies that the red geodesic from $\lambda+x$, with boundary data $\{a_i^{\lambda+x}\}_{i\in\mathbb{N}}$, to $(\lambda+z,1)$ starts at a line above the j^{th} one.

for otherwise we could replace π', as portion of the path π, by ρ' and obtain a path from the vertical line λ to $(\lambda+z,1)$ of greater weight (taking into account the boundary data at λ).

However, (40) contradicts the definition of k as the minimum index achieving a certain maximum. This is because we can consider the path ρ'' obtained by replacing ρ', as a portion of the path ρ, by π'; this path goes from $(\lambda, i^\lambda(z))$ to $(\lambda+x,j)$ and, by (40), satisfies

$$\mathcal{P}[\rho''] + a^\lambda_{i^\lambda(z)} \geq \mathcal{P}[\rho] + a^\lambda_\ell;$$

but then ℓ is not the minimum index achieving the supremum in the definition of $Z_j^{\lambda,a}(x)$ since $\ell > i^\lambda(z)$, whence the contradiction. Thus the proof of Lemma 5.15 is complete. □

Lemma 5.16. *Fix $\lambda\in\mathbb{R}$ and $0\le x\le z$, and let π^z be the geodesic implicit in the definition (36) of $Z_1^{\lambda,a}(z)$. Let j be the index of the line that π^z is on at $\lambda+x$. Then $j\geq i^{\lambda+x}(z-x)$.*

Proof. For ease of notation, let $\pi^z=\pi$, $i^{\lambda+x}=i^{\lambda+x}(z-x)$, and $i^\lambda=i^\lambda(z)$. Recall from Lemma 2.8 that $i^{\lambda+x}$ and i^λ are respectively the minimum indices which achieve the supremums in

$$\begin{aligned}\mathcal{S}(y_a,\lambda+z) &= \sup_{i\in\mathbb{N}}\left\{a_i^{\lambda+x} + \mathcal{P}[(\lambda+x,i)\to(\lambda+z,1)]\right\}\\ &= \sup_{i\in\mathbb{N}}\left\{a_i^{\lambda} + \mathcal{P}[(\lambda,i)\to(\lambda+z,1)]\right\}.\end{aligned} \tag{41}$$

We *claim* that

$$a_j^{\lambda+x} + \mathcal{P}[(\lambda+x,j)\to(\lambda+z,1)] \geq a^\lambda_{i^\lambda} + \mathcal{P}[(\lambda,i^\lambda)\to(\lambda+z,1)]; \tag{42}$$

though we will not need this, we note that combining this inequality with (41) implies that we actually have equality.

We first prove Lemma 5.16 given this claim. By the definition of i^λ and since both supremums in (41) equal $\mathcal{S}(y_a,\lambda+z)$, (42) implies that the first supremum in (41) is achieved at $i=j$. Then by the minimality in the definition of $i^{\lambda+x}$, it follows that $i^{\lambda+x}\le j$, which is the claim of Lemma 5.16.

It remains to prove (42). Since j is the index of the line π (the geodesic from (λ, i^λ) to $(\lambda+z,1)$) is on at $\lambda+x$, we see that

$$\mathcal{P}[(\lambda, i^\lambda) \to (\lambda+z,1)] = \mathcal{P}[(\lambda, i^\lambda) \to (\lambda+x,j)] + \mathcal{P}[(\lambda+x,j) \to (\lambda+z,1)].$$

Thus verifying (42) reduces to showing

$$a_j^{\lambda+x} \geq a_{i^\lambda}^\lambda + \mathcal{P}[(\lambda, i^\lambda) \to (\lambda+x,j)].$$

To show this, observe that, by the definition (11) of a_i^λ for $i \in \mathbb{N}$ and since $\mathcal{P}[x \to z] \geq \mathcal{P}[x \to y] + \mathcal{P}[y \to z]$ for any coordinates $x, y, z \in \mathbb{R} \times \mathbb{N}$,

$$\begin{aligned}
&a_{i^\lambda}^\lambda + \mathcal{P}[(\lambda, i^\lambda) \to (\lambda+x,j)] \\
&\quad= \lim_{k\to\infty} \Big(\mathcal{P}[(y_a)_k \to (\lambda, i^\lambda)] + \mathcal{P}[(\lambda, i^\lambda) \to (\lambda+x,j)] - \mathcal{P}[(y_a)_k \to (\lambda,1)] + \mathcal{S}(y_a,\lambda)\Big) \\
&\quad\leq \lim_{k\to\infty} \Big(\mathcal{P}[(y_a)_k \to (\lambda+x,j)] - \mathcal{P}[(y_a)_k \to (\lambda,1)] + \mathcal{S}(y_a,\lambda)\Big) \\
&\quad= \lim_{k\to\infty} \Big(\mathcal{P}[(y_a)_k \to (\lambda+x,j)] - \mathcal{P}[(y_a)_k \to (\lambda+x,1)] + \mathcal{S}(y_a,\lambda+x)\Big) \\
&\qquad\quad+ \lim_{k\to\infty} \Big(\mathcal{P}[(y_a)_k \to (\lambda+x,1)] - \mathcal{P}[(y_a)_k \to (\lambda,1)] + \mathcal{S}(y_a,\lambda) - \mathcal{S}(y_a,\lambda+x)\Big).
\end{aligned}$$

Now, the first limit is $a_j^{\lambda+x}$ by definition, while the second is zero by item (ii) of Definition 2.6. This completes the verification of (42) and thus the proof of Lemma 5.16. ☐

The next lemma says that the geodesic from the vertical line at λ, with boundary data $\{a_i^\lambda\}_{i\in\mathbb{N}}$, to $(\lambda+z,1)$ must have starting line index go to ∞ as $z\to\infty$. We have till now adopted the notation $i^\lambda(z)$ for the starting line index of this geodesic (recall from before Lemma 2.9). Here we slightly augment the notation: for $y>0$, $\lambda\in\mathbb{R}$, and $z\geq\lambda$, $i_y^\lambda(z)$ will be the starting line index of the geodesic from vertical line at λ to $(\lambda+z,1)$, where the boundary data at λ is given by the analogue of $\{a_i^\lambda\}_{i\in\mathbb{N}}$ obtained by replacing y_a by y in the definition of $\{a_i^\lambda\}_{i\in\mathbb{N}}$ (11). In this notation, Lemma 2.9 says that, if $y<y'$, then $i_y^\lambda(z) \leq i_{y'}^\lambda(z)$ for all $z\geq 0$, and we will make use of this.

Lemma 5.17. *Fix $y>0$ and $\lambda\in\mathbb{R}$. Then, almost surely, $\lim_{z\to\infty} i_y^\lambda(z)=\infty$.*

Proof. Suppose not. Then with positive probability, there exists K such that

$$i_y^\lambda(z) \leq K$$

for all $z\geq 0$. By Lemma 2.9, we have that $i_{y'}^\lambda(z) \leq i_y^\lambda(z)$ for all $z\geq 0$ and $0<y'<y$. Thus, on the same event, by the pigeonhole principle, there exist positive rationals $y_1<y_2$ such that, for all large z, $i_{y_1}^\lambda(z) = i_{y_2}^\lambda(z)$.

But then, by Lemma 2.8, we see that for all such large z,

$$\mathcal{S}(y_2,\lambda+z) - \mathcal{S}(y_1,\lambda+z) = c,$$

where c is a (random) finite constant. This contradicts Lemma 4.5, which, by a union bound over positive rational starting points, implies that $\lim_{z\to\infty}(\mathcal{S}(y_2,\lambda+z) - \mathcal{S}(y_1,\lambda+z)) = \infty$ almost surely. ☐

We may now give the proof of Proposition 5.14. It may be useful to refer to Figure 6 once again.

Proof of Proposition 5.14. The case of $j=1$ is asserted by Lemma 2.8 (which is being cited from [SV21]), and we will rely on this input to prove the general case.

Let us denote the smallest index achieving the maximum for $Z_j^{\lambda,a}(x)$ by $\ell_j^\lambda(x)$, which is infinite in the case that the maximum is not achieved at any finite index. By a planarity argument as in

Lemma 2.9 (which asserts the following for $j = 1$), we see that $\ell_j^\lambda(x)$ is non-decreasing in x. Thus it is sufficient to prove that, almost surely, $\ell_j^\lambda(x)$ is finite for each $x \in \mathbb{N}$. Fix an $x \in \mathbb{N}$ now.

By Lemma 5.15, we see that $\ell_j^\lambda(x) \le i^\lambda(z)$ whenever z is such that the geodesic π^z implicit in the definition (36) of $Z_1^{\lambda,a}(z)$ exits line j after $\lambda + x$. Since $i^\lambda(z) < \infty$ for all z by the $j = 1$ case, we are done if we can show that, almost surely, for all large enough z, π^z exits line j after $\lambda + x$.

Suppose not. Then on a positive probability event the index of the line that π^z is on at $\lambda + x$ is at most $j - 1$ for all $z \ge x$. Then on the same event, by Lemma 5.16, for all $z \ge x$, $i^{\lambda+x}(z - x) \le j - 1$. This contradicts Lemma 5.17, which asserts that, with probability one, $i^{\lambda+x}(z - x) \to \infty$ as $z \to \infty$. This completes the proof of Proposition 5.14. ☐

Equipped with Lemma 5.14 we now provide the outstanding proof of Proposition 5.2.

5.5. **Proving Proposition 5.2.** As mentioned, the proof relies on the fact that two-dimensional Brownian motion almost surely never hits any given point in the plane, which we recall precisely.

Lemma 5.18 (Corollary 2.26 of [MP10])**.** *Let $x, y \in \mathbb{R}^2$ and $B : [0, \infty) \to \mathbb{R}^2$ be two-dimensional Brownian motion begun at x. Almost surely, $\mathbb{P}(y \in \{B(t) : t > 0\}) = 0$.*

The proof of Proposition 5.2 will convert the condition that $Z_1^{\lambda,a}$ and $Z_3^{\lambda,a}$ meet into one on $Z_3^{\lambda,a}$, $\mathcal{P}_1$, and $\mathcal{P}_2$. To do this, we will prove a recursive formula for $Z_i^{\lambda,a}$ in terms of $\mathcal{P}$. To explain this, recall the definition (36) of $Z_i^{\lambda,a}$:

$$Z_i^{\lambda,a}(x) = \sup_{j \ge i} \left\{ a_j^\lambda + \mathcal{P}[(\lambda, j) \to (\lambda + x, i)] \right\}.$$

Essentially, $Z_i^{\lambda,a}(x)$ is the best weight from λ to (x, i) with the boundary data $\{a_i^\lambda\}_{i\in\mathbb{N}}$. Thus heuristically, it should be expressible as a two-line LPP problem with top line $\mathcal{P}_i$ and bottom line $Z_{i+1}^{\lambda,a}$. However, observe that $Z_{i+1}^{\lambda,a}(0) = a_{i+1}^\lambda$, not zero as required by the definition (16) of the Pitman transform PT; and, indeed, we should not expect to be able to express $Z_i^{\lambda,a}$ in terms of $Z_{i+1}^{\lambda,a}$ if we subtract off the value of $Z_{i+1}^{\lambda,a}(0)$ as we would if it was the bottom line in an LPP problem, as doing so would be throwing away the information of the weight of paths starting at $(\lambda, i + 1)$.

Thus, we need to modify the definition of $\mathsf{PT}(f_1, f_2)$ to allow non-zero values at the origin for f_1 and f_2. In fact, we will only need a variant of $(\mathsf{PT}(f_1, f_2))_1$. Our variant is really just the second equality in the definition (16) of the latter, which is no longer equivalent to the first equality expressing it as an LPP problem in a two-line environment. But to emphasize the distinction, we label our variant $\mathsf{PT}_*(f_1, f_2)$. So for continuous functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$, define $\mathsf{PT}_*(f_1, f_2)$ by

$$(\mathsf{PT}_*(f_1, f_2))(x) = f_1(x) + \max_{0\le s\le x} (f_2(s) - f_1(s)). \tag{43}$$

This works better with boundary data, for example, by allowing a convenient behaviour with additive constants.

Lemma 5.19. *Let $a \in \mathbb{R}$. Let $f_1, f_2, f_2^i : [0, \infty) \to \mathbb{R}$ be continuous for $i \in \mathbb{N}$. Then,*

(i) $a + \mathsf{PT}_*(f_1, f_2) = \mathsf{PT}_*(f_1, a + f_2)$, *where* $(a + f_2)(x) = a + f_2(x)$
(ii) $\max_{i\in\mathbb{N}} \mathsf{PT}_*(f_1, f_2^i) = \mathsf{PT}_*(f_1, \max_{i\in\mathbb{N}} f_2^i)$.

The proof of Lemma 5.19 is trivial and we omit it.

Now we may state the recursive formula for $Z_i^{\lambda,a}$. Essentially, $Z_i^{\lambda,a}$ is $\mathcal{P}_i$ reflected off of $Z_{i+1}^{\lambda,a}$ (in the sense of Skorohod reflection; see the earlier mentioned [War07] for work within KPZ on Brownian motions reflecting off of other Brownian objects, namely Dyson Brownian motion).

Lemma 5.20. *Let $\lambda \in \mathbb{R}$ and recall that $\mathcal{P}_j^\lambda(x) = \mathcal{P}_j(\lambda + x) - \mathcal{P}_j(\lambda)$ for all $j \in \mathbb{N}$. Almost surely, for all $i \in \mathbb{N}$ and $x \geq 0$,*

$$Z_i^{\lambda,a}(x) = \max\left\{(\mathsf{PT}_*(\mathcal{P}_i^\lambda, Z_{i+1}^{\lambda,a}))(x), a_i^\lambda + \mathcal{P}_i^\lambda(x)\right\}.$$

Proof. From the definition (36) of Z_i^λ,

$$Z_i^\lambda(x) = \max_{j \geq i+1}\left\{a_j^\lambda + \mathcal{P}[(\lambda, j) \to (\lambda + x, i)]\right\} \vee (a_i^\lambda + \mathcal{P}_i^\lambda(x)). \tag{44}$$

So it is enough to show that the first maximization is $\mathsf{PT}_*(\mathcal{P}_i^\lambda, Z_{i+1}^\lambda)$. For this, observe that

$$\begin{aligned}\mathcal{P}[(\lambda, j) \to (\lambda + x, i)] &= \mathcal{P}_i(\lambda + x) + \max_{0 \leq z \leq x}\left(\mathcal{P}[(\lambda, j) \to (\lambda + z, i+1)] - \mathcal{P}_i(\lambda + z)\right)\\ &= \Big(\mathsf{PT}_*\big(\mathcal{P}_i^\lambda, \mathcal{P}[(\lambda, j) \to (\lambda + \bullet, i+1)]\big)\Big)(x).\end{aligned}$$

Putting this into (44), applying Lemma 5.19, and recalling the definition (36) of Z_{i+1}^λ completes the proof of Lemma 5.20. ☐

The proof of Proposition 5.2 will show that any point where $Z_1^{\lambda,a}$ and $Z_2^{\lambda,a}$ agree is a running maximizer of $Z_2^{\lambda,a} - \mathcal{P}_1^\lambda$, and a similar statement for points of agreement of $Z_2^{\lambda,a}$ and $Z_3^{\lambda,a}$. Thus we introduce the following notation. For a stochastic process $X : [0, \infty) \to \mathbb{R}$ and $\varepsilon > 0$, let $\mathrm{rec}_\varepsilon(X) \subseteq [\varepsilon, \infty)$ be the set of running maximizers of $X|_{[\varepsilon,\infty)}$, i.e., the points $x \geq \varepsilon$ such that $X(x) = \max_{\varepsilon \leq s \leq x} X(s)$.

Proof of Proposition 5.2. It is enough to show that for every $0 < \varepsilon < T$ rationals, there almost surely does not exist $\tau \in [\varepsilon, T]$ such that $Z_1^{\lambda,a}(\tau) = Z_3^{\lambda,a}(\tau)$, since $Z_1^{\lambda,a}(x) \geq Z_3^{\lambda,a}(x)$ for all $x \geq 0$. Fix such ε and T now.

Suppose to the contrary that $\tau \in [\varepsilon, T]$ is such that $Z_1^{\lambda,a}(\tau) = Z_2^{\lambda,a}(\tau) = Z_3^{\lambda,a}(\tau)$, since always $Z_1^\lambda(x) \geq Z_2^\lambda(x) \geq Z_3^\lambda(x)$. By Lemma 5.20 and the definition (43) of PT_*, $Z_2^\lambda(\tau) = Z_3^\lambda(\tau)$ implies that

$$Z_3^{\lambda,a}(\tau) - \mathcal{P}_2^\lambda(\tau) = a_2 \vee \max_{0 \leq s \leq \tau}\left(Z_3^{\lambda,a}(s) - \mathcal{P}_2^\lambda(s)\right) \geq \max_{\varepsilon/2 \leq s \leq \tau}\left(Z_3^{\lambda,a}(s) - \mathcal{P}_2^\lambda(s)\right).$$

Similarly, $Z_1^{\lambda,a}(\tau) = Z_3^{\lambda,a}(\tau)$ implies that

$$Z_3^{\lambda,a}(\tau) - \mathcal{P}_1^\lambda(\tau) = a_1 \vee \max_{0 \leq s \leq \tau}\left(Z_2^{\lambda,a}(s) - \mathcal{P}_1^\lambda(s)\right) \geq \max_{\varepsilon/2 \leq s \leq \tau}\left(Z_3^{\lambda,a}(s) - \mathcal{P}_1^\lambda(s)\right),$$

the inequality using also that $Z_2^{\lambda,a}(x) \geq Z_3^{\lambda,a}(x)$ for all x. Thus, we see that $Z_1^{\lambda,a}(\tau) = Z_2^{\lambda,a}(\tau) = Z_3^{\lambda,a}(\tau)$ implies that

$$\tau \in \mathrm{rec}_{\varepsilon/2}(Z_3^{\lambda,a} - \mathcal{P}_1^\lambda) \cap \mathrm{rec}_{\varepsilon/2}(Z_3^{\lambda,a} - \mathcal{P}_2^\lambda) \cap [\varepsilon, T]. \tag{45}$$

We claim that the set on the righthand side is almost surely empty.

First, observe that $(\mathcal{P}_1^\lambda, \mathcal{P}_2^\lambda, Z_3^{\lambda,a})|_{[\varepsilon/2,T]} \ll (B_1, B_2, B_3)_{[\varepsilon/2,T]}$, where the B_i are independent rate two Brownian motions. To see this, note that Proposition 5.14 implies that the index achieving the supremum in the definition (36) of $Z_i^{\lambda,a}(x)$ is at most a random finite constant K for all $x \in [\varepsilon/2, T]$. On the event that $K = k$, we use that $(\mathcal{P}_1^\lambda, \ldots, \mathcal{P}_k^\lambda)$ is absolutely continuous to k independent rate two Brownian motions. Now, on $\{K = k\}$, $Z_3^{\lambda,a}$ is a function of $\mathcal{P}_3^\lambda, \ldots, \mathcal{P}_k^\lambda$: it is equal to the function $x \mapsto a_i^\lambda + \mathcal{P}[(\lambda, i) \to (x, 3)]$ for some $i \in [\![3, k]\!]$.

Recall from Section 3.2 that functions of the form $x \mapsto \mathcal{P}[(\lambda, i) \to (x, 3)]$ can be written as a sequence of Pitman transforms on the environment $\mathcal{P}_3^\lambda, \ldots, \mathcal{P}_i^\lambda$. Also recall Lemma 3.5, which says that Pitman transforms preserve local absolute continuity away from zero to Brownian motion. Thus by induction and Lemma 3.5, each of the functions $x \mapsto \mathcal{P}[(\lambda, i) \to (x, 3)]$ is locally absolutely continuous away from zero to rate two Brownian motion; a union bound over $i \in [\![1, k]\!]$ and $k \in \mathbb{N}$ therefore implies that the same is true of $Z_3^{\lambda,a}$.

That $(\mathcal{P}_1^\lambda, \mathcal{P}_2^\lambda, Z_3^{\lambda,a})|_{[\varepsilon/2,T]} \ll (B_1, B_2, B_3)_{[\varepsilon/2,T]}$ implies that $(Z_3^{\lambda,a} - \mathcal{P}_1^\lambda, Z_3^{\lambda,a} - \mathcal{P}_2^\lambda)|_{[\varepsilon/2,T]}$ is absolutely continuous to the restriction to $[\varepsilon/2, T]$ of a pair of (non-trivially) correlated Brownian motions, which is in turn absolutely continuous to $(B, B')|_{[\varepsilon/2,T]}$, where B and B' are a pair of independent Brownian motions on $[0, \infty)$.

Since the set on the righthand side of (45) is a function of $(Z_3^{\lambda,a} - \mathcal{P}_1^\lambda)|_{[\varepsilon/2,T]}$ and $(Z_3^{\lambda,a} - \mathcal{P}_2^\lambda)|_{[\varepsilon/2,T]}$, to prove that that set is empty almost surely, it is sufficient to show that, almost surely,

$$\mathrm{rec}_{\varepsilon/2}(B) \cap \mathrm{rec}_{\varepsilon/2}(B') \cap [\varepsilon, T] = \varnothing.$$

Now by the Markov property of Brownian motion, and since $\mathrm{rec}_{\varepsilon/2}(X)$ is unaffected by vertical shifts to X, $\mathrm{rec}_{\varepsilon/2}(B) \cap \mathrm{rec}_{\varepsilon/2}(B')$ has the same distribution as $\varepsilon/2 + \mathrm{rec}_0(B) \cap \mathrm{rec}_0(B')$, where $x + A$ for $x \in \mathbb{R}$ and a set A is the set $\{x + y : y \in A\}$.

By Lévy's identity (Proposition 3.2), $(\varepsilon/2 + \mathrm{rec}_0(B) \cap \mathrm{rec}_0(B')) \cap [\varepsilon, \infty) = \varnothing$ almost surely is equivalent to two independent Brownian motions almost surely not sharing a zero after time $\varepsilon/2$. This is because, if M is the running maximum of B, then $\mathrm{rec}_0(B)$ is the set of points where $M = B$, i.e., $M - B = 0$; by Lévy's identity, these have the distribution of the set of points where $|B| = 0$. That independent Brownian motions almost surely do not share a zero after time $\varepsilon/2$ is implied by Lemma 5.18, thus completing the proof of Proposition 5.2. □

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