OPTIMAL TAIL EXPONENTS IN GENERAL LAST PASSAGE PERCOLATION VIA BOOTSTRAPPING & GEODESIC GEOMETRY

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Abstract. We consider last passage percolation on $\mathbb{Z}^2$ with general weight distributions, which is expected to be a member of the Kardar-Parisi-Zhang (KPZ) universality class. In this model, an oriented path between given endpoints which maximizes the sum of the i.i.d. weight variables associated to its vertices is called a geodesic. Under natural conditions of curvature of the limiting geodesic weight profile and stretched exponential decay of both tails of the point-to-point weight, we use geometric arguments to upgrade the assumptions to prove optimal upper and lower tail behavior with the exponents of $3/2$ and 3 for the weight of the geodesic from $(1,1)$ to (r,r) for all large finite r. The proofs merge several ideas, including the well known super-additivity property of last passage values, concentration of measure behavior for sums of stretched exponential random variables, and geometric insights coming from the study of geodesics and more general objects called geodesic watermelons. Previously such optimal behavior was only known for exactly solvable models, with proofs relying on hard analysis of formulas from integrable probability, which are unavailable in the general setting. Our results illustrate a facet of universality in a class of KPZ stochastic growth models and provide a geometric explanation of the upper and lower tail exponents of the GUE Tracy-Widom distribution, the conjectured one point scaling limit of such models. The key arguments are based on an observation of general interest that super-additivity allows a natural iterative bootstrapping procedure to obtain improved tail estimates.

Contents

1. Introduction, main results, and key ideas

The 1+1 dimensional Kardar-Parisi-Zhang (KPZ) universality class includes a wide range of models of interfaces suspended over a one-dimensional domain, in which growth in a direction normal to the surface competes with a smoothening surface tension, in the presence of a local randomizing force that roughens the surface. Such interfaces are expected to exhibit characteristic exponents dictating one-point height fluctuations and correlation length. While the class is predicted to describe a plethora of models including first passage percolation, last passage percolation, the KPZ stochastic PDE, and the totally asymmetric simple exclusion process, among others, the above predictions have been rigorously proven only for a very small subset of them.

A canonical object in this context is the Airy_2 process [PS02], a stationary continuous stochastic process, and its one point distribution, the GUE Tracy-Widom distribution, first discovered in random matrix theory [TW94]. The Airy_2 process is the universal limit that many of the interfaces in the above class are expected to converge to, under proper centering and scaling and appropriate boundary conditions.

We now give a brief description of the model of last passage percolation, an important member of this class and the model in consideration in this article.

In last passage percolation (LPP) one assigns i.i.d. non-negative weights $\{\xi_v : v \in \mathbb{Z}^2\}$ to the vertices of $\mathbb{Z}^2$ and studies the weight and geometry of weight-maximising directed paths. The weight of a given up-right nearest neighbor path γ is $\ell(\gamma) := \sum_{v\in\gamma} \xi_v$. For given vertices $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{Z}^2$ with $u_i \leq v_i$ for $i = 1$ and 2 (i.e., the natural partial order), the last passage value $X_{u,v}$ is defined by $X_{u,v} = \max_{\gamma:u\to v} \ell(\gamma)$, where the maximization is over the set of up-right paths from u to v; maximizing paths are called *geodesics*. For $r \in \mathbb{N}$, we adopt the shorthand $X_r := X_{(1,1),(r,r)}$.

A few special distributions of the vertex weights $\{\xi_v : v \in \mathbb{Z}^2\}$ render the model integrable, i.e., admitting exact connections to algebraic objects such as random matrices and Young diagrams. This allows exact computations which lead to the appearance of the Airy$_2$ process and hence the GUE Tracy-Widom distribution. Most of the progress in understanding the KPZ universality class has relied primarily on such exactly solvable features. For concreteness, we highlight next the special case of exponentially distributed (with rate one) vertex weights. In this case, Johansson proved the following [Joh00] via the development of the aforementioned connections to representation theory.

Theorem 1.1 (Theorem 1.6 of [Joh00]). *Let $\{\xi_v : v \in \mathbb{Z}^2\}$ be i.i.d. exponential rate one random variables. As $r \to \infty$ it holds that*

$$\frac{X_r - 4r}{2^{4/3} r^{1/3}} \xrightarrow{d} F_{\mathrm{TW}},$$

where F_{TW} is the GUE Tracy-Widom distribution, and $\xrightarrow{d}$ denotes convergence in distribution.

This one-point convergence was later upgraded to convergence to the Airy$_2$ process in the sense of finite dimensional distributions by considering a suitable observable (see for example [BF08]).

An important feature of the GUE Tracy-Widom distribution is the "non-Gaussian" behavior of its upper and lower tails. In particular, it is known, for example from [RRV11, Theorem 1.3], that as $\theta \to \infty$,

$$\begin{aligned} F_{\mathrm{TW}}\big([\theta,\infty)\big) &= \exp\left(-\frac{4}{3}\theta^{3/2}\,(1+o(1))\right) \qquad \text{and} \\ F_{\mathrm{TW}}\big((-\infty,-\theta]\big) &= \exp\left(-\frac{1}{12}\theta^{3}\,(1+o(1))\right). \end{aligned} \tag{1}$$

In fact, these tail exponents of 3/2 and 3 are more universal in KPZ than just the GUE Tracy-Widom distribution. The latter distribution and the Airy$_2$ process are only expected to arise under what is called the *narrow-wedge* initial data; this is seen in Theorem 1.1 by the imposition that X_r be the weight of the best path from the *fixed* starting point of $(1, 1)$. But the same tail exponents are expected for a much wider class of initial data. For example, the results of [CG18] assert that the (suitably scaled) solution to the KPZ stochastic PDE has upper bounds on the one-point upper and lower tails with the same tail exponents (up to a certain depth into the tail) under a wide class of *general* initial data. Similarly, the same exponents are known from [RRV11] for the entire class of Tracy-Widom(β) distributions (with the GUE case corresponding to $\beta = 2$).

Given the distributional convergence asserted by Theorem 1.1, it is natural to ask whether tail bounds similar to (1) are satisfied by X_r at the finite r level. Indeed, again in the case of exponential weights, estimates along these lines have been attained which achieve the correct upper and lower tail exponents of 3/2 and 3. Johansson proved in [Joh00, Remark 1.5] via representation theoretic techniques that X_r is equal in distribution to the top eigenvalue of the Laguerre Unitary Ensemble, and upper bounds on the upper and lower tails on this eigenvalue were proved in [LR10, Theorem 2].

[LR10] remarks, but does not prove, that a lower bound on the upper tail should be achievable by methods in the paper, but not a lower bound on the lower tail; the latter was proved very recently in [BGHK19, Theorem 2]. This discussion may be summarized as the following theorem.

Theorem 1.2 ([Joh00, LR10, BGHK19])**.** *Let $\{\xi_v : v \in \mathbb{Z}^2\}$ be i.i.d. exponential random variables. There exist positive finite constants c_1, c_2, c_3, θ_0, and r_0 such that, for $r > r_0$ and $\theta_0 < \theta < r^{2/3}$,*

$$\mathbb{P}\left(X_r > 4r + \theta r^{1/3}\right) \le \exp\left(-c_1\theta^{3/2}\right) \qquad \text{and}$$
$$\exp\left(-c_2\theta^3\right) \le \mathbb{P}\left(X_r < 4r - \theta r^{1/3}\right) \le \exp\left(-c_3\theta^3\right).$$

Remark 1.3. In fact, the missing lower bound on the upper tail is a straightforward consequence of one of our results (Theorem 2) along with with the distributional convergence in Theorem 1.1 and an application of the Portmanteau theorem.

That the above bounds hold only for $\theta \le r^{2/3}$ is an important fact because one should not expect universality beyond this threshold. The lower tail is trivially zero for $\theta > 4r^{2/3}$ since the vertex weights are non-negative; for the upper tail, beyond this level, we enter the large deviation regime, where the tail behavior is dictated by the individual vertex distribution. Thus in the case of exponential LPP, the upper tail decays exponentially in $\theta r^{1/3}$ for $\theta > r^{2/3}$.

Similar bounds as Theorem 1.2 are available in only a handful of other LPP models; these are when the vertex weights are geometric [Joh00, BDM$^+$01], and the related models of Poissonian LPP [LM01, LMR02] and Brownian LPP [OY02, LR10]. In these models as well, the bounds are proven using powerful identities with random matrix theory and connections to representation theory, combined with precise analysis of the resulting formulas.

However, the conjectured universality of KPZ behavior suggests that similar bounds should hold under rather minimal assumptions on the vertex weight distribution, i.e., even when special connections to random matrix theory and representation theory are unavailable. Thus it is an important goal to develop more robust methods of investigation that may apply to a wider class of models, an objective that has driven a significant amount of work in this field, with the eventual aim to go beyond integrability. The present work is a continuation of this program.

Nonetheless, despite various attempts, so far only a few results are known to be true in a universal sense. These include the existence of a limiting geodesic weight profile (i.e., the expected geodesic weight as the endpoint varies) and its concavity under mild moment assumptions on the vertex weights [Mar06]. This is a relatively straightforward consequence of super-additivity properties exhibited by the geodesic weights, as we elaborate on later. This and certain general concentration estimates based on martingale methods were first developed in Kesten's seminal work on first passage percolation (FPP) [Kes86]; FPP is a notoriously difficult to analyze and canonical non-integrable model in the KPZ class where the setting is the same as that of LPP, but one instead minimizes the weight among all paths between two points, without any orientation constraint. Similar arguments extend to the case of general LPP models. Note that while the precise limiting profile is expected to depend on the model, properties such as concavity as well as local fluctuation behavior are predicted to be universal.

Following Kesten's work, there has been significant progress in FPP in providing rigorous proofs assuming certain natural conditions, such as strong curvature properties of limit shapes and the existence of critical exponents dictating fluctuations. Thus an important broad goal is to extract the minimal set of key properties of such models that govern other more refined behavior. The recent work of the authors with Riddhipratim Basu and Alan Hammond in [BGHH20], as well as the present work, are guided by the same philosophy. We will revisit this discussion in more detail after the statements of our main results.

To initiate the geometric perspective of the present paper, we point out the disparity in the upper and lower tail exponents in Theorem 1.2. This is not surprising since, while the upper tail event enforces the existence of a *single* path of high weight, the lower tail event is global and forces *all* paths to have low weight.

However, the precise exponents of 3/2 and 3 might appear mysterious, and it is natural to seek a geometric explanation for them. This is the goal of this work. More precisely, we establish bounds with optimal exponents in the nature of Theorem 1.2, starting from certain much weaker tail bounds as well as local strong concavity assumptions on the limit shape. In particular, we do not make use of any algebraic techniques in our arguments. Instead, our methods are strongly informed by an understanding of the geometry of geodesics and other weight maximising path ensembles in last passage percolation.

We next set up precisely the framework of last passage percolation on $\mathbb{Z}^2$, describe our assumptions, and state our main results.

1.1. **Model and notation.** We denote the set $\{1, 2, \ldots\}$ by $\mathbb{N}$, and, for $i, j \in \mathbb{Z}$, we will denote the integer interval $\{i, i+1, \ldots, j\}$ by $[\![i, j]\!]$.

We start with a random field $\left\{\xi_v : v \in \mathbb{Z}^2\right\}$ of i.i.d. random variables following a distribution ν supported on $[0, \infty)$. We consider up-right nearest neighbor paths, which we will refer to as *directed* paths. For a directed path γ, the associated *weight* is denoted $\ell(\gamma)$ and is defined by

$$\ell(\gamma) := \sum_{u \in \gamma} \xi_u.$$

For $u, v \in \mathbb{Z}_+^2$, with $u \preceq v$ in the natural partial order mentioned earlier, we denote by $X_{u,v}$ the *last passage value* or *weight* from u to v, i.e.,

$$X_{u,v} := \max_{\gamma : u \to v} \ell(\gamma),$$

where the maximization is over all directed paths from u to v; for definiteness, when u and v are not ordered in this way and there is no directed path from u to v, we define $X_{u,v} = -\infty$. Now for ease of notation, for sets $A, B \subseteq \mathbb{Z}^2$, we also adopt the intuitive shorthand

$$X_{A,B} := \sup_{u \in A, v \in B} X_{u,v}.$$

For $v \in \mathbb{Z}_+^2$, X_v will denote $X_{(1,1),v}$, and for $r, z \in \mathbb{Z}$, we will denote $X_{(1,1),(r-z,r+z)}$ by X_r^z. We will also denote the case of $z = 0$ by X_r, as above. Notational confusion between X_v and X_r is avoided in practice in this usage as v will always be represented by a pair of coordinates, while r is a scalar. Recall that a path (which may not be unique) which achieves the last passage value is called a geodesic.

For an up-right path γ from $(1, 1)$ to $(r - z, r + z)$, we define the *transversal fluctuation* of γ by

$$\mathrm{TF}(\gamma) := \min\left\{w : \gamma \subseteq U_{r,w,z}\right\},$$

where $U_{r,w,z}$ is the strip of width w around the interpolating line, i.e., the set of vertices $v \in \mathbb{Z}^2$ such that $v + t \cdot (-1, 1)$ lies on the line $y = \frac{r+z}{r-z} \cdot x$ for some $t \in \mathbb{R}$ with $|t| \le w/2$.

1.2. **Assumptions.** The general form of our assumptions is quite similar to the ones in the recent work [BGHH20] devoted to the study of geodesic watermelons, a path ensemble generalizing the geodesic. We start by recalling that ν is the distribution of the vertex weights and has support contained in $[0, \infty)$. The limit shape is the map $[-r, r] \to \mathbb{R} : z \mapsto \lim_{r\to\infty} r^{-1}\mathbb{E}[X_r^z]$. It follows from standard super-additivity arguments that this limit exists (though possibly infinite if the upper tail of ν is too heavy) for each $z \in [-r, r]$ and that this map is *concave* [Mar06, Proposition 2.1]. Let

$$\mu = \lim_{r\to\infty} r^{-1}\mathbb{E}[X_r]$$

be this map evaluated at zero. Also note from Theorems 1.1 and 1.2 that the fluctuations of X_r around μ can be expected to be on scale $r^{1/3}$. Finally, we point out that the normalized limit shape map in the exactly solvable models of Exponential, Geometric, Brownian, and Poissonian LPP is, up to translation and scaling by constants,

$$\sqrt{r^2 - z^2} = r - \frac{z^2}{2r} - O\left(\frac{z^4}{r^3}\right); \tag{2}$$

this will be relevant in motivating the form of our second assumption. Note that the first term of the right hand side of (2) denotes the expected linear growth of the model, while the second encodes a form of strong concavity of the limit shape. Also, the non-random fluctuation, i.e., how much the mean of X_r^z falls below (2), is expected to be $\Theta(r^{1/3})$, which is known in the aforementioned exactly solvable models.

Given the setting, we state our assumptions; not all the assumptions are required for each of the main results, and we will specify which ones are in force in each case. We will elaborate more on the content of each assumption following their statements.

(1) **Limit shape existence:** The vertex weight distribution ν is such that $\mu < \infty$.
(2) **Strong concavity of limit shape and non-random fluctuations:** There exist positive finite constants ρ, G, H, g_1, and g_2 such that, for large enough r and $z \in [-\rho r, \rho r]$,

$$\mathbb{E}[X_r^z] \in \mu r - G\frac{z^2}{r} + \left[-H\frac{z^4}{r^3}, 0\right] + \left[-g_1 r^{1/3}, -g_2 r^{1/3}\right].$$

The first three terms on the right hand side encode the limit shape and its strong concavity as in (2), while the final interval captures the non-random fluctuation.
(3) **Upper bound on moderate deviation probabilities, uniform in direction:** There exists $\alpha > 0$ such that the following hold. Fix any $\varepsilon > 0$, and let $|z| \in [0, (1-\varepsilon)r]$. Then, there exist positive finite constants c, θ_0, and r_0 (all depending on only ε) such that, for $r > r_0$ and $\theta > \theta_0$,
   (a) $\mathbb{P}\left(X_r^z - \mathbb{E}[X_r^z] > \theta r^{1/3}\right) \leq \exp(-c\theta^\alpha)$,
   (b) $\mathbb{P}\left(X_r^z - \mathbb{E}[X_r^z] < -\theta r^{1/3}\right) \leq \exp(-c\theta^\alpha)$.
(4) **Lower bound on diagonal moderate deviation probabilities:** There exist constants $\delta > 0$, $C > 0$, r_0 such that, for $r > r_0$,
   (a) $\mathbb{P}\left(X_r - \mu r > C r^{1/3}\right) \geq \delta$,
   (b) $\mathbb{P}\left(X_r - \mu r < -C r^{1/3}\right) \geq \delta$.

These will be respectively referred to as Assumptions 1–4 in this paper. Assumption 1, which is known to be true under mild moment conditions on ν, is stated to avoid any pathologies and will be in force throughout the rest of the paper without us explicitly mentioning it further. Assumption 3 is the a priori tail assumption that our work seeks to improve on. We will refer to the tail bounds as *stretched exponential* though this term usually refers to $0 < \alpha \leq 1$, (which is the case of primary interest for us).

Assumption 1 in fact follows from Assumption 3a, for the latter implies that $\nu([\theta, \infty)) \leq \exp(-c\theta^\alpha)$, for a possibly smaller c and sufficiently large θ (see Remark 1.5).

Observe that Assumption 2 is a mild relaxation of the form of the weight profile in all known integrable models, as we do not impose a lower order term of order $-z^4/r^3$ in the upper bound. Our arguments would also work if we replaced the third term $[-Hz^4/r^3, 0]$ of Assumption 2 with $[-Hz^4/r^3,\, Hz^4/r^3]$, but we have not included this relaxation so as to not introduce further complexity.

The additional translation by $-\Theta(r^{1/3})$ in Assumption 2 for the non-random fluctuation is an important ingredient (note that $\mathbb{E}[X_r] \le \mu r$ by super-additivity). Non-random fluctuations are an important object of study as will be evident from their role in the arguments in this paper (in particular that they are the same scale as the random fluctuations) as well as in past work: see, for example, [BGHH20, BHS18]. For applications in FPP, see [Cha13] and [AD14]. A powerful general theory to control such objects for general sub-additive sequences, particularly in the context of FPP, was developed in [Ale97].

Note that the upper and lower bounds on $\mathbb{E}[X_r^z]$ in Assumption 2 are within the order of the fluctuations $r^{1/3}$ for z up till order $r^{5/6}$; this fact will be relevant several times in this paper, and explains the future recurrence of the 5/6 exponent. However, the precise form of Assumption 2 should not be essential, and we expect our arguments to go through under reasonable relaxations. For example, a polynomial lower order term in (2), say of the form $|z|^{2+\delta}/r^{1+\delta}$ for some $\delta > 0$, or the related assumption of local uniform strong concavity of the limit shape may be sufficient.

We end this discussion by pointing out that Assumption 4b follows from Assumptions 2 and 3b; see Lemma 4.2. This is essentially because by assumption $\mu r > \mathbb{E}[X_r] + \Theta(r^{1/3})$ and we have assumed deviation bounds from the expectation. However, this style of argument does not work to derive Assumption 4a from Assumptions 2 and 3, and this task seems more difficult.

1.3. **Main results.** The main contribution of this paper is to obtain the optimal upper and lower tail exponents for X_r in terms of upper and lower bounds, starting from a selection of the assumptions just stated. Here are the precise statements.

Theorem 1. *Under Assumptions 2 and 3a, there exist constants c, $\zeta \in (0, \frac{2}{25}]$, r_0, and θ_0 (all depending on α) such that, for $\theta_0 \le \theta \le r^{\zeta}$ and $r > r_0$,*

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \ge \theta r^{1/3}\right) \le \exp\left(-c\theta^{3/2}(\log\theta)^{-1/2}\right).$$

Further, $\zeta(\alpha) \to 0$ as $\alpha \to 0$, and $\zeta(\alpha) = \frac{2}{25}$ if $\alpha \ge 1$.

Theorem 2. *Under Assumptions 2 and 4a (the former only at $z = 0$), there exist constants $c > 0$, $\eta > 0$ and r_0 such that for $r > r_0$ and $\theta_0 < \theta < \eta r^{2/3}$,*

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \ge \theta r^{1/3}\right) \ge \exp\big(-c\theta^{3/2}\big).$$

Theorem 3. *Under Assumptions 2 and 3, there exist constants $c > 0$, r_0, and θ_0 (all depending on α) such that, for $\theta > \theta_0$ and $r > r_0$,*

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \le -\theta r^{1/3}\right) \le \exp\left(-c\theta^{3}\right).$$

Theorem 4. *Under Assumptions 2, 3, and 4b, there exist constants $c > 0$, $\eta > 0$, θ_0, and r_0 (all depending on α) such that for $r > r_0$ and $\theta_0 < \theta < \eta r^{2/3}$,*

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \le -\theta r^{1/3}\right) \ge \exp\left(-c\theta^{3}\right).$$

The constants θ_0 and r_0 in the theorems should not be confused with the ones appearing in the assumptions.

Next we make some remarks and observations on the results, focusing mainly on aspects of Theorem 1.

Remark 1.4 (Suboptimal log factor in Theorem 1)**.** The reader would have noticed that the tail in Theorem 1 is not optimal, due to the appearance of $(\log\theta)^{-1/2}$. This arises due to the lack of sub-additivity of the sequence $\{X_r\}_{r\in\mathbb{N}}$ (which is super-additive instead), which necessitates considering a certain union bound; coping with the entropy from the union bound leads to the introduction of the factor of $(\log\theta)^{-1/2}$ in the exponent. We discuss this further in Section 1.4.

Remark 1.5 ($\zeta(\alpha) \to 0$ as $\alpha \to 0$)**.** The tail exponent claimed in Theorem 1 holds only for $\theta \le r^{\zeta}$ for a positive $\zeta = \zeta(\alpha)$ with $\lim_{\alpha\to 0}\zeta(\alpha) = 0$, and as we will see now, this is indeed necessary. First note that Assumption 3 implies that the vertex weight distribution's upper tail decays with exponent at least α; to see this, observe that $\mathbb{P}(X_{(r-1,r)} - \mathbb{E}[X_{(r-1,r)}] > -0.5tr^{1/3}) > 1/2$ for all large enough t by Assumption 3b, and so

$$\begin{aligned}\frac{1}{2}\cdot\mathbb{P}\left(\xi_{(r,r)} \ge tr^{1/3}\right) &\le \mathbb{P}\left(X_{(r-1,r)} - \mathbb{E}[X_{(r-1,r)}] > -0.5tr^{1/3}, \xi_{(r,r)} \ge tr^{1/3}\right)\\ &\le \mathbb{P}\left(X_r - \mathbb{E}[X_r] \ge 0.25tr^{1/3}\right) \le \exp(-ct^{\alpha}),\end{aligned}$$

using Assumption 3a in the last inequality, and bounding $\mathbb{E}[X_r] - \mathbb{E}[X_{(r-1,r)}]$ by $0.25tr^{1/3}$. This holds for all r and t large enough; taking $r = r_0$ large enough for the bound to hold and letting ξ be any random variable distributed according to ν shows that, for all large enough t,

$$\mathbb{P}\big(\xi \ge tr_0^{1/3}\big) \le \exp(-ct^{\alpha}) \implies \mathbb{P}\big(\xi \ge t\big) \le \exp(-\tilde{c}t^{\alpha}).$$

Conversely, assuming that $\mathbb{P}\big(\xi \ge t\big) \ge \exp(-\tilde{c}t^{\alpha})$, it follows that Assumption 3a cannot hold with any power $\beta > \alpha$ for the entire tail. Now recall, as mentioned after Theorem 1.2, that after a certain point the behavior of individual vertex weights is expected to govern the tail of point-to-point weights. So under the aforementioned assumption on ξ, an upper bound for $\zeta(\alpha)$ could be obtained by considering the value of ζ which solves

$$\exp(-c\theta^{3/2}) = \exp(-c(\theta r^{1/3})^{\alpha})$$

for $\theta = r^{\zeta}$, which is $\zeta = 2\alpha/(9-6\alpha)$. This goes to zero as $\alpha \to 0$, as in Theorem 1.

Remark 1.6 (Intermediate regimes for upper tail)**.** While Theorem 1 asserts the $3/2$ tail exponent up till r^{ζ}, its proof will also show the existence of a number of ranges of θ in the interval $[r^{\zeta}, \infty)$ in which the tail exponent transitions from $3/2$ to α. More precisely, there exists a finite n and numbers $\alpha = \beta_1 < \beta_2 < \ldots < \beta_n = 3/2$ and $\infty = \zeta_1 > \zeta_2 > \ldots > \zeta_n = \zeta$ such that, for $j \in [\![1, n-1]\!]$ and $\theta \in [r^{\zeta_{j+1}}, r^{\zeta_j}]$,

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \ge \theta r^{1/3}\right) \le \exp\left(-c\theta^{\beta_j}\right).$$

Recursive expressions are also derived for the β_j and ζ_j quantities; see Remark 3.2.

However, we believe that these intermediate regimes are an artifact of our proof, and that the true behavior is that the tail $\exp(-c\theta^{3/2})$ holds for θ till $r^{2\alpha/(9-6\alpha)}$, and $\exp(-c(\theta r^{1/3})^{\alpha})$ after (as in Remark 1.5). Note also that for $\alpha = 1$, $r^{2\alpha/(9-6\alpha)} = r^{2/3}$, matching Theorem 1.2.

Remark 1.7 (Extending to other values of z)**.** We have stated our results for the last passage value to (r, r), but some also extend to $(r - z, r + z)$ for certain ranges of z. For the upper tail the argument of Theorem 1 also applies for $|z| = O(r^{2/3})$, while Theorem 2 extends to all $|z| = O(r^{5/6})$; as mentioned after the assumptions, the source of the $5/6$ is that for z of this order, the upper and lower bounds of Assumption 2 differ by the weight fluctuation order, i.e., $r^{1/3}$. Regarding the upper bound on the lower tail, the argument for Theorem 3 does not conceptually rely on $z = 0$, but formally uses a result from [BGHH20] which is not proven for $z \neq 0$. The latter result can be extended to larger z without much difficulty, but we have not pursued this here. Finally, the argument for Theorem 4 applies for $|z| \le r^{5/6}$.

The set of assumptions we adopt bears similarities to the ones that have appeared in the past literature on FPP. The most prominent of these include the work of Newman and coauthors (see e.g. [NP95, ADH17]) which investigated the effect of limit shape curvature assumptions on the geometry of geodesics and the fluctuation exponents. More recently, the work [Cha13] of Chatterjee assumed a strong form of existence of the exponents governing geometric and weight fluctuations of the geodesics and verified the KPZ relation between them; see also [AD14]. Subsequently [DH14] and [Ale20] studied geodesics and bi-geodesics under related assumptions.

Inspired by this, recently, results in the exactly solvable cases of LPP have been obtained, relying merely on inputs analogous to the ones stated in the assumptions. See for example the very recent work [BGHH20] which develops the theory of *geodesic watermelons* under a similar set of assumptions to deduce properties of all known integrable lattice models. Other examples include [BHS18, BSS14, FO18], which work in the specific case of LPP with exponential weights; and [Ham17a, Ham17b, Ham17c], in which geometric questions in the semi-discrete model of Brownian LPP are studied.

An intriguing aspect of our arguments is their use of the concentration of measure phenomena for sums of independent stretched exponential random variables, which is in fact at the heart of this paper. General concentration results have, of course, been widely investigated in recent times [BLM13]. A common theme is that sums of independent random variables have behavior which transitions, as we extend further into the tail, from being sub-Gaussian to being governed by the tail decay of the individual variables. When the variables have stretched exponential tails, a precise form of this is a bound that is a generalization of Bernstein's inequality for subexponential random variables. Though such results are not unexpected, the recent article [KC18] explicitly records many extensions of concentration results for sums of sub-Gaussian or subexponential random variables to the stretched exponential case with a high dimensional statistics motivation, in a form particularly convenient for our application.

We next move on to an outline of the key ideas driving our proofs.

1.4. **A brief discussion of the arguments.** Before turning to the ideas underlying our arguments, we deal with some matters of convention. We will use the words "width" and "height" to refer to measurements made along the antidiagonal and diagonal respectively. So, for example, the set of $(x, y) \in \mathbb{Z}^2$ such that $2 \leq x + y \leq 2r$ and $|x - y| \leq \ell r^{2/3}$ is a parallelogram of height r and width $\ell r^{2/3}$. This usage will continue throughout the article.

In the overview we will at certain moments make use of a few refined tools, which have appeared previously in [BGHH20], and whose content is explained informally in this section; their precise statements are gathered in Section 1.8 ahead.

Now we turn to the mathematical discussion. The flavors of our arguments are different for the upper and lower bounds on the two tails. Super-additivity, in various guises, plays a recurring role in all except the upper bound on the lower tail. In all the arguments a parameter k appears which plays different roles, but is essentially always finally set to be a multiple of $\theta^{3/2}$, where θ measures the depth into the tail we are considering. The reader should keep in mind this value of k in the discussion. Also, we assume without loss of generality that $\alpha \leq 1$ in this section.

We briefly give a version of a common theme which underlies the different arguments, namely of looking at smaller scales, which further explains why we take $k = \Theta(\theta^{3/2})$. Consider a geodesic path from $(1, 1)$ to (r, r) which attains a weight of $\mu r + \theta r^{1/3}$ for large θ (the following also makes sense for $-\theta$). If we look at a given $1/k$-fraction of the geodesic, that fraction's weight should be close to $\mu r/k + \theta r^{1/3}/k$ if the geodesic gains weight roughly uniformly across its journey; but on the other hand, KPZ fluctuation dictates that the fraction's weight should typically be $\mu r/k + C(r/k)^{1/3}$. So we look for a scale at which the typical behavior is not in tension with the notion of the geodesic's weight being spread close to uniformly over much of its journey. This means finding k such that $\theta r^{1/3}/k$ and $C(r/k)^{1/3}$ are of the same order, which occurs if $k = \Theta(\theta^{3/2})$.

Now we come to the detailed descriptions.

Upper bound on upper tail. We start by discussing a simplified argument for the upper tail of the upper bound to illustrate the idea of bootstrapping. The starting point is a concentration of measure phenomenon for stretched exponential random variables alluded to before. More precisely, sums of independent stretched exponential random variables have the same qualitative tail decay deep in the tail as that of a single one (see Proposition 2.1 ahead). Not so deep in the tail lies a regime of Gaussian decay, but we will never be in this regime in our arguments.

Let $X^{(i)}_{r/k}$ be the last passage value from $i(r/k, r/k) + (1, 0)$ to $(i + 1)(r/k, r/k)$. Suppose, for purposes of illustration, that we actually had that X_r are *sub*-additive rather than super-additive, i.e., we had that $X_r \leq \sum_{i=1}^{k} X^{(i)}_{r/k}$. Each $X^{(i)}_{r/k}$ fluctuates at scale $(r/k)^{1/3}$, and

$$\Big|\sum_{i=1}^{k} \mathbb{E}[X^{(i)}_{r/k}] - \mathbb{E}[X_r]\Big| \leq k \cdot C(r/k)^{1/3} = Ck^{2/3}r^{1/3}, \tag{3}$$

using Assumption 2. So under this illustrative sub-additive assumption we would have

$$\begin{aligned}\mathbb{P}\left(X_r - \mathbb{E}[X_r] > \theta r^{1/3}\right) &\leq \mathbb{P}\left(\sum_{i=1}^{k}(X^{(i)}_{r/k} - \mathbb{E}[X^{(i)}_{r/k}]) > \theta r^{1/3} - Ck^{2/3}r^{1/3}\right)\\ &\leq \mathbb{P}\left(\sum_{i=1}^{k}(X^{(i)}_{r/k} - \mathbb{E}[X^{(i)}_{r/k}]) > \frac{1}{2}\theta k^{1/3}(r/k)^{1/3}\right), \end{aligned}\tag{4}$$

the last inequality for $k \leq (2C)^{-3/2}\theta^{3/2}$, which dictates our choice of k. Now by Assumption 3a we know that

$$\begin{aligned} &\mathbb{P}\left(X_{r/k,i} - \mathbb{E}[X_{r/k,i}] > \theta(r/k)^{1/3}\right) \leq \exp(-c\theta^{\alpha})\\ \implies &\mathbb{P}\left(X_{r/k,i} - \mathbb{E}[X_{r/k,i}] > \theta k^{1/3}(r/k)^{1/3}\right) \leq \exp(-c\theta^{\alpha}k^{\alpha/3}).\end{aligned}$$

Because sums of stretched exponentials have the same deep tail decay as a single one, (4) shows that the probability that $X_r - \mathbb{E}[X_r]$ is greater than $\theta r^{1/3}$ is essentially like that of $X_{r/k} - \mathbb{E}[X_{r/k}]$ being greater than $\theta k^{1/3}(r/k)^{1/3}$, which is at most $\exp(-c\theta^{\alpha}k^{\alpha/3})$. This gives an improved tail exponent of $3\alpha/2$ for the point-to-point's upper tail, compared to the input of α, since k can be at most $O(\theta^{3/2})$.

We can now repeat this argument, with the improved exponent as the input, and obtain an output exponent which is greater by a factor of $3/2$, and we can continue doing so as long as the input exponent is at most 1. If we perform the argument one last time with the input exponent as 1, we obtain the optimal exponent of $3/2$.

The reason we require the input exponent to be at most 1 is that, beyond this point, the concentration behavior changes: for $\alpha \leq 1$ the deep tail behavior of a sum of independent stretched exponentials is governed by the event that a single variable is large, while for $\alpha > 1$ the behavior is governed by the event that the deviation is roughly equidistributed among all the variables. This is a result of the change of the function x^{α} from being concave to convex as α increases beyond 1. More precisely, suppose $\alpha \in (1, 3/2]$ is the point-to-point tail exponent and let us accept the equidistributed characterization of the deep tail (as is proved in [KC18]). Then the probability (4) would be at most the probability that each of the k variables $X^{(i)}_{r/k} - \mathbb{E}[X^{(i)}_{r/k}]$ is at least $\theta k^{1/3}/k = \theta k^{-2/3}$, which is in turn bounded by

$$\exp\left(-ck \cdot (\theta k^{-2/3})^{\alpha}\right) = \exp\left(-c\theta^{\alpha}k^{1-2\alpha/3}\right).$$

By taking $k = \eta\theta^{3/2}$, which, as mentioned earlier, is the largest possible value we can take, we see that this final expression is $\exp(-c\theta^{3/2})$. In other words, the exponent of $3/2$ is a natural fixed point for the bootstrapping procedure.

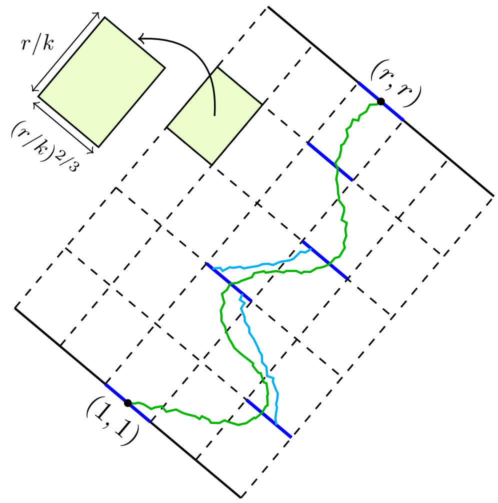

Figure 1. In green is depicted the heaviest path which passes through the selection of intervals in blue. The cyan curve between the second and third (similarly the third and fourth) blue intervals is the heaviest path with endpoints on those intervals. Because these consecutive cyan paths do not need to share endpoints, the weight of the green path is at most the sum of the interval-to-interval weights defined by the blue intervals, which provides the substitute sub-additive relation.

Now we turn to addressing the simplifications we made in the above discussion. Handling them correctly makes the argument significantly more complicated and technical, and reduces the tail from $\theta^{3/2}$ to $\theta^{3/2}(\log\theta)^{-1/2}$.

One simplification we skipped over is that the improvement in the tail bound after one iteration only holds for $\theta \leq r^{2/3}$ and not the entire tail (since k, the number of parts that the geodesic to (r,r) is divided into, can be at most r, and $k = \Theta(\theta^{3/2})$), which is a slight issue for the next round of the iteration. This is handled by a simple truncation.

But the main difficulty is that the X_r are super-additive, not sub-additive. To handle this, we consider a grid of height r and width $\mathrm{poly}(\theta)\cdot r^{2/3}$. This width is set such that, with probability at most $\exp(-c\theta^{3/2})$, the geodesic exits the grid, using the bound recorded in Proposition 1.9 ahead on the transversal fluctuation; this allows us to restrict to the event that the geodesic stays within the grid. Intervals in the grid have width $(r/k)^{2/3}$ and are separated by a height of r/k.

The utility of the grid is that X_r can be bounded by a sum of interval-to-interval weights in terms of the intervals of the grid that the geodesic passes through; this bound can play the role of a sub-additive relation. See Figure 1. Then, just as we had a tail bound above for $X^{(i)}_{r/k}$ to bootstrap, a requisite step is to obtain an upper bound on the upper tail of the interval-to-interval weight, using only the point-to-point estimate available. We do this in Lemma 3.5 with the basic idea that the interval-to-interval weight being high will cause a point-to-point weight, from "backed up" points, to also be high; see Figure 5 for a more detailed illustration of the argument (such an argument of backing up has previously been implemented in [BSS14, BGHH20]).

With the interval-to-interval tail bound, we discretize the geodesic by considering which sequence of intervals it passes through, and bound the highest weight through a given sequence by the sum of interval-to-interval weights. This uses the bootstrapping idea and yields an improved tail estimate for the highest weight through a given sequence. Later we will take a union bound over all possible sequences of intervals; this union bound is what leads to the appearance of the suboptimal $(\log\theta)^{-1/2}$ in the bound as mentioned in Remark 1.4.

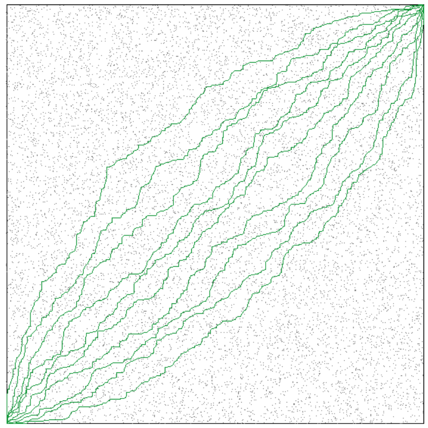

FIGURE 2. A simulation of the k-geodesic watermelon in the related model of Poissonian last passage percolation for $k = 10$.

This strategy requires handling paths which are extremely "zig-zaggy"; to show that these paths are not competitive, we need upper bounds on upper tails of point-to-point weights, i.e. X_r^z, in a large number of directions indexed by z, though we are only ultimately proving a bound for paths ending at (r, r). (Recall that X_r^z is the weight to $(r - z, r + z)$ from $(1, 1)$.) Further, in order to repeat the iterations of the bootstrap, the bounds in other directions must also be improving with each iteration. To achieve this, we in fact bound the deviations not from $\mathbb{E}[X_r^z]$ (which to second order is $\mu r - Gz^2/r$) in the j^{th} round of iteration, but from the bigger $\mu r - \lambda_j Gz^2/r$, for a $\lambda_j \le 1$ which decreases with the iteration number j. By adopting this relaxation we are able to obtain the improvement in the tail for all the required z with each iteration, which appears to be difficult if one insists that $\lambda_j = 1$ for all j.

A similar grid construction has been used previously, for example to obtain certain tail bounds in [BGHH20], to bound the number of disjoint geodesics in a parallelogram in [BHS18], and to study coalescence of geodesics in [BSS17].

Lower bound on upper tail. This is the easiest of the four arguments. Recall that we have C and δ from Assumption 4a such that $\mathbb{P}(X_{r/k} > \mu r/k + C(r/k)^{1/3}) \ge \delta$, and let $X_{r/k}^{(i)}$ be as in (3). By the super-additivity that the X_r genuinely enjoy, for any k it holds that $X_r \ge \sum_{i=1}^k X_{r/k}^{(i)}$. Choosing k to be an appropriate multiple of $\theta^{3/2}$, we obtain

$$\mathbb{P}\left(X_r > \mu r + \theta r^{1/3}\right) \ge \prod_{i=1}^{k} \mathbb{P}\left(X_{r/k}^{(i)} > \mu r/k + C(r/k)^{1/3}\right) \ge \delta^k = \exp(-c\theta^{3/2}).$$

Replacing μr by $\mathbb{E}[X_r]$ is a simple application of Assumption 2.

Upper bound on lower tail. The illustrative argument using sub-additivity given above for the upper bound on the upper tail is actually correct for the upper bound on the lower tail, as the super-additivity of X_r is in the favorable direction in this case. But, as we saw there, the approach can only bring the tail exponent up to 3/2, and not 3. This is essentially because that argument focuses on the weight of a single path, while the exponent of 3 for the lower tail is a result of *all* paths having low weight. Thus our strategy to prove the stronger bound is to construct $\theta^{3/2}$ disjoint paths moving through independent parts of the space, each suffering a weight loss of $\theta r^{1/3}$. By the discussion above and independence, the probability of each of them being small can be bounded by $\exp(-c\theta^{3/2} \cdot \theta^{3/2}) = \exp(-c\theta^3)$.

To do this formally, we rely on an important ingredient from [BGHH20], which studies the weight and geometry of maximal weight collections of k disjoint paths in $[\![1, r]\!]^2$, called *k-geodesic watermelons*. See Figure 2. It is shown that these k paths typically are each of weight $\mu r - Ck^{2/3}r^{1/3}$, and that they have a collective transversal fluctuation of order $k^{1/3}r^{2/3}$. In fact, the following quantitative bound on the weight X_r^k of the k-geodesic watermelon is proved there via a direct multi-scale construction of disjoint paths with correct order collective weight, which we formally state ahead as Theorem 1.10:

$$\mathbb{P}\left(X_r^k \leq \mu k r - Ck^{5/3}r^{1/3}\right) \leq \exp(-ck^2), \tag{5}$$

for all $k \leq \eta r$ for a small constant $\eta > 0$. We give a brief overview of the construction in Section 4.2 due to its conceptual importance in the argument for the lower tail bound.

For our purposes we observe that, for any $k \in \mathbb{N}$,

$$\mathbb{P}\left(X_r < \mu r - \theta r^{1/3}\right) \leq \mathbb{P}\left(X_r^k < \mu k r - \theta k r^{1/3}\right).$$

Taking $k = \eta\theta^{3/2}$ and noting that then $k\theta$ is of order $k^{5/3}$ and that $\theta < r^{2/3} \implies k < \eta r$ shows that

$$\mathbb{P}\left(X_r < \mu r - \theta r^{1/3}\right) \leq \exp(-ck^2) = \exp(-c\theta^3),$$

and it is a simple matter to replace μr by $\mathbb{E}[X_r]$ by possibly reducing the constant c.

However the framework in [BGHH20] works with strong tail bounds on the one point weight of the kind we are seeking to prove in this paper. So in order to access the bound (5) from [BGHH20] we need to deliver the required inputs starting from our assumptions. There are three inputs required. The first is the following:

(1) Limit shape bounds, which we have by Assumption 2.

The next two inputs concern the maximum weight over all midpoint-to-midpoint paths constrained to lie in a given parallelogram $U = U_{r,\ell,z}$ of height r, width $\ell r^{2/3}$, and opposite side midpoints $(1, 1)$ and $(r - z, r + z)$. We will call such weights "constrained weights".

(2) An exponential upper bound on the constrained weight's lower tail, which we will arrive at by bootstrapping. To elaborate, by using Assumption 3b and the previously mentioned Proposition 1.9 on the transversal fluctuation of the unconstrained geodesic, we can obtain an initial stretched exponential upper bound ((7) of Proposition 1.11 ahead) on the constrained weight's lower tail. Then, via a bootstrapping argument as above, we can upgrade this to a tail with exponent $3/2$ (see Proposition 2.3).
(3) A lower bound on the mean of constrained weights using the above tail, provided by (8) of Proposition 1.11.

Lower bound on lower tail. A detail about the construction described, which is captured in its formal statement Theorem 1.10, is that it fits inside a strip of width $4k^{1/3}r^{2/3}$ around the diagonal. To lower bound the lower tail probability, this suggests that we need to focus on paths which remain in the strip of this width (again we will be setting k to be a constant times $\theta^{3/2}$). Essentially this is because a consequence of the parabolic weight loss of Assumption 2 is that *any* path (not just the geodesic) which exits the strip of width $k^{1/3}r^{2/3}$ suffers a loss of $(k^{1/3}r^{2/3})^2/r = k^{2/3}r^{1/3}$, which is of order $\theta r^{1/3}$, with high probability. This is captured more precisely in Theorem 1.8 ahead.

Similar to the argument for the upper bound on the upper tail, we consider a grid where each cell has height r/k and width $(r/k)^{2/3}$, but with overall width $k^{1/3}r^{2/3}$. This gives k cells in each column and in each row, for a total of k^2 cells. See Figure 3

Now consider the event that, for each interval in the grid, the maximum weight from that interval to the next row of intervals is less than $\mu r/k - C(r/k)^{1/3}$, and that the maximum weight of a path

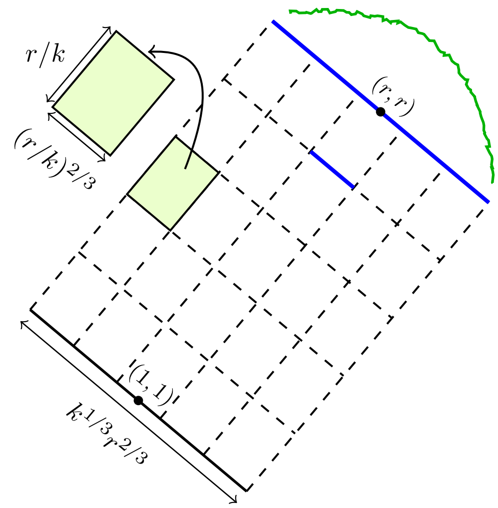

FIGURE 3. The grid of k^2 intervals for the lower bound of the lower tail. An interval and the following row of intervals are blue: consider the event that the heaviest path from the former to the latter is at most $\mu r/k - C(r/k)^{1/3}$. To prove that this has positive probability, we make use of parabolic curvature of the weight profile (shown in green) to argue that if the endpoint on the row is too extreme, it will typically suffer the loss we want; a separate backing up argument is employed for when the endpoint is near the center where the parabolic weight loss is not significant.

which exits the grid is at most $\mu r - Ck^{2/3}r^{1/3}$. This is an intersection of decreasing events, and on this event X_r is at most $\mu r - Ck^{2/3}r^{1/3}$: if it exits the grid it suffers a loss of $Ck^{2/3}r^{1/3}$ and if it stays in the grid it undergoes a loss of at least $C(r/k)^{1/3}$ for each of the k rows. Now if we know that there is a constant probability (say $\delta > 0$) lower bound on the event that a single interval-to-line weight is low, the FKG inequality (along with Theorem 1.8 to lower bound the probability of parabolic weight loss when exiting the grid) provides a lower bound of order δ^{k^2} on the described event's probability; setting k to be a multiple of $\theta^{3/2}$ will complete the proof.

To implement this we need a lower bound on the probability that the interval-to-line weight is small using the point-to-point lower bound. This is Lemma 4.4. The proof proceeds in two steps. First, a stepping back strategy as earlier gives a constant lower bound on the interval-to-interval weight's lower tail for intervals of size $\varepsilon r^{2/3}$, for some small $\varepsilon > 0$ (as will be clear from the precise argument, the smallness of ε is crucial for this). By the FKG inequality, this is upgraded to a bound for intervals of length $r^{2/3}$; essentially, if each of the intervals are divided into ε^{-1} intervals of size $\varepsilon r^{2/3}$, and all ε^{-2} pairs of intervals have small weight (which is an intersection of decreasing events), then so must the original intervals. To get from this to an interval-to-line bound we again argue based on FKG. We divide the line into $r^{1/3}$ many intervals of size $r^{2/3}$ each. We can ensure that the weight is low whenever the destination interval is one of a constant number near (r, r) using the previous bound, and for the rest the parabolic curvature ensures that it is so likely to be low that the FKG inequality gives a positive lower bound independent of r, in spite of considering an intersection of $r^{1/3}$ many events; see Figure 3.

1.5. **Tails for constrained weight.** The ideas described in the previous section can be applied slightly more generally to yield the following theorem on the lower tail of the constrained weight X_r^U of the best path from $(1, 1)$ to (r, r) constrained to stay inside a parallelogram U.

Recall that $U = U_{r,\ell,z}$ denotes a parallelogram of height r, width $\ell r^{2/3}$, and opposite side midpoints $(1, 1)$ and $(r - z, r + z)$, defined to be the set of vertices $v = (v_x, v_y) \in \mathbb{Z}^2$ such that $v + t(-1, 1)$

lies on the line $y = \frac{r+z}{r-z} \cdot x$ for some $t \in \mathbb{R}$ with $|t| \leq \ell r^{2/3}/2$, and $2 \leq v_x + v_y \leq 2r$. Let X_r^U be the maximum weight among all paths from $(1,1)$ to $(r-z, r+z)$ constrained to be inside U. The notation $U_{r,\ell,z}$ will be used for parallelograms throughout this article. Here we take $z = 0$.

Estimates on constrained weights have been crucial in several recent advances, see [BSS14]. The following theorem proves a sharp estimate on the tail as a function of the aspect ratio of the parallelogram, measured on the characteristic KPZ scale.

Theorem 5. *Under Assumptions 2, 3 and 4b, there exist finite positive constants c_1, c_2, η, C, θ_0, and r_0 (all independent of ℓ) such that, for $z = 0$, ranges of θ to be specified, $r > r_0$, and $C\theta^{-1} \leq \ell \leq 2r^{1/3}$,*

$$\exp\left(-c_1 \min(\ell\theta^{5/2}, \theta^3)\right) \leq \mathbb{P}\left(X_r^U - \mu r \leq -\theta r^{1/3}\right) \leq \exp\left(-c_2 \min(\ell\theta^{5/2}, \theta^3)\right);$$

the second inequality holds for $\theta > \theta_0$ while the first holds for $\theta_0 < \theta < \eta r^{2/3}$. If ℓ is bounded below by a constant $\varepsilon > 0$ independent of θ, we can replace μr by $\mathbb{E}[X_r^U]$ for $r > \tilde{r}_0(\varepsilon)$ and with c_1 depending on ε.

We note that Theorem 3 and Theorem 4 are implied by Theorem 5 by taking $\ell = 2r^{1/3}$.

We also remark that the transition from $\ell\theta^{5/2}$ to θ^3 occurs when ℓ becomes of order $\theta^{1/2}$; this matches the belief (which comes from the parabolic curvature) that the geodesic, conditioned on its weight being less than $\mu r - \theta r^{1/3}$, will have typical transversal fluctuations of order $\theta^{1/2} r^{2/3}$.

The proof idea of Theorem 5 is a refinement of those of Theorems 3 and 4 described above, by picking the number of paths of average separation $k^{-2/3} = \theta^{-1}$ to be packed inside U, which turns out to be $\min(\ell k^{2/3}, k)$ (rather than k as before). We omit further outline to avoid repetition.

1.6. **Related work.** The main tools we use in our arguments are the super-additivity of the X_r (i.e., $X_{r+j} \geq X_r + X_{(r+1,r),(r+j,r+j)}$), geodesic watermelons, and concentration of measure results for sums of independent stretched exponential random variables. We have discussed aspects of the latter two that have appeared in various works, and here we briefly overview the first, i.e., super-additivity.

Not surprisingly, super-additivity of the weight has been an important tool in other investigations of non-integrable models; for example, the proof of the almost sure existence of a deterministic limit for X_r/r as $r \to \infty$ under a wide class of vertex distributions goes via Kingman's sub-additive theorem. Super-additivity was also crucial in [Led18], where a law of iterated logarithm for X_r was proved. More precisely, for exponential weights, $\limsup_{r\to\infty}(X_r - 4r)/r^{1/3}$ was shown to almost surely exist and be a finite, positive, deterministic constant. Super-additivity only aids in proving a result for the lim sup, and so the result on the lim inf in [Led18] is weaker. This was addressed in [BGHK19], where the lack of sub-additivity was handled by shifting perspective to also consider point-to-line passage times, which, as we have outlined, we will do in the present article as well.

An example closer in spirit to our usage of super-additivity was made by Johansson in [Joh00], where, from a limiting large deviation theorem for the upper tail, it was pointed out that the same gives an explicit bound for finite r by a super-additivity argument. Briefly, and again in the context of Exponential LPP, the observation is that for every r and every $N \geq 1$,

$$\mathbb{P}\big(X_r > \theta\big)^N \leq \mathbb{P}\big(X_{Nr} > N\theta\big) \implies \mathbb{P}\big(X_r > \theta\big) \leq \lim_{N\to\infty} \left[\mathbb{P}\big(X_{Nr} > N\theta\big)\right]^{1/N}, \tag{6}$$

and the latter limit was shown to exist and explicitly identified in [Joh00]. In a sense our arguments are dual to that of (6); while (6) uses super-additivity to go to *larger* r in order to obtain a bound, our arguments use super-additivity to reason about *smaller* r to obtain a bound.

Finally, we mention the recent work [EJS20] which proves a sharp upper bound (i.e., with the correct coefficient of 4/3 as in (1)) on the right tail of X_r (centred by $\mu r = 4r$ and appropriately scaled) in exponential LPP via more probabilistic arguments, rather than precise analysis of integrable formulas. The technique utilizes calculations in an increment-stationary version of exponential LPP (where the vertex weight on the boundaries of $\mathbb{Z}^2_{\geq 0}$ differ in distribution from the rest) and a moment generating function identity specific to this model—features absent in the general setting under consideration in this article.

1.7. **Future directions.** This work leads to several research directions, some of which we outline below. A natural question is whether the stretched exponential tails of Assumption 3 may be weakened, for example to polynomially decaying tails. An important aspect of our arguments is that we are able to reach the exponent of 3/2 in finitely many iterations of the bootstrap. While it should not be difficult to show via a bootstrapping argument that, starting from a polynomial decay of a given degree, one can reach a polynomial decay of any given higher degree after finitely many rounds, it appears to us non-trivial to bridge the gap and reach a superpolynomial decay, such as the stretched exponential decay of Assumption 3, in finitely many steps.

Another interesting direction of inquiry is whether these methods can be used to study directed last passage percolation model tails in higher dimensions. This appears to us doable in principle, given suitable assumptions on fluctuation scales and the limit shape.

It is worthwhile to point out that the basic super-additive argument for the lower bound on the upper tail explained above does not have any dependencies on the limit shape or dimension, and, assuming a weight fluctuation exponent of χ, yields a lower bound of $\exp(-c\theta^{1/(1-\chi)})$ for the probability that X_r is at least $\mu r + \theta r^\chi$ under an analogous assumption to Assumption 4a. It is an interesting question whether there is a matching upper bound and what is the exponent of the lower tail. A natural guess for the latter, by considering disjoint paths packed optimally, would be $2/(1-\chi)$, which, given the KPZ scaling relation $\chi = 2\xi - 1$, is the same as $1/(1-\xi)$ (ξ being the exponent of transversal fluctuations, which, for example, is $2/3$ in two dimensions).

Finally, we comment on the possibility of applying these techniques to first passage percolation, perhaps the most canonical non-integrable model expected to be in the KPZ class (and hence have weight and transversal fluctuation exponents of $1/3$ and $2/3$ in two dimensions). In principle many of our arguments should apply, as FPP enjoys a natural sub-additive structure analogous to the super-additive structure of LPP. But one technical difference that arises is that the paths in FPP are not directed and can backtrack, and this would require changes in the grid based discretizations employed in this paper for several of the main results. This will be pursued in future work.

1.8. **A few important tools.** In this section we collect some refined tools for last passage percolation which we will use for our arguments as outlined in Section 1.4. There are four statements: the first asserts that it is typical for a path to suffer a weight loss which is quadratic in its transversal fluctuation, measured in the characteristic scalings of $r^{1/3}$ and $r^{2/3}$; the second is a related transversal fluctuation bound, but for paths with endpoint $(r-z, r+z)$ for $|z| \leq r^{5/6}$; the third is a high probability construction of a given number of disjoint paths which achieve a good collective weight; and the fourth provides bounds on the lower tail and mean of constrained weights.

We will import the proof ideas from [BGHH20] where similar statements have appeared. Our proofs are essentially the same but adapted suitably to work under the weaker tail exponent α assumed here; for this reason, we only explain the modifications that need to be made for the first, second and fourth tools in Appendix A. The proof of the third tool is discussed in Section 4.2 in slightly more detail.

1.8.1. *Parabolic weight loss for paths with large transversal fluctuation.* The following is the precise statement of the first tool.

Theorem 1.8 (Refined transversal fluctuation loss)**.** *Let $X_r^{s,t}$ be the maximum weight over all paths Γ from the line segment joining $(-tr^{2/3}, tr^{2/3})$ and $(tr^{2/3}, -tr^{2/3})$ to the line segment joining $(r - tr^{2/3}, r + tr^{2/3})$ and $(r + tr^{2/3}, r - tr^{2/3})$ such that $\mathrm{TF}(\Gamma) > (s+t)r^{2/3}$, with $t \leq s$. Under Assumptions 2 and 3a, there exist absolute constants r_0, s_0, $c > 0$ and $c_2 > 0$ such that, for $s > s_0$ and $r > r_0$,*

$$\mathbb{P}\left(X_r^{s,t} > \mu r - c_2 s^2 r^{1/3}\right) < \exp\left(-cs^{2\alpha}\right).$$

The proof of this follows that of [BGHH20, Theorem 3.6]. We explain the necessary modifications in the appendix.

An important feature of Theorem 1.8 is that it bounds the probability of a *decreasing* event, which is useful as it allows the application of the FKG inequality.

1.8.2. *Transversal fluctuation bound for $|z| \leq r^{5/6}$.* The second tool is a result on the transversal fluctuation of geodesics to $(r-z, r+z)$ (note that Theorem 1.8 is related but only for $z = 0$), which is the following. We note in passing that the event of the geodesic having large transversal fluctuation is neither increasing nor decreasing.

Proposition 1.9 (Transversal fluctuations)**.** *For given z, let Γ_r^z be the geodesic from $(1,1)$ to $(r-z, r+z)$ with maximum transversal fluctuation. Under Assumptions 2 and 3, there exist constants $c > 0$, s_0, and r_0 such that, for $r > r_0$, $s > s_0$, and $|z| \leq r^{5/6}$,*

$$\mathbb{P}\left(\mathrm{TF}(\Gamma_r^z) > sr^{2/3}\right) \leq \exp\left(-cs^{2\alpha}\right).$$

The proof of this is similar to that of [BSS14, Theorem 11.1] and appears in the appendix.

1.8.3. *A high probability construction of disjoint paths with good collective weight.* Here is the statement of our third tool.

Theorem 1.10 (Theorem 3.1 of [BGHH20])**.** *Under Assumptions 2 and 3, there exist $c, C_1 > 0, k_0 \in \mathbb{N}$ and $\eta > 0$ such that for all $k_0 \leq k \leq \eta r$ and $m \in [\![1, k]\!]$, with probability $1 - e^{-ckm}$, there exist m disjoint paths $\gamma_1, \ldots, \gamma_m$ in the square $[\![1, r]\!]^2$, with γ_i from $(1, i)$ to $(r, r-i+1)$ and $\max_i \mathrm{TF}(\gamma_i) \leq 2mk^{-2/3}r^{2/3}$, such that*

$$\sum_{i=1}^m \ell(\gamma_i) \geq \mu rm - C_1 mk^{2/3} r^{1/3}.$$

The proof of Theorem 1.10 will be discussed in some detail in Section 4.2 and will require as input our fourth tool on bounds for the lower tail and mean of constrained weights.

1.8.4. *Bounds for constrained weights.* To state our fourth and final tool, recall from Section 1.5 the notation for parallelograms $U_{r,\ell,z}$ of height r, width $\ell r^{2/3}$ and opposite midpoints $(1,1)$ and $(r-z, r+z)$ as well as that for maximum weight of paths constrained inside U, X_r^U.

Proposition 1.11 (Lower tail & mean of constrained point-to-point, Proposition 3.7 of [BGHH20])**.** *Let positive constants L_1, L_2, and $K > 0$ be fixed. Let z and ℓ be such that $|z| \leq Kr^{2/3}$ and $L_1 \leq \ell \leq L_2$, and let $U = U_{r,\ell,z}$. There exist positive constants $r_0 = r_0(K, L_1, L_2)$ and $\theta_0 = \theta_0(K, L_1, L_2)$, and an absolute positive constant c, such that, for $r > r_0$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(X_r^U \leq \mu r - \theta r^{1/3}\right) \leq \exp\left(-c\ell^{2\alpha/3}\theta^{2\alpha/3}\right). \tag{7}$$

As a consequence, there exists $C = C(K, L_1, L_2)$ such that, for $r > r_0$,

$$\mathbb{E}[X_r^U] \geq \mu r - Gz^2/r - Cr^{1/3}. \tag{8}$$

To be consistent with previous expressions we have included the parabolic term $-Gz^2/r$ in the previous, but note that for the ranges of z mentioned we can absorb it into the $Cr^{1/3}$ term.

1.9. **Organization of the article.** In Section 2 we collect the concentration statements for stretched exponential random variables and prove an abstracted version of the bootstrap. In Section 3 we prove Theorems 1 and 2 which respectively concern upper and lower bounds on the upper tail. In Section 4 we address Theorems 3 and 4 on the corresponding bounds for the lower tail, as well as Theorem 5 on the lower tails of the constrained geodesic weight. Finally in Appendix A we explain how the proofs of the first and fourth tools (Theorem 1.8 and Proposition 1.11) of Section 1.8 follow from the proofs of analogous results in [BGHH20] by replacing the use of tail bounds with exponent 3/2 with the stretched exponential tails assumed here; provide the proofs of Lemmas 3.5, 4.5, and 4.2 from the main text; and prove the second tool of Section 1.8, Proposition 1.9.

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2. Concentration tools and the bootstrap

In this section we collect the concentration inequality for stretched exponential random variables from [KC18] and prove a slightly more flexible version which is more suitable for our applications. We then move to stating a general version of one iteration of the bootstrap, which will both illustrate the basic mechanism and be used later in Section 4.

To set the stage, let $\alpha \in (0,1]$ and suppose Y_i are independent mean zero random variables which satisfy, for some $L, M < \infty$,

$$\inf\left\{\eta > 0 : \mathbb{E}\left[g_{\alpha,L}\left(\frac{|Y_i|}{\eta}\right)\right] \leq 1\right\} \leq M, \tag{9}$$

where $g_{\alpha,L}(x) = \exp\left(\min\{x^2, (x/L)^\alpha\}\right) - 1$. The above condition is equivalent to the finiteness of a certain Orlicz norm introduced in [KC18]; see Definition 2.3 and Proposition A.1 therein. The use of Orlicz norms to prove concentration inequalities is well known; see for example [Ver18, Wai19]. The reader not familiar with this notion can keep in mind mean zero random variables Y_i with the property that, for some $c > 0$ and C, and all $t \geq 0$,

$$\mathbb{P}(|Y_i| \geq t) \leq C\exp(-ct^\alpha), \tag{10}$$

which are known to satisfy (9).

Proposition 2.1. *Given the above setting, there exists $c = c(M, L) > 0$ such that for all $t \geq 0$ and all $k \in \mathbb{N}$,*

$$\mathbb{P}\left(\left|\sum_{i=1}^k Y_i\right| \geq t\right) \leq \begin{cases} 2\exp\left(-\dfrac{ct^2}{k}\right) & 0 \leq t \leq k^{1/(2-\alpha)} \\ 2\exp\left(-ct^\alpha\right) & t \geq k^{1/(2-\alpha)}. \end{cases}$$

These two regimes capture the transition from the Gaussian behavior in the immediate tail to stretched exponential behavior deep into the tail.

Proof of Proposition 2.1. [KC18, Theorem 3.1] and the discussion after Remark 2.1 therein imply that, for some constants C and $c > 0$ (depending on M and L), for all $t \geq 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^k Y_i\right| \geq C(\sqrt{kt} + t^{1/\alpha})\right) \leq 2\exp(-ct).$$

Evaluating the transition point where $\sqrt{kt} = t^{1/\alpha}$ yields the statement of Proposition 2.1 by modifying the value of c in the previous display. ∎

In our applications, we will only have an upper tail bound and hence not a direct verification of the hypothesis (9) which needs two sided bounds as in (10). It will also at times be convenient to center the variables not by their expectation but by some other constant for which a tail bound is available. These two aspects are handled in the next lemma.

Lemma 2.2. *Suppose $k \in \mathbb{N}$, $\{Y_i : i \in [\![1,k]\!]\}$ are independent, and there exist constants ν_i, $\alpha \in (0,1]$, and $c > 0$ such that, for $t > t_0$, and $i \in [\![1,k]\!]$,*

$$\mathbb{P}\left(Y_i - \nu_i \geq t\right) \leq \exp(-ct^\alpha). \tag{11}$$

Then there exist constants $c_1 = c_1(c, \alpha, t_0)$ and $c' = c'(c, \alpha) > 0$ such that, for $t \geq 0$ and all $k \in \mathbb{N}$,

$$\mathbb{P}\left(\sum_{i=1}^{k}(Y_i - \nu_i) > t + kc_1\right) \leq \begin{cases} 2\exp\left(-\dfrac{c't^2}{k}\right) & 0 \leq t \leq k^{1/(2-\alpha)} \\ 2\exp\left(-c't^\alpha\right) & t \geq k^{1/(2-\alpha)}. \end{cases}$$

Proof. Let W_i be independent positive random variables whose distribution is defined by $\mathbb{P}(W_i > t) = \exp(-ct^\alpha)$ for $t \geq 0$. Then the hypothesis on Y_i implies that $Y_i - \nu_i$ is stochastically dominated by $W_i + t_0$, and hence there is a coupling of the Y_i and W_i over all i simultaneously such that

$$Y_i - \nu_i \leq W_i + t_0,$$

by standard coupling arguments. It is a calculation that $\mathbb{E}[W_i] = \alpha c^{-1/\alpha}\Gamma(\alpha)$, where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$ is the gamma function. Thus we get

$$P\left(\sum_{i=1}^{k}(Y_i - \nu_i) > t + kc_1\right) \leq \mathbb{P}\left(\sum_{i=1}^{k}(W_i - \mathbb{E}[W_i]) > t + k(c_1 - t_0 - \alpha c^{-1/\alpha}\Gamma(\alpha))\right).$$

Setting $c_1 = t_0 + \alpha c^{-1/\alpha}\Gamma(\alpha)$ and applying Proposition 2.1 completes the proof of Lemma 2.2, under the condition that $W_i - \mathbb{E}[W_i]$ satisfies (9) for some L and M depending only on α and c. We verify this next. [KC18, Proposition A.3] asserts that for any random variable Y satisfying, for all $t \geq 0$,

$$\mathbb{P}(|Y| \geq t) \leq 2\exp(-\tilde{c}t^\alpha), \tag{12}$$

there exist M and L, depending on α and $\tilde{c}$, such that (9) holds with Y in place of Y_i. Therefore it is sufficient to verify (12) for $Y = W_i - \mathbb{E}[W_i]$ for some $\tilde{c}$ depending on α and c. Since W_i is positive for each i, we have the bound

$$\mathbb{P}\Big(|W_i - \mathbb{E}[W_i]| \geq t\Big) \leq \begin{cases} 1 & 0 \leq t \leq \mathbb{E}[W_i] \\ \exp(-ct^\alpha) & t > \mathbb{E}[W_i] \end{cases},$$

which implies that (12) holds with $\tilde{c} = \min(c, \log 2)\cdot(\mathbb{E}[W_i])^{-\alpha}$, since $2\exp(-\tilde{c}t^\alpha) \geq 1$ for $0 \leq t \leq \mathbb{E}[W_i]$. Note that $\tilde{c}$ depends on only α and c. This completes the proof of Lemma 2.2. ☐

With the concentration tool Lemma 2.2 in hand, we next present the driving step of the bootstrapping argument. It is the formal statement and proof of one step of the iteration under a *sub*-additive assumption. As indicated in the outline of proof section, since X_r are *super*-additive, this will not be of use for the upper bound on the upper tail; but it will find application in the upper bound on the lower tail, where super-additivity is the favourable direction.

Proposition 2.3. *Suppose that for each $r, k \in \mathbb{N}$ with $k \leq r$, $\{Y_{r,i}^{(k)} : i \in [\![1,k]\!]\}$ is a collection of independent random variables. Suppose also that there exist $\alpha \in (0,1]$, $c > 0$, r_0, and θ_0 such that, for $r \in \mathbb{N}$, $k \in \mathbb{N}$, $i \in [\![1,k]\!]$, and $\theta \in \mathbb{R}$ such that $r/k > r_0$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(Y_{r,i}^{(k)} > \theta(r/k)^{1/3}\right) \leq \exp\left(-c\theta^\alpha\right). \tag{13}$$

Finally, let Y_r be a random variable such that $Y_r \le \sum_{i=1}^k Y_{r,i}^{(k)}$ for any $k \in \mathbb{N}$ satisfying $r/k > r_0$. Then there exist $\widetilde{\theta}_0 = \widetilde{\theta}_0(c, \alpha, \theta_0, r_0)$ and $c' = c'(c, \alpha, \theta_0, r_0)$ such that, for $\widetilde{\theta}_0 < \theta < r^{2/3}$ and $r > r_0$,

$$\mathbb{P}\left(Y_r > \theta r^{1/3}\right) \le \exp\left(-c'\theta^{3\alpha/2}\right).$$

Proposition 2.3 is written in a slightly more general way, without explicit reference to the LPP context it will be applied, to highlight the features of LPP that are relevant. In its application Y_r will be the weight of the heaviest path constrained to be in a certain parallelogram of height r, centred by μr, and $Y_{r,i}^{(k)}$ will be weights when constrained to be in disjoint subparallelograms of height r/k, centred by $\mu r/k$.

Finally, we mention a rounding convention we will adopt for the rest of the paper: the quantities k and r/k should always be integers and, when expressed as real numbers, will be rounded down without comment. The discrepancies of ± 1 which so arise will be absorbed into universal constants.

Proof of Proposition 2.3. By the bound $Y_r \le \sum_{i=1}^k Y_{r,i}^{(k)}$, for every $r, k \in \mathbb{N}$ with $k \le r$,

$$\mathbb{P}\left(Y_r > \theta r^{1/3}\right) \le \mathbb{P}\left(\sum_{i=1}^k Y_{r,i}^{(k)} > \theta r^{1/3}\right). \tag{14}$$

We will choose $k = \eta\theta^{3/2}$ for some $\eta \in (0,1)$, a form which is guided by our desire to apply the concentration bound Lemma 2.2 with its input bound (11) provided by the hypothesis (13) of Proposition 2.3; we also need $k \ge 1$. The first two considerations will determine an acceptable value for η via their development as the following two constraints:

(1) Lemma 2.2 introduces a linear term kc_1, which, when multiplied by the scale $(r/k)^{1/3}$ of the $Y_{r,i}^{(k)}$ indicated by (13), is $c_1 k^{2/3} r^{1/3}$; we want this to be smaller than a constant, say $\frac{1}{2}$, times $\theta r^{1/3}$. Note that c_1 depends on α, c, and θ_0.
(2) We require $r/k > r_0$ to apply the hypothesis of Proposition 2.3.

These two constraints, and that $\theta < r^{2/3}$ by hypothesis, force η to be smaller than r_0^{-1} and $2^{-3/2}c_1^{-3/2}$. We pick an η which satisfies these inequalities; thus η depends on c_1 and r_0. Set $\tilde{\theta}_0 = \eta^{-2/3}$; then $\theta \ge \tilde{\theta}_0$ implies $k \ge 1$. We will apply Lemma 2.2 with $Y_i = Y_{r,i}^{(k)}(r/k)^{-1/3}$, $\nu_i = \mu(r/k)^{2/3}$, and $t = \frac{1}{2}\theta k^{1/3}$. For $\theta \ge \tilde{\theta}_0$ and for a $\tilde{c}$ depending on only c and α,

$$\begin{aligned}
\mathbb{P}\left(Y_r > \theta r^{2/3}\right) &\le \mathbb{P}\left(\sum_{i=1}^k Y_{r,i}^{(k)} > \frac{1}{2}\theta k^{1/3}\left(\frac{r}{k}\right)^{1/3} + kc_1\left(\frac{r}{k}\right)^{1/3}\right) \\
&\le \begin{cases} 2\exp\left(-\tilde{c}\theta^2 k^{-1/3}\right) & \tilde{\theta}_0 k^{1/3} \le \theta k^{1/3} \le k^{1/(2-\alpha)} \\ 2\exp\left(-\tilde{c}\theta^\alpha k^{\alpha/3}\right) & \theta k^{1/3} \ge \max(\tilde{\theta}_0, k^{1/(2-\alpha)}) \end{cases} && \text{(applying Lemma 2.2)} \\
&\le 2\exp\left(-\tilde{c}\eta^{\alpha/3}\theta^{3\alpha/2}\right);
\end{aligned}$$

in the final line we have taken the second case of the preceding line. This is because $\alpha \le 1$ implies $k^{1/(2-\alpha)} \le k$, and the choice of k (and that $\eta < 1$) ensures that $\theta k^{1/3} \ge k$; so the second case holds, since we have already assumed $\theta > \tilde{\theta}_0$.

The proof of Proposition 2.3 is complete by absorbing the factor of 2 in the final display into the exponential, which we do by setting c' to $\tilde{c}\eta^{\alpha/3}/2$ and increasing $\tilde{\theta}$ (if needed), depending on c', so that $\exp(-c'(\tilde{\theta}_0)^{3\alpha/2}) \le 1/2$. ∎

3. Upper tail bounds

In this section we prove Theorems 1 and 2, respectively the upper and lower bounds on the upper tail.

3.1. **Upper bound on upper tail.** As mentioned in Section 1.4, for the argument for the upper bound on the upper tail, we need a sub-additive relation, instead of the natural super-additive properties that point-to-point weights exhibit. To bypass this issue, we discretize the geodesic and bound the weights of the discretizations by interval-to-interval weights, which do have a sub-additive relation with the point-to-point weight; this allows us to appeal to a form of the basic bootstrapping argument outlined around (3); Then performing a union bound over all possible discretizations will complete the proof.

We next state a version of one iteration of the bootstrap for the upper bound on the upper tail. There are a number of parameters which we will provide more context for after the statement.

Proposition 3.1. *Let* $\lambda_j = \frac{1}{2} + \frac{1}{2^j}$. *Suppose there exist* $\alpha \in (0,1]$, $\beta \in [\alpha, 1]$, $\zeta \in (0,\infty]$, $j \in \mathbb{N}$ *and constants* $c > 0$, θ_0, *and* r_0 *such that, for* $\theta > \theta_0$, $r > r_0$, *and* $|z| \le r^{5/6}$,

$$\mathbb{P}\left(X_r^z \ge \mu r - \lambda_j \frac{Gz^2}{r} + \theta r^{1/3}\right) \le \begin{cases} \exp\left(-c\theta^{\beta}\right) & \theta_0 < \theta < r^{\zeta} \\ \exp\left(-c\theta^{\alpha}\right) & \theta \ge r^{\zeta}. \end{cases} \tag{15}$$

Let $\zeta' = \min\left(\frac{\alpha\zeta}{1+\alpha\zeta} \cdot \frac{3-\beta}{3\beta}, \frac{2\alpha}{9+16\alpha}\right)$, *with* $\frac{\alpha\zeta}{1+\alpha\zeta}$ *interpreted as* 1 *if* $\zeta = \infty$. *There exist* $c' = c'(c,\alpha,\beta,j) > 0$, $\theta_0' = \theta_0'(\theta_0, c, \alpha, \beta, j)$, *and* $r_0' = r_0'(\alpha, j, r_0)$ *such that, for* $\theta > \theta_0'$, $r > r_0'$, *and* $|z| \le r^{5/6}$,

$$\mathbb{P}\left(X_r^z \ge \mu r - \lambda_{j+1} \frac{Gz^2}{r} + \theta r^{1/3}\right) \le \begin{cases} \exp\left(-c'\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}}\right) & \theta_0 < \theta < r^{\zeta'} \\ \exp(-c'\theta^{\alpha}) & \theta \ge r^{\zeta'}. \end{cases}$$

In particular, the input (15) *with parameters* (α,β,ζ,j) *gives as output the same inequality with parameters* $(\alpha,\beta',\zeta',j+1)$, *where* $\beta' > \beta$ *may be taken to be* $\frac{3-\beta/2}{3-\beta}\cdot\beta$ *in order to absorb the logarithmic factor.*

We first explain in words the content of the above result and describe the role of the various quantifiers appearing in the statement.

The range of z. Though Theorem 1 is stated only for $z = 0$, the discretization of the geodesic we adopt demands that we have the bootstrap improve the tail bound in a number of directions, defined by $|z| \le r^{5/6}$, in order to handle the potential "zig-zaggy" nature of the geodesic. Here we choose to consider $|z|$ till $r^{5/6}$ as till this level the second order term Hz^4/r^3 in Assumption 2 is at most of the order of fluctuations, namely $r^{1/3}$.

The role of λ_j. One may expect to be able to obtain an improved tail for deviation from the expectation, which is $\mu r - Gz^2/r$ up to smaller order terms. However, for technical reasons, this proves to be difficult; we say a little more about this in the caption of Figure 5. Instead, Proposition 3.1 proves a bound for the deviation only from a point away from the expectation, reflected by the factor λ_j in front of the parabolic term, which decreases as j increases. Nonetheless, this weaker bound suffices for our application: the relaxation has no effect for the $z = 0$ direction asserted by Theorem 1 since the parabolic term is always zero in that case.

The role of ζ. Notice that in the hypothesis (15) we allow two tail behaviors (with tail exponents α and β) for X_r^z in different regimes, with boundary at r^{ζ}. This is to allow the use of the conclusion of Proposition 3.1, which only improves the tail exponent for θ up to $r^{\zeta'}$, as input for subsequent applications of the same proposition. Theorem 1 will be obtained by applying Proposition 3.1 a finite number of times, with the output bound (with an increased exponent) of one application being the input for the next, till the exponent is raised from the initial value of $\beta = \alpha$ to a value greater than one for θ in the appropriate range of the tail. Then the same proposition will be applied one final time with $\beta = 1$; at this value of β,

$$\theta^{3\beta/(3-\beta)}(\log\theta)^{-\beta/(3-\beta)} = \theta^{3/2}(\log\theta)^{-1/2},$$

which will yield Theorem 1. The quantity

$$\zeta' = \min\left(\frac{\alpha\zeta}{1+\alpha\zeta}\cdot\frac{3-\beta}{3\beta},\ \frac{2\alpha}{9+16\alpha}\right)$$

measures how far into the tail each improved exponent holds via our arguments. The above explicit expression we obtain is perhaps hard to parse and is not of great importance for our conclusions. Nonetheless, we point out two basic properties of ζ': (i) it is smaller than ζ, as may be seen by algebraic manipulations of the first of the two expressions being minimized in its definition (along with $\beta \geq \alpha$); and (ii) it decays to zero as $\alpha \to 0$ linearly.

We next prove Theorem 1 given Proposition 3.1, before turning to the proof of Proposition 3.1.

Proof of Theorem 1. First, if $\alpha \geq 1$, we apply Proposition 3.1 with $\alpha = \beta = 1$, $\zeta = \infty$, $j = 1$, and the hypothesis (15) provided by Assumption 3a. This yields Theorem 1 by taking $z = 0$.

If $\alpha \in (0,1)$, we will apply Proposition 3.1 iteratively finitely many times. Let α_j, β_j, and ζ_j be values which we will specify shortly. We will select these values such that the hypothesis (15) of Proposition 3.1 holds with parameters $(\alpha_1, \beta_1, \zeta_1, 1)$ for all $|z| \leq r^{5/6}$, and, knowing that (15) holds with parameters $(\alpha_j, \beta_j, \zeta_j, j)$ for all $|z| \leq r^{5/6}$ and applying Proposition 3.1 will imply that (15) holds with parameters $(\alpha_{j+1}, \beta_{j+1}, \zeta_{j+1}, j+1)$ for all $|z| \leq r^{5/6}$.

We set $\alpha_j = \alpha$ for all j, and adopt the initial settings $\beta_1 = \alpha$ and $\zeta_1 = \infty$; so again (15) is provided by Assumption 3a when $j = 1$. The subsequent values are read off of Proposition 3.1 as follows for $j \geq 2$:

$$\beta_j = \min\left(\frac{3-\frac{1}{2}\beta_{j-1}}{3-\beta_{j-1}}\cdot\beta_{j-1},\ 1\right) \quad\text{and}\quad \zeta_j = \min\left(\frac{\alpha\zeta_{j-1}}{1+\alpha\zeta_{j-1}}\cdot\frac{3-\beta_{j-1}}{3\beta_{j-1}},\ \frac{2\alpha}{9+16\alpha}\right), \tag{16}$$

where $\alpha\zeta_{j-1}/(1+\alpha\zeta_{j-1})$ in the definition of ζ_j is interpreted as 1 when $\zeta_{j-1} = \infty$. We adopt the previous expression for β_j instead of the one given by Proposition 3.1 in order to absorb the log factor in the denominator of the exponent furnished by that proposition. Observe that $\beta_j > \beta_{j-1}$ whenever $\beta_{j-1} < 1$.

We define $n \in \mathbb{N}$ by

$$n := \min\left\{j : \beta_j = 1\right\}; \tag{17}$$

it can be checked that n is finite since, if $\beta_j < 1$,

$$\frac{\beta_j}{\beta_{j-1}} = \frac{3-\frac{1}{2}\beta_{j-1}}{3-\beta_{j-1}} = 1 + \frac{\beta_{j-1}}{2(3-\beta_{j-1})} \geq 1 + \frac{\alpha}{2(3-\alpha)},$$

as $\beta_{j-1} > \beta_{j-2} > \ldots > \beta_1 = \alpha$.

By the previous discussion, we know that (15) holds with parameters $(\alpha_n, \beta_n = 1, \zeta_n, n)$. Applying Proposition 3.1 with these parameters and taking $z = 0$ gives the statement of Theorem 1 with $\zeta = \zeta_{n+1} = \min(\frac{2}{3}\cdot\frac{\alpha\zeta_n}{1+\alpha\zeta_n}, \frac{2\alpha}{9+16\alpha})$. It is clear from this expression that $\zeta \to 0$ as $\alpha \to 0$, and, since $2\alpha/(9+16\alpha)$ achieves a maximum value of $2/25$ for all $\alpha \in (0,1]$, that $\zeta \in (0, 2/25]$. ☐

Remark 3.2. We can now specify more precisely the regimes of θ provided by the proof of Theorem 1 where the tail exponent transitions from $3/2$ to α, as mentioned in Remark 1.6. That is, for $j = [\![1, n]\!]$ with n as in (17) and β_j and ζ_j as in (16), it holds for $\theta \in [r^{\zeta_{j+1}}, \zeta_j]$ that

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \geq \theta r^{1/3}\right) \leq \exp\left(-c\theta^{\beta_j}\right) \quad\text{for}\quad \theta \in [r^{\zeta_{j+1}}, r^{\zeta_j}).$$

It remains to prove Proposition 3.1. A roadmap for the proof is as follows.

(1) As indicated immediately before the statement of the proposition, to achieve a stochastic domination of the geodesic weight by a sum, we specify a grid-based discretization of the geodesic, and Lemma 3.3 bounds the cardinality of the number of possible discretizations.
(2) Lemma 3.4 provides an improved tail bound (compared to the hypothesis (15)) for the weight of a given discretization, using the bootstrapping idea of looking at smaller scales. This makes use of Lemma 3.5, which takes the point-to-point tail available from (15) and gives an interval-to-interval bound with the same tail.
(3) When Lemmas 3.3 and 3.4 are in hand, the proof of Proposition 3.1 will be completed by taking a union bound.

We address each of the above three steps in turn in the next three subsections.

3.1.1. *Step 1: The discretization scheme.* We will define a grid $\mathbb{G}^z$ of intervals through which any geodesic from $(1,1)$ to $(r-z, r+z)$, on the event that it is typical, must necessarily pass through; see Figure 4.

We recall from Section 1.4 that "width" refers to measurement along the anti-diagonal and "height" to measurement along the diagonal. For $k \in \mathbb{N}$ to be set, the width of a cell in the grid will be $(r/k)^{2/3}$, and the height r/k. The number of cells in a column of the grid is k, and the number of cells in a row is $M = 2\theta^{3/4\alpha}k^{2/3}$ as we want the width of $\mathbb{G}^z$ to be $2\theta^{3/4\alpha}r^{2/3}$. The width of $\mathbb{G}^z$ is set to this value because, by Proposition 1.9 on the probability of any geodesic having large transversal fluctuations, $\mathbb{P}(\mathrm{TF}(\Gamma_r^z) > \theta^{3/4\alpha}r^{2/3}) \le \exp(-c\theta^{3/2})$; note that this is smaller than the bound we are aiming to prove in Proposition 3.1 and so we may essentially ignore the event that any geodesic exits the grid.

We now move to the formal definition. We assume k is small enough that $(r/k)^{2/3} \ge 1$, i.e., $k \le r$ (as the minimum separation of points in $\mathbb{Z}^2$ is 1). The grid $\mathbb{G}^z$ consists of intervals $\mathbb{L}_{ij}^z$ as follows:

$$\mathbb{G}^z = \left\{\mathbb{L}_{ij}^z : i \in [\![0,k]\!], j \in [\![0,M]\!]\right\},$$

where M is defined as

$$M = 2\cdot\lceil \theta^{\frac{3}{4\alpha}}k^{2/3}\rceil. \tag{18}$$

Let $v_i = \lfloor ir/k \rfloor$ and $h_{i,j}^z = \lfloor iz/k + (\theta^{\frac{3}{4\alpha}} - jk^{-2/3})r^{2/3}\rfloor$. For $i \in [\![0,k]\!]$ and $j \in [\![0,M]\!]$, the line segment $\mathbb{L}_{ij}^z$ will connect the points

$$\left(v_i - h_{i,j}^z, v_i + h_{i,j}^z\right) \quad \text{and} \quad \left(v_i - h_{i,j+1}^z, v_i + h_{i,j+1}^z\right)$$

In words, the grid $\mathbb{G}^z$ is contained in the rectangle $\{|y - \frac{r+z}{r-z}\cdot x| \le \theta^{\frac{3}{4\alpha}}r^{2/3}, 0 \le x+y \le 2r\}$. Grid lines along the anti-diagonal will be called $\mathbb{G}_i^z$, i.e., for $i = 0, 1, \ldots, k$,

$$\mathbb{G}_i^z = \left\{\mathbb{L}_{ij}^z : j \in [\![0,M]\!]\right\}.$$

We call $\mathcal{L}^z = (L_0, \ldots, L_k)$ a discretization, where $L_i \in \mathbb{G}_i^z$ is an interval on the i^{th} grid line. We impose that L_0 and L_k are the intervals whose midpoints are $(1,1)$ and $(r-z, r+z)$ respectively.

Lemma 3.3. *The set of discretizations has size at most* $\exp\left\{k\left(\log k + \frac{3}{4\alpha}\log\theta + \log 2\right)\right\}$.

Proof. This follows from the observation that there are $M = 2\theta^{\frac{3}{4\alpha}}k^{2/3} \le 2\theta^{\frac{3}{4\alpha}}k$ intervals on each grid line $\mathbb{G}_i^z$, and there are $k-1$ grid lines in total where there is a choice of interval (as the intervals from $\mathbb{G}_0^z$ and $\mathbb{G}_k^z$ are fixed), giving $(2\theta^{\frac{3}{4\alpha}}k^{2/3})^{k-1}$ discretizations. ∎

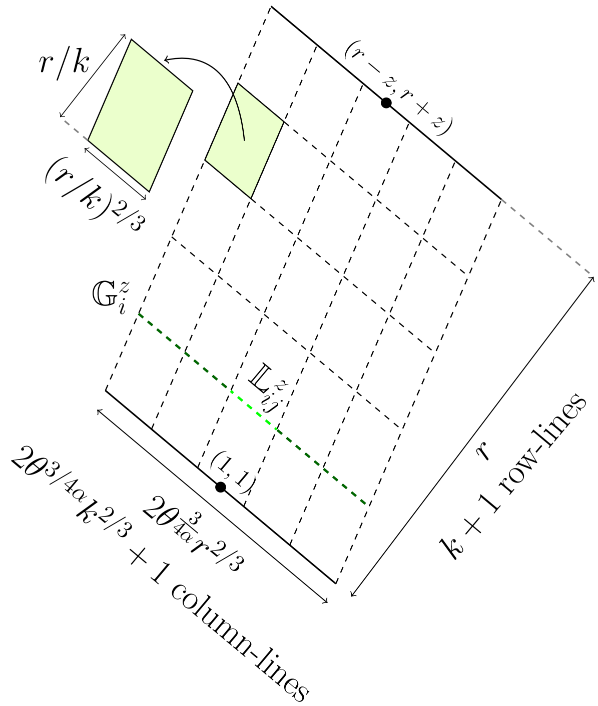


FIGURE 4. The grid utilized for the discretization in Step 1 of the proof of Proposition 3.1. Note that measurements are made along the antidiagonal and diagonal only, with the diagonal chosen over the line with the slope of the left or right boundary of the grid. The lower boundary of the grid $\mathbb{G}^z$ is centered at $(1,1)$ and the upper boundary at $(r-z, r+z)$. From each grid line $\mathbb{G}_i^z$, one interval L_i is picked to form a discretization $\mathcal{L}^z = (L_0, \ldots, L_k)$ with the constraint that L_0 is fixed to be the interval on $\mathbb{G}_0^z$ whose midpoint is $(1,1)$ and L_k to be the interval on $\mathbb{G}_k^z$ whose midpoint is $(r-z, r+z)$. On the high probability event that all geodesics passes through the grid, its weight is upper bounded by the maximum, over all discretizations $\mathcal{L}^z$, of the sum of interval-to-interval weights of the intervals in $\mathcal{L}^z$. These weights are independent and have fluctuations of scale $(r/k)^{1/3}$, which allows us to use the idea of bootstrapping.

For a given discretization $\mathcal{L}^z = (L_0, \ldots, L_k)$, let $X_{\mathcal{L}^z}$ be the maximum weight of all paths which pass through all intervals of $\mathcal{L}^z$. The discretization described above implies that, on the event that $\mathrm{TF}(\Gamma_r^z) \leq \theta^{\frac{3}{4\alpha}} r^{2/3}$,

$$X_r^z \leq \max_{\mathcal{L}^z} X_{\mathcal{L}^z},$$

where the maximization is over all discretizations $\mathcal{L}^z$. So to prove Proposition 3.1, we need a tail bound on $X_{\mathcal{L}^z}$ for a fixed discretization $\mathcal{L}^z$; this is Step 2 and is done in the next subsection, where the hypothesis (15) and bootstrapping are used to provide an improved tail bound on $X_{\mathcal{L}^z}$.

3.1.2. *Step 2: An improved tail bound on $X_{\mathcal{L}^z}$.* Because θ is a global parameter which affects the set of discretizations, we will use the symbol t as in (19) ahead to denote the scaled deviation when considering the weight associated to a fixed discretization, though we will eventually set $t = \theta$. The following lemma uses the idea of moving to lower scales to obtain an improved tail bound for $X_{\mathcal{L}^z}$ for a fixed discretization $\mathcal{L}^z$.

Lemma 3.4. *Under the hypotheses of Proposition 3.1 there exist* $c' = c'(c, \alpha, \beta, j) > 0$, $\delta = \delta(c, \beta, j, \theta_0) > 0$, *and* $t_0 = t_0(c, \beta, j)$ *such that the following holds. Let* $t > t_0$, $r > r_0$, $2^6 \leq k \leq \min(\delta t^{3/2}, r_0^{-1} r)$, $\theta \geq \theta_0$, *and* $z \in [-r, r]$ *be such that* $|z| \leq r^{5/6}$ *and* $(r/k)^{5/6} > 4\theta^{3/4\alpha} r^{2/3}$. *Let* $\mathcal{L}^z = (L_0, \ldots, L_k)$ *be a fixed discretization. Then*

$$P\left(X_{\mathcal{L}^z} > \mu r - \lambda_{j+1} \frac{Gz^2}{r} + tr^{1/3}\right) \leq \exp\left(-c' t^{\beta} k^{\beta/3}\right) + k \cdot \exp\left(-c'(r/k)^{\alpha\zeta}\right), \tag{19}$$

with the second term interpreted as zero if $\zeta = \infty$.

The basic tool in the proof of Lemma 3.4 is to bound $X_{\mathcal{L}^z}$ by the sum of the interval-to-interval weights defined by the intervals in $\mathcal{L}^z$. So given a point-to-point upper tail bound, as in the hypothesis of Proposition 3.1, we will first need to obtain an upper tail bound for interval-to-interval weights.

We define the relevant intervals to state the interval-to-interval bound next. For r fixed, and $|w| \leq r^{5/6}$, let $\mathbb{L}_{\text{low}}$ be the line segment joining $(-r^{2/3}, r^{2/3})$ and $(r^{2/3}, -r^{2/3})$ and let $\mathbb{L}_{\text{up}}$ be the line segment joining $(r - w - r^{2/3}, r + w + r^{2/3})$ and $(r - w + r^{2/3}, r + w - r^{2/3})$. Thus w is the midpoint displacement of the intervals, and note that their height difference is r. Define Z by

$$Z = X_{\mathbb{L}_{\text{low}}, \mathbb{L}_{\text{up}}}.$$

The content of the next lemma is a tail bound on Z.

Lemma 3.5. *Suppose* (15) *holds as in Proposition 3.1. Then there exist* $\tilde{c} = \tilde{c}(c, j)$, $\tilde{t}_0 = \tilde{t}_0(\theta_0, j)$, *and* $\tilde{r}_0 = \tilde{r}_0(r_0, j)$ *such that, for* $r > \tilde{r}_0$, $|w| \leq r^{5/6}$, *and* $t > \tilde{t}_0$

$$\mathbb{P}\left(Z > \mu r - \lambda_{j+1} \frac{Gw^2}{r} + tr^{1/3}\right) \leq \begin{cases} \exp\left(-\tilde{c} t^{\beta}\right) & \tilde{t}_0 < t < r^{\zeta} \\ \exp\left(-\tilde{c} t^{\alpha}\right) & t \geq r^{\zeta}. \end{cases} \tag{20}$$

We note that the hypothesis (15) of Proposition 3.1 is a point-to-point tail bound from $\mu r - \lambda_j G z^2/r$, whereas the conclusion of Lemma 3.5 has the weaker λ_{j+1} in place of λ_j (recall $\lambda_j = 1/2 + 2^{-j}$). This reduction in the coefficient of the parabolic term is the previously mentioned relaxation which allows the bootstrap to proceed to the next iteration.

The proof of Lemma 3.5 relies on the geometric idea of stepping back from the two intervals and considering a proxy point-to-point weight. Similar arguments have appeared in the literature previously (see e.g., [BSS14]), but for completeness we give a self-contained proof of Lemma 3.5 in Appendix A. However, we highlight the main idea in Figure 5 where we also say a few words on why it is difficult to avoid the relaxation in the parabolic loss.

Proof of Lemma 3.4. Observe the following stochastic domination

$$X_{\mathcal{L}^z} \preceq \sum_{i=1}^{k} Z_i,$$

where Z_i are independent random variables distributed as the weight of the best path from L_{i-1} to L_i. Apart from possible rounding, because Z_i and Z_{i-1} are independent versions of weights which overlap on the interval L_{i-1}, it is possible that the linear term in Z_i is $\mu r/k + O(1)$ rather than $\mu r/k$. We handle this discrepancy by absorbing it into the term $tr^{1/3}$ of Lemma 3.5, which is the only situation where it arises, without further comment.

We note that the diagonal separation between the sides of Z_i is r/k, instead of r as in the definition of Z. We denote the anti-diagonal displacement of the midpoints of the corresponding intervals of Z_i by z_i. We want to eventually apply Lemma 2.2 to $\sum Z_i$, appropriately centred, with its input

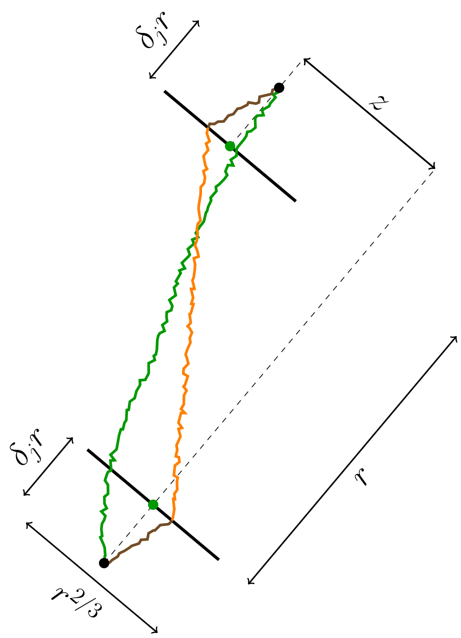


FIGURE 5. The argument for Lemma 3.5. The two black intervals have midpoint separation of z in the antidiagonal direction. The orange path is the heaviest path between the two intervals (so has weight Z), and the brown paths are geodesics connecting the black points to the endpoints of the red path. The green path is a geodesic between the two black points. With positive probability the two brown paths each have weight greater than $\mu\delta_j r - \frac{1}{3}\theta r^{1/3}$, and so, on the intersection of those events with $\{Z > \mu r - \lambda_{j+1}Gz^2/r + \theta r^{1/3}\}$, it holds that the green path has weight at least $\mu(1+2\delta_j)r - \lambda_{j+1}Gz^2/r + \frac{1}{3}\theta r^{1/3}$. We choose δ_j such that the parabolic term in this expression is $\lambda_j Gz^2/(1+2\delta_j)r$ and apply the point-to-point bound we have. It is because the antidiagonal separation between each pair of black and green points is zero that we have a decrease in the parabolic term. If we make this separation proportional to z, then there is no decrease in the parabolic term, but for large z the gradient of the limit shape from Assumption 2 causes issues. This can be more carefully handled if we instead consider the supremum of *fluctuations* of point-to-point weights from their expectation, and we will have need to do this on one occasion in the appendix.

tail bound (11) provided by Lemma 3.5. To reach a form of the probability where Lemma 2.2 is applicable, we observe that

$$
\begin{aligned}
\mathbb{P}\left(X_{\mathcal{L}^z} > \mu r - \lambda_{j+1}\frac{Gz^2}{r} + tr^{1/3}\right) &\leq \mathbb{P}\left(\sum_{i=1}^{k} Z_i \geq \mu r - \lambda_{j+1}\frac{Gz^2}{r} + tr^{1/3}\right) \\
&= \mathbb{P}\left(\sum_{i=1}^{k} (Z_i - \nu_i) \geq \mu r - \lambda_{j+1}\frac{Gz^2}{r} - \sum_{i=1}^{k}\nu_i + tr^{1/3}\right) \\
&\leq \mathbb{P}\left(\sum_{i=1}^{k} (Z_i - \nu_i) \geq tr^{1/3}\right), \qquad (21)
\end{aligned}
$$

where $\nu_i = \mu r/k - \lambda_{j+1} G z_i^2 k/r$. The choice of ν is dictated by the desire to apply (20) with r replaced by r/k. All the steps before the last inequality are straightforward consequences of definitions. To see the last inequality, note that since $\sum z_i = z$, the Cauchy-Schwarz inequality implies that $\sum \nu_i$ is smaller than $\mu r - \lambda_{j+1} G z^2/r$.

We will soon apply Lemma 3.5, which will yield a tail bound for $Z_i - \nu_i$; the tail bound is (20) with r replaced by r/k. However, this tail bound has two regimes with different exponents, α and β, while the basic concentration result we seek to apply, i.e., Theorem 2.2, assumes the same tail exponent throughout.

Thus to have variables that have the larger exponent β in the entire tail, we will apply a simple truncation on Z_i: define

$$\overline{Z}_i = \begin{cases} Z_i & \text{if } Z_i - \nu_i \le \left(\frac{r}{k}\right)^{\zeta+1/3} \\ \nu_i & \text{if } Z_i - \nu_i > \left(\frac{r}{k}\right)^{\zeta+1/3}. \end{cases}$$

Now following (21), we get

$$\begin{aligned}\mathbb{P}\left(X_{\mathcal{L}^z} > \mu r - \lambda_{j+1}\frac{Gz^2}{r} + tr^{1/3}\right) &\le \mathbb{P}\left(\sum_{i=1}^k (\overline{Z}_i - \nu_i) \ge tr^{1/3}\right) \\ &\qquad + \mathbb{P}\bigg(\bigcup_{i=1}^k \left\{Z_i - \nu_i > (r/k)^{\zeta+1/3}\right\}\bigg).\end{aligned} \tag{22}$$

We will apply the concentration bound Lemma 2.2 to bound the first term. We first want to apply Lemma 3.5, with r/k in place of r, in order to get the tail bound on each individual $Z_i - \nu_i$, which will act as input for Lemma 2.2. Two hypotheses of Lemma 3.5, namely (15) and that $r/k > r_0$, are available here by the hypotheses of Lemma 3.4.

But Lemma 3.5 has the additional hypothesis that the anti-diagonal displacement $|w|$ is at most $(r/k)^{5/6}$, which must also be checked. The verification of this follows from the hypothesis in Lemma 3.4 that $(r/k)^{5/6} > 4\theta^{3/4\alpha} r^{2/3}$, as the maximum anti-diagonal displacement possible in a single row of the grid is at most $2\theta^{3/4\alpha} r^{2/3} + |z|/k$, where the first term is the grid width $2\theta^{3/4\alpha} r^{2/3}$, and the second term is the shift caused by the overall slope of the grid. Now since $|z| \le r^{5/6}$ and $k \ge 2^6$, we see that $|z|/k$ is at most $\frac{1}{2}(r/k)^{5/6}$, and some simple algebra completes the verification.

Thus, applying Lemma 3.5 with r/k in place of r, we use the first case of (20) (since Z_i has been appropriately truncated to give $\overline{Z}_i$) as the input tail bound with exponent β on $\overline{Z}_i - \nu_i$ required for Lemma 2.2. Finally, with $c_1 = c_1(c, \beta, \theta_0, j)$ as in the statement of the latter, let $\delta = \min(1, (2c_1)^{-3/2})$ where recall we have the hypothesis that $k \le \min(\delta t^{3/2}, r_0^{-1} r)$; δ depends on c, β, j and θ_0. With this preparation, we see

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^k (\overline{Z}_i - \nu_i) \ge tr^{1/3}\right) &= \mathbb{P}\left(\sum_{i=1}^k (\overline{Z}_i - \nu_i)(r/k)^{-1/3} \ge (tk^{1/3} - kc_1) + kc_1\right) \\ &\le \mathbb{P}\left(\sum_{i=1}^k (\overline{Z}_i - \nu_i)(r/k)^{-1/3} \ge \frac{1}{2} tk^{1/3} + kc_1\right) \\ &\qquad\qquad [\text{since by hypothesis } k \le (2c_1)^{-3/2} t^{3/2}] \\ &\le \begin{cases} 2\exp\left(-\tilde{c} t^2 k^{-1/3}\right) & 0 \le tk^{1/3} < k^{1/(2-\beta)} \\ 2\exp\left(-\tilde{c} t^\beta k^{\beta/3}\right) & tk^{1/3} \ge k^{1/(2-\beta)}; \end{cases} \qquad [\text{by Lemma 2.2}]\end{aligned}$$

we have applied Lemma 2.2 with $tk^{1/3}$ in place of t and $\alpha = \beta$. Here $\tilde{c}$ is a function of c (as given in the hypothesis (15)), β, and j. We now claim that the second case of the last display dictates the fluctuation behavior under our hypotheses. To see this, note that since $\beta \le 1$, $k^{1/(2-\beta)} \le k$. Thus the first case in the last display holds only if $k > t^{3/2}$ while by hypothesis $k \le t^{3/2}$ since $\delta \le 1$. Further, since $k \ge 1$, we may set the lower bound t_0 on t high enough that $\exp(-\frac{1}{2}\tilde{c}t^{\beta}) \le 1/2$ so as to absorb the pre-factor of 2 in the last display; t_0 depends on $\tilde{c}$ and β. We have hence bounded the first term of (22).

To bound the second term when $\zeta < \infty$, we take a union bound and apply Lemma 3.5, where the latter's hypotheses are satisfied by the same reasoning as used above in the application for the first term. This yields that the second term of (22) is bounded by $k \cdot \exp\left(-\tilde{c}(r/k)^{\alpha\zeta}\right)$, using the second case of (20) with r/k in place of r. Here $\tilde{c}$ is a function of c, α, and j. When $\zeta = \infty$, the second term of (22) is clearly zero.

Returning to (22) with these two bounds completes the proof of Lemma 3.4, taking $c' = \tilde{c}$. □

3.1.3. *Step 3: Handling all the discretizations.* With the improved tail bound for a fixed discretization provided by Lemma 3.4, we can implement Step 3 and complete the proof of Proposition 3.1, essentially via a union bound.

Proof of Proposition 3.1. Recall that θ_0' is the lower bound on θ under which the conclusions of Proposition 3.1 must be shown to hold, and that we have the freedom to set it. We will increase its value as needed as the proof proceeds. We will be explicit about the dependencies θ_0' takes on at each such time. We start with $\theta_0' = e$ so that $\log\theta \ge 1$. Also, in this proof, c is reserved for the constant in the point-to-point tail hypothesis (15).

Lemma 3.3 says that the entropy from the union bound we will soon perform will be $\exp\{\Theta(k\log k + k\log\theta)\}$, which needs to be counteracted by the bound from Lemma 3.4. Anticipating this we take, in Lemma 3.4,

$$t = \theta \qquad \text{and} \qquad k = \varepsilon \cdot \theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{3}{3-\beta}}, \tag{23}$$

for $\varepsilon = \varepsilon(c,\alpha,\beta,j) \in (0,1)$ a sufficiently small constant, to be set shortly. At this point we will ensure that the hypotheses of Lemma 3.4 hold. We set θ_0' larger if needed so that it is at least t_0 as in Lemma 3.4, so that the value of t above satisfies $t > t_0$. Additionally we have to verify that, with δ as provided by Lemma 3.4,

- $4\theta^{3/4\alpha}r^{2/3} < (r/k)^{5/6}$.
- $k \in [\![2^6, \min(\delta t^{3/2}, r_0^{-1}r)]\!]$

For the first condition, the fact that $k \le \theta^{3/2}$ (since $\varepsilon, \beta \le 1$ and $\log\theta \ge 1$), and some algebraic manipulation, implies that it is sufficient if $\theta \le \frac{1}{4}r^{(2\alpha)/(9+15\alpha)}$; to avoid carrying forward the factor of 4, we instead reduce the exponent of r to absorb it and impose that

$$\theta \le r^{\frac{2\alpha}{9+16\alpha}}; \tag{24}$$

this implies $\theta \le \frac{1}{4}r^{(2\alpha)/(9+15\alpha)}$ (and hence the first condition above) when r_0', which is the lower bound on r that we are free to set, is large enough. The value of r_0' depends only on α.

For the second condition, note that $2\alpha/(9+16\alpha) < 2/3$, and that $\beta \le 1$ implies $3\beta/(3-\beta) \le 3/2$. Combining this latter inequality with the value (23) of k, and that $\theta \le r^{2/3}$ from (24), ensures that $l \in [\![2^6, \min(\delta\theta^{3/2}, r_0^{-1}r)]\!]$ by setting θ_0' large enough, depending on β, δ, and ε; so the second condition holds.

Thus applying Lemma 3.4 with values of t and k as in (23) we obtain that, for $\theta_0' < \theta < r^{2\alpha/(9+16\alpha)}$,

$$\begin{aligned}\mathbb{P}&\left(X_{\mathcal{L}^z} > \mu r - \lambda_{j+1}\frac{Gz^2}{r} + \theta r^{1/3}\right)\\ &\qquad\le \exp\left(-c'\cdot\varepsilon^{\beta/3}\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}}\right) + \theta^{\frac{3\beta}{3-\beta}}\exp\left(-c' r^{\alpha\zeta}\theta^{-\frac{3\alpha\beta\zeta}{3-\beta}}\right),\end{aligned} \tag{25}$$

with the second term equal to zero if $\zeta = \infty$, and with c' as in Lemma 3.4; thus c' depends on c, α, β, and j. In substituting k in the second term of (19) we have used that $k \leq \theta^{3\beta/(3-\beta)}$ since $\varepsilon < 1$ and $\log\theta \geq 1$. When $\zeta < \infty$, we would like the exponential factor of the second term to be smaller than the first term; i.e., it is sufficient if

$$r^{\alpha\zeta}\theta^{-\frac{3\alpha\beta\zeta}{3-\beta}} \geq \theta^{\frac{3\beta}{3-\beta}}.$$

We will soon absorb the polynomial-in-θ factor in the second term by reducing the constant c. Simple algebraic manipulations show that the inequality of the last display is implied by the condition

$$\theta \leq r^{\bar\zeta} \quad \text{with} \quad \bar\zeta = \frac{\alpha\zeta}{1+\alpha\zeta}\cdot\frac{3-\beta}{3\beta}.$$

For simplicity, we impose this last condition on θ even when $\zeta = \infty$, even though in this case the second term of (25) is zero (and so smaller than the first term) for all θ up to $r^{2\alpha/(9+16\alpha)}$ (when $\zeta = \infty$, we interpret the first factor of $\bar\zeta$ to be one, i.e., $\bar\zeta = (3-\beta)/3\beta$).

To handle both the condition in the last display and (24), we impose $\theta_0' < \theta < r^{\zeta'}$, with

$$\zeta' = \min\left(\bar\zeta, \frac{2\alpha}{9+16\alpha}\right).$$

So far we have shown that, for $r > r_0'$ and $\theta_0' < \theta < r^{\zeta'}$,

$$\mathbb{P}\left(X_{\mathcal{L}^z} > \mu r - \lambda_{j+1}\frac{Gz^2}{r} + \theta r^{1/3}\right) \leq 2\exp\left(-\tfrac{1}{2}c'\cdot\varepsilon^{\beta/3}\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}}\right); \tag{26}$$

where, for all $\theta > \theta_0'$, the $\theta^{\frac{3\beta}{3-\beta}}$ polynomial factor coming from the second term of (25) has been absorbed by the reduction of c' to $c'/2$. To do this we may also need to increase the value of θ_0'; this choice of θ_0' can be made depending only on c' since we only need $\theta^{3\beta/(3-\beta)}\exp(-c'\theta^{3\beta/(3-\beta)}) \leq \exp(-0.5c'\theta^{3\beta/(3-\beta)})$ and the same function of θ is in the exponent and as the polynomial-factor.

Now we observe that on the event that any geodesic stays within the grid $\mathbb{G}^z$, X_r^z is dominated by $\max_{\mathcal{L}^z} X_{\mathcal{L}^z}$. This yields

$$\begin{aligned}\mathbb{P}&\left(X_r^z > \mu r - \lambda_{j+1}\frac{Gz^2}{r} + \theta r^{1/3}\right)\\ &\leq \mathbb{P}\left(\max_{\mathcal{L}^z} X_{\mathcal{L}^z} > \mu r - \lambda_{j+1}\frac{Gz^2}{r} + \theta r^{1/3}\right) + \mathbb{P}\left(\mathrm{TF}(\Gamma_r^z) > \theta^{\frac{3}{4\alpha}}r^{2/3}\right).\end{aligned} \tag{27}$$

The second term is bounded by $\exp(-c'\theta^{3/2})$ by Proposition 1.9 for all θ such that $\theta^{3/4\alpha} > s_0$, with s_0 an absolute constant as given in the statement of the corollary. We increase θ' if needed to meet this condition; this increase can be done in a way that depends only on s_0 as since $\alpha \leq 1$, it is sufficient if $\theta_0 \geq s_0^{4/3}$.

We want to bound the first term of(27) by a union bound over all discretizations $\mathcal{L}^z$. First we bound the cardinality of the set of discretizations using Lemma 3.3. Note that the definition of k in (23) implies that $\log k \leq \frac{3\beta}{3-\beta}\log\theta$ as $\varepsilon < 1$. Lemma 3.3 asserts that the set of discretizations has cardinality at most $\exp\{k(\log k + \frac{3}{4\alpha}\log\theta + \log 2)\}$. The just mentioned bound on $\log k$ and the value of k from (23) shows that this cardinality is at most

$$\exp\left(\tilde c\varepsilon\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{3}{3-\beta}+1}\right) = \exp\left(\tilde c\varepsilon\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}}\right),$$

with $\tilde c$ a constant which depends on only α and β. Given this and the bound in (26), we apply a union bound. This yields that, for $\theta_0' < \theta < r^{\zeta'}$, the first term of (27) is at most

$$2\exp\left(-\tfrac{1}{2}c'\cdot\varepsilon^{\beta/3}\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}} + \tilde c\cdot\varepsilon\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}}\right).$$

Now since $\beta \le 1$, for sufficiently small ε it holds that $\tilde{c}\varepsilon \le \frac{1}{4}c' \cdot \varepsilon^{\beta/3}$, and we fix ε to such a value; note that ε does not depend on θ and only on c, α, β, and j. This can be seen since ε depends on $\tilde{c}$ and c', which respectively depend on α and β only, and c, α, β, and j.

For this value of ε and for $\theta_0' < \theta < r^{\zeta'}$, the previous display is bounded above by

$$\exp\left(-\tfrac{1}{4}c' \cdot \varepsilon^{\beta/3}\theta^{\frac{3\beta}{3-\beta}}(\log\theta)^{-\frac{\beta}{3-\beta}}\right).$$

Putting this bound into (27) completes the proof of Proposition 3.1 for $\theta_0' < \theta < r^{\zeta'}$ after relabeling c' in its statement by $\frac{1}{4}c' \cdot \varepsilon^{\beta/3}$. For when $\theta > r^{\zeta'}$, the hypothesis (15) provides the bound when $r > r_0$, which we ensure by raising r_0' (if necessary) to be at least r_0. This completes the proof of Proposition 3.1 by relabeling c' in its statement to be less than c if needed. □

3.2. **Lower bound on upper tail.** We prove the lower bound on the upper tail, i.e., Theorem 2.

Proof of Theorem 2. Assumption 2 implies that $\mathbb{P}(X_r \ge \mu r + \theta r^{1/3}) \le \mathbb{P}\left(X_r \ge \mathbb{E}[X_r] + \theta r^{1/3}\right)$, and we prove the stronger bound that $\mathbb{P}(X_r \ge \mu r + \theta r^{1/3}) \ge \exp(-c\theta^{3/2})$ for appropriate θ.

Observe that $X_r \ge \sum_{i=0}^{n} X_{r/k}^{(i)}$ where $X_{r/k}^{(i)} = X_{i(r/k,r/k)+(1,0),(i+1)(r/k,r/k)}$. Now by Assumption 4a at $z = 0$ we have that

$$\mathbb{P}\left(X_{r/k}^{(i)} \ge \mu r/k + C(r/k)^{1/3}\right) \ge \delta$$

for each $i \in [\![0, k-1]\!]$, as long as $r/k > r_0$. Since

$$\left\{\sum_{i=0}^{k} X_{r/k}^{(i)} \ge \mu r + Ck^{2/3}r^{1/3}\right\} \supseteq \bigcap_{i=0}^{k-1}\left\{X_{r/k}^{(i)} \ge \mu r/k + C(r/k)^{1/3}\right\},$$

we have

$$\mathbb{P}\left(X_r \ge \mu r + Ck^{2/3}r^{1/3}\right) \ge \delta^k,$$

using the independence of the $X_{r/k}^{(i)}$ across i. Now we set $k = C^{-3/2}\theta^{3/2}$, giving

$$\mathbb{P}\left(X_r \ge \mu r + \theta r^{1/3}\right) \ge \exp(-c\theta^{3/2})$$

for some $c > 0$ and for all θ satisfying $1 \le C^{-3/2}\theta^{3/2} \le r/r_0$, which is equivalent to $C \le \theta \le Cr_0^{-2/3} \times r^{2/3}$. Thus the proof of Theorem 2 is completed by setting $\theta_0 = C$ and $\eta = Cr_0^{-2/3}$. □

4. Lower tail and constrained lower tail bounds

In this section we prove Theorems 3, 4, and 5. In fact Theorem 5 implies both of the other two, but we prove Theorem 3 first separately to aid in exposition.

4.1. **Upper bound on lower tail.** Note that the abstracted bootstrap statement Proposition 2.3 is applicable with $Y_r = -(X_r - \mu r)$ and $Y_{r,i}^{(k)} = -(X_{r/k}^{(i)} - \mu r/k)$, where $X_{r/k}^{(i)}$ is the last passage value from $(i-1)/k \cdot (r,r) + (1,0)$ to $i/k \cdot (r,r)$ for $i = 1, \ldots, k$. Iterating this would yield a lower tail exponent of 3/2 (a similar argument for the upper tail under *sub*-additivity was outlined in the beginning of Section 1.4) but will not be able to reach the optimal exponent of 3.

Recall from Section 1.4 that our argument relies on a high-probability construction from [BGHH20] of k disjoint paths with good collective weight, here Theorem 1.10. Thus the probability the construction fails is an upper bound on the probability that many disjoint curves have small weight, which in turn bounds the probability that the geodesic has small weight, as we seek.

As outlined before, the construction relies on three inputs: the first is the parabolic curvature on the limit shape, provided by Assumption 2; the second is an exponential upper bound on the lower tail of the maximum weight among all paths constrained to stay within a given parallelogram; and the third is a lower bound on the mean of such weights. Recall that we call such weights "constrained weights". Like the first input, the third input is available to us already, and is the content of (8) of Proposition 1.11. So only the second input needs to be attained via bootstrapping.

From here on the argument has two broad steps.

(1) Use our assumptions to obtain the exponential bound (in fact, we obtain an exponent of $3/2$) on the constrained weight's lower tail that can be used as an input for the construction in [BGHH20]. This is Proposition 4.1. The argument uses bootstrapping as in Proposition 2.3, and applies that proposition iteratively.
(2) Relate the lower tail event of X_r to the event of the existence of k disjoint paths constructed in [BGHH20] (Theorem 1.10 here).

We will implement these two steps in turn next, and then, in Section 4.2, we provide an overview of the main ideas of the construction from [BGHH20] that we are invoking. We start by specifying some notation for constrained weights.

Recall the notation for parallelograms introduced in Section 1.8.4, where $U = U_{r,\ell,z}$ is a parallelogram of height r, width $\ell r^{2/3}$, and opposite side midpoints $(1,1)$ and $(r-z, r+z)$. Recall also that X_r^U is the maximum weight over all paths from $(1,1)$ to $(r-z, r+z)$ which are constrained to stay in U.

Proposition 1.11 provides a stretched exponential lower tail for X_r^U from our assumptions. The following upgraded tail obtained via bootstrapping will suffice for our purpose; note that the bound is still not the optimal one stated in Theorem 5, which we prove later.

Proposition 4.1. *Let L_1, L_2, and K be such that $L_1 < \ell < L_2$ and $|z| \leq Kr^{2/3}$. Under Assumptions 2 and 3, there exist constants r_0, θ_0, and $c > 0$, all depending on only L_1, L_2, K, and α, such that, for $r > r_0$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(X_r^U \leq \mu r - \theta r^{1/3}\right) \leq \exp(-c\theta^{3/2}).$$

This is the first step outlined above. The proof is similar to that outlined at the beginning of this section for X_r, and involves using bootstrapping for a number of iterations, with the exponent increasing by the end of each iteration to $3/2$ times its value at the start of it. Once the exponent passes 1, a final iteration brings it to $3/2$.

Proof of Proposition 4.1. Consider the k subparallelograms U_i, $i \in [\![1,k]\!]$, where U_i is defined as the parallelogram with height r/k, width $\min(\ell r^{2/3}, (r/k)^{2/3})$, and opposite side midpoints $(r-z, r+z)\cdot(i-1)/k + (1,0)$ and $(r-z,r+z)\cdot i/k$. Let $Y_r = -(X_r^U - \mu r)$ and $Y_{r,i}^{(k)} = -(X_{r/k}^{U_i} - \mu r/k)$. We want to apply Proposition 2.3 to these variables. By the definition of X_r^U and $X_{r/k}^{U_i}$, we have that $Y_r \leq \sum_{i=1}^k Y_{r,i}^{(k)}$ for all $k \leq r$. The variables $\{Y_{r,i}^{(k)} : i \in [\![1,k]\!]\}$ are independent for each k as they are defined by the randomness in disjoint parts of the environment, and (7) of Proposition 1.11 provides a stretched exponential tail (of exponent $\alpha' = 2\alpha/3$) for each $Y_{r,i}^{(k)}$. Since the constants c, θ_0, and r_0 of Proposition 1.11 depend on only K, L_1, L_2, we obtain from Proposition 2.3 that there exist $\tilde{c} = \tilde{c}(K, L_1, L_2)$, $\tilde{r}_0$, and $\tilde{\theta}_0$ such that, for $r > \tilde{r}_0$ and $\tilde{\theta}_0 < \theta < r^{2/3}$,

$$\mathbb{P}\left(X_r^U \leq \mu r - \theta r^{1/3}\right) \leq \exp(-\tilde{c}\theta^{3\alpha'/2}).$$

Note that if $\mu > 1$, the constraint $\theta < r^{2/3}$ can be extended to $\theta < \mu r^{2/3}$ by reducing the constant c if needed, in a way that depends on only α and μ. Beyond $\mu r^{2/3}$, the probability on the left side of the last display is zero since the vertex weights are non-negative, and so the last displayed inequality actually holds for all $\theta > \tilde{\theta}_0$.

We may iterate the above argument, such that at the end of each iteration the tail exponent is $3/2$ times its value at the beginning, till the tail exponent exceeds 1. Then we may apply the above argument one last time with $\alpha' = 1$, and this completes the proof. Since the finite number of iterations is only a function of α, the proposition follows. □

We may now formally prove Theorem 1.10 under the weaker point-to-point tail assumptions of this paper (as compared to [BGHH20]) by detailing which statements of the latter paper need to be replaced by statements proved in this paper; an overview of the construction given in [BGHH20] will be discussed shortly in Section 4.2.

Proof of Theorem 1.10. As mentioned, [BGHH20, Theorem 3.1] has three inputs: (i) parabolic curvature of the weight profile; (ii) an exponential upper bound on the constrained weight lower tail; and (iii) a lower bound on the expected constrained weight. [BGHH20, Assumption 2] provides (i), while (ii) and (iii) are provided by [BGHH20, Proposition 3.7].

In this paper, Assumption 2 implies [BGHH20, Assumption 2] and provides (i). The item (ii) is provided by Proposition 4.1, and (iii) by (8) of Proposition 1.11. The proof of [BGHH20, Theorem 3.1] applies verbatim after making these replacements. □

Next we prove Theorem 3 using Theorem 1.10.

Proof of Theorem 3. Since $\mathbb{E}[X_r] \le \mu r$ by Assumption 2 (this is also implied directly by the super-additivity of $\{X_r\}_{r\in\mathbb{N}}$), it is sufficient to upper bound the probability $\mathbb{P}\left(X_r \le \mu r - \theta r^{1/3}\right)$. Let η be as given in Theorem 1.10, and denote the event whose probability is lower bounded there by $E_{m,k,r}$, i.e., $E_{m,k,r}$ is the event that there exist m disjoint paths $\gamma_1, \ldots, \gamma_m$ with prescribed endpoints, $\max_i \mathrm{TF}(\gamma_i) \le 2mk^{-2/3}r^{1/3}$, and $\sum_{i=1}^m \ell(\gamma_i) \ge \mu rm - C_1 mk^{2/3}r^{1/3}$. Observe that any of these paths γ_i can be extended to a path from $(1,1)$ to (r,r) without decreasing its weight. Now for $\theta \le C_1\eta^{2/3}r^{2/3}$, set $m = k = C_1^{-3/2}\theta^{3/2}$, and observe that $C_1 mk^{2/3} = m\theta$. Thus,

$$\mathbb{P}\left(X_r \le \mu r - \theta r^{1/3}\right) \le \mathbb{P}\left(E_{m,k,r}^c\right) \le \exp(-cmk) = \exp(-c\theta^3),$$

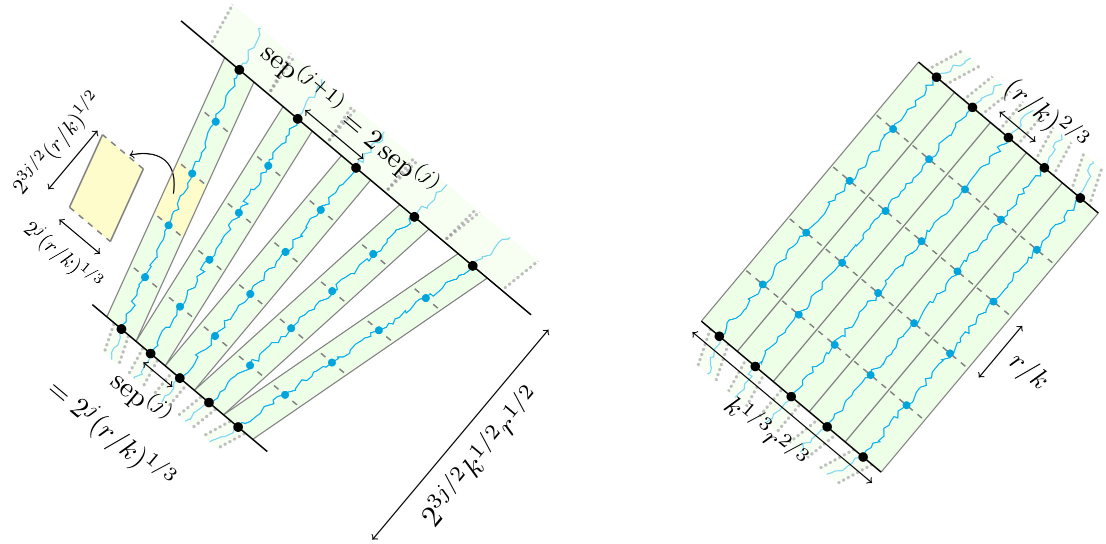
the second inequality by Theorem 1.10 since the value of k satisfies $k \le \eta r$; this latter inequality is implied by the condition that $\theta \le C_1\eta^{2/3}r^{2/3}$. The inequality for $\theta \in [C_1\eta^{2/3}r^{2/3}, \mu r^{2/3}]$ can be handled be reducing the value of c (if $C_1\eta^{2/3} < \mu$), and for $\theta > \mu r^{2/3}$, the probability being bounded is trivially zero. This completes the proof of Theorem 3. □

We next present a brief outline of the construction from [BGHH20] before going into the proof of Theorem 5, which then also implies Theorem 4.

4.2. **An overview of the proof of Theorem 1.10.** Here we give a brief overview of the high-probability construction that proves Theorem 1.10, with the help of Figure 6. A detailed description and proof appears in [BGHH20, Section 8].

Recall that we have to construct m disjoint paths, each with weight loss at most of order $k^{2/3}r^{1/3}$, in a strip of width $4mk^{-2/3}r^{2/3}$. In the bulk of the environment, this is straightforward: for each curve, we set up order k many parallelograms of width $(r/k)^{2/3}$ and height r/k sequentially and consider the path obtained by concatenating together the heaviest midpoint-to-midpoint path constrained to remain in the corresponding parallelogram; see Figure 6b. The weight loss in each parallelogram is on scale, i.e., of order $(r/k)^{1/3}$, and so the total loss across the m curves is $m\cdot k\cdot(r/k)^{1/3} = mk^{2/3}r^{1/3}$. The total transversal fluctuation is of order $m(r/k)^{2/3}$, as required, and it is in this phase of the construction that the transversal fluctuation is maximum.

But the previous description is only possible in the bulk, and if the curves have already been brought to a separation of $(r/k)^{2/3}$. Since the curves start and end at a microscopic separation of 1 at the corners of $[\![1, r]\!]^2$, the difficult part of the construction is there, where the curves must be coaxed apart while not sacrificing too much weight. Here the construction proceeds in a dyadic fashion, doubling the separation between curves as the scale increases, while ensuring that the



(A) One scale of construction near corners

(B) Bulk phase of construction

FIGURE 6. Panel A is a depiction of one scale (indexed by j) of the construction near the corners of $[\![1, r]\!]^2$ when $m = k = 5$. The paths which form the construction are in blue, and the separation on the j^{th} scale is denoted $\mathrm{sep}^{(j)}$, which is also the width of the green parallelograms that the paths are constrained to pass through. Note that each individual green parallelogram has on-scale dimensions. Depicted in lower opacity is how the construction continues on the succeeding (larger) and preceding (smaller) scale. In panel B is the bulk phase of the construction, where the separation between curves is maintained for a distance of order r; thus there are order k cells in a single column. Also depicted in lower opacity on either side are the largest scales of the second phase.

antidiagonal displacement borne by the curves is not too high, so as to not incur a high weight loss due to parabolic curvature. Again the idea is to construct a sequence of parallelograms for each curve that it is constrained to remain within; see Figure 6a. It is to estimate the weight loss in this phase of the construction, where antidiagonal displacement is increasing, that we require a curvature assumption such as Assumption 2, and a calculation shows that the weight loss is again of order $mk^{2/3}r^{1/3}$.

The other two inputs, namely a lower bound on the means of constrained weights and exponential decay of the lower tails of the same, are needed to control the probability that the paths constructed by concatenating together these constrained paths have the requisite weight; in particular, the mean bound is used to show that the expected weight is correct, while the lower tail bound is used to control the deviation below the mean of the total construction weight after expressing it as a sum of independent subexponential variables and invoking a concentration inequality.

4.3. **Bounds on constrained lower tail.** In this section we will prove Theorem 5; this will also imply Theorem 4. We start with the short proof of the upper bound of Theorem 5 which is a straightforward consequence of Theorem 1.10 and is a refinement of the argument for Theorem 3.

Proof of upper bound of Theorem 5. We prove a stronger bound with μr in place of $\mathbb{E}[X_r^U]$ (since $\mathbb{E}[X_r^U] \le \mu r$).

Recall that the width of U is $\ell r^{2/3}$. On the event that $X_r^U \le \mu r - \theta r^{1/3}$, it follows that any m disjoint paths which lie inside U must have total weight at most $\mu m r - m\theta r^{1/3}$. But for any m

which satisfies $2mk^{-2/3}r^{2/3} \leq \ell r^{2/3}$, m disjoint paths which lie inside U with total weight at least $\mu mr - m\theta r^{1/3}$ are provided by Theorem 1.10 by setting $k = C_1^{-3/2}\theta^{3/2}$, with probability at least $1 - \exp(-cmk) = 1 - \exp(-cm\theta^{3/2})$.

Thus for any m and ℓ satisfying $m \leq k$ and $m \leq \frac{1}{2}\ell k^{2/3}$ we have

$$\mathbb{P}\left(X_r^U \leq \mu r - \theta r^{1/3}\right) \leq \exp(-cm\theta^{3/2}).$$

Taking $m = \min\left(\frac{1}{2}\ell k^{2/3}, k\right)$ with k as above completes the proof if we verify that $m \geq 1$. This follows by setting $C = 2C_1$ in the assumed lower bound $\ell \geq C\theta^{-1}$ and setting $\theta_0 > C_1$: the bound on ℓ and the value of k implies that $\frac{1}{2}\ell k^{2/3} \geq 1$, and the bound on θ and the value of k implies that $k \geq 1$. ☐

The rest of this section is devoted to assembling the tools to prove, and then proving, the lower bound on the lower tail of Theorem 5. To start with, we need a constant lower bound on the lower tail of the point-to-point weight for a range of directions. This is a straightforward consequence of the assumed mean behavior in Assumption 2 and the lower tail bound in Assumptions 3b, and its proof is deferred to the appendix. In fact, we only need $X_r^z < \mu r - Cr^{1/3}$ with positive probability in our application, but we prove a stronger statement with a parabolic loss.

Lemma 4.2. *Let ρ be as given in Assumption 2. Under Assumptions 3b and 2, there exist $C > 0$ and $\delta > 0$ such that, for $r > r_0$ and $|z| \leq \rho r$,*

$$\mathbb{P}\left(X_r^z < \mu r - \frac{Gz^2}{r} - Cr^{1/3}\right) \geq \delta.$$

The argument for the lower bound of Theorem 5, however, will require moving from the above lower bound on the point-to-point lower tail to a similar lower bound on the interval-to-line lower tail. Note that although we had previously encountered interval-to-interval weights, this is the first time in our arguments that we are seeking to bound interval-to-line weights. This is the content of the next lemma. The proof will entail a few steps which we will describe soon. For the precise statement recall that for two sets of vertices A and B in $\mathbb{Z}^2$, $X_{A,B}$ is the maximum weight of all up-right paths starting in A and ending in B.

Lemma 4.3. *Let $I \subseteq \mathbb{Z}^2$ be the interval of lattice points connecting the coordinates $(-r^{2/3}, r^{2/3})$ and $(r^{2/3}, -r^{2/3})$ on the line $x + y = 0$, and let $\mathbb{L}_r \subseteq \mathbb{Z}^2$ be the lattice points on the line $x + y = 2r$. Under Assumptions 3b and 4b, there exist C', $\delta' > 0$, and r_0' such that, for $r > r_0'$,*

$$\mathbb{P}\left(X_{I,\mathbb{L}_r} \leq \mu r - C' r^{1/3}\right) \geq \delta'.$$

Before turning to the proof of Lemma 4.3, we finish the proof of the lower bound of Theorem 5 and hence also the proof of Theorem 4.

Proof of Theorem 4 and lower bound of Theorem 5. Recall the lower bound statement of Theorem 5 that, for $\theta_0 \leq \theta \leq \eta r^{2/3}$,

$$\mathbb{P}\left(X_r^U - \mu r \leq -\theta r^{1/3})\right) \geq \exp\left(-c_1 \min(\ell\theta^{5/2}, \theta^3)\right);$$

note that Theorem 4 is implied by the case that $\ell = r^{1/3}$, since by choosing $\delta < 1$ in the statement of the latter, we assume $\theta \leq r^{2/3}$, and hence $\min(\ell\theta^{5/2}, \theta^3) = \theta^3$. We now proceed to proving the bound for general ℓ.

Let k and m be positive integers whose values will be specified shortly. We will define a grid similar to the one in Section 3.1.1 that was depicted in Figure 4, but of width $mk^{-2/3}r^{2/3}$. For $i \in [\![1, k]\!]$

and $j \in [\![1, m]\!]$, let $v_{i,j}$ be the point

$$\left(i\frac{r}{k} - \frac{m}{2}\left(\frac{r}{k}\right)^{2/3},\ i\frac{r}{k} + \frac{m}{2}\left(\frac{r}{k}\right)^{2/3}\right) + j\left(\left(\frac{r}{k}\right)^{2/3},\ -\left(\frac{r}{k}\right)^{2/3}\right),$$

and let $I_{i,j}$ be the interval with endpoints $v_{i,j}$ and $v_{i,j+1}$ (of width $\left(\frac{r}{k}\right)^{2/3}$). As in Section 3.1.1, we will collectively refer to these intervals as a grid, and so the rows of the grid are indexed by i and the columns by j. Note that the grid lies inside U if $m \le \ell k^{2/3}$ and covers the breadth of U if $m = \ell k^{2/3}$, since the total breadth of the grid is $mk^{-2/3}r^{2/3}$.

This is what dictates the choice of m, although for technical reasons, we set

$$m = \min(\ell k^{2/3}, k), \tag{28}$$

where our choice for k later (of order $\theta^{3/2}$) will ensure that indeed for all interesting values of ℓ (i.e., $\ell = O(\theta^{1/2})$), we would have $m = \ell k^{2/3}$.

The idea now is to construct an event on which $X_r^U \le \mu r - \theta r^{1/3}$. Let $X_{I,\mathbb{L}}^{i,j}$ be the maximum weight among all paths with starting point on $I_{i,j}$ and ending point on the line $x + y = 2(i+1)\frac{r}{k}$. The event will be defined by forcing

(1) all the $X_{I,\mathbb{L}}^{i,j}$ to be small, i.e., $X_{I,\mathbb{L}}^{i,j} \le \mu(r/k) - C'(r/k)^{1/3}$ for a constant C'; and
(2) any path which has transversal fluctuation greater than $k^{1/3}r^{2/3}$ to suffer a parabolic weight loss of order $k^{2/3}r^{1/3}$.

Before proceeding, we let Y_r^k be the maximum weight among all paths Γ from $(1,1)$ to (r,r) with transversal fluctuation satisfying $\mathrm{TF}(\Gamma) \ge k^{1/3}r^{2/3}$. Thus the second condition above says Y_r^k falls below μr by at least order $k^{2/3}r^{1/3}$.

We claim that, on the event described, $X_r^U \le \mu r - \Omega(k^{2/3}r^{1/3})$. This is due to the following. First, any path within the grid must pass through one of $\{I_{i,j} : j \in [\![1, m]\!]\}$ for every $i \in [\![1, k]\!]$ and so has weight at most $k \cdot \max_{i,j} X_{I,\mathbb{L}}^{i,j} \le \mu r - C'k^{2/3}r^{1/3}$. Second, any path which exits the grid, by our choice of m, either exits U and may be ignored or has transversal fluctuation greater than $k^{1/3}r^{2/3}$ and so suffers a weight loss of at least order $k^{2/3}r^{1/3}$.

Finally, we will show that this event has probability at least $\exp(-cmk)$ (since there are mk values of (i,j) for which $X_{I,\mathbb{L}}^{i,j}$ is made small) and set k to be a multiple of $\theta^{3/2}$.

A more precise form of the above discussion starts with the following inclusion, where c_2 is as in Theorem 1.8 and C' is as in Lemma 4.3:

$$\begin{aligned}&\left\{X_r^U \le \mu r - \min(c_2, C')k^{2/3}r^{1/3}\right\}\\ &\qquad\supseteq \left\{Y_r^k \le \mu r - c_2 k^{2/3}r^{1/3}\right\} \cap \bigcap_{\substack{1\le i\le k\\ 1\le j\le m}} \left\{X_{I,\mathbb{L}}^{i,j} \le \mu r/k - C'(r/k)^{1/3}\right\}.\end{aligned}$$

Note that all the events on the right hand side are decreasing events. Hence, by the FKG inequality,

$$\begin{aligned}\mathbb{P}\left(X_r^U \le \mu r - \min(c_2, C')k^{2/3}r^{1/3}\right) &\ge \mathbb{P}\left(Y_r^k \le \mu r - c_2 k^{2/3}r^{1/3}\right)\\ &\qquad \times \prod_{\substack{1\le i\le k\\ 1\le j\le m}} \mathbb{P}\left(X_{I,\mathbb{L}}^{i,j} \le \mu r/k - C'(r/k)^{1/3}\right)\\ &\ge (1 - \exp(-\tilde{c}k^{2\alpha/3})) \cdot (\delta')^{mk}.\end{aligned} \tag{29}$$

The final inequality was obtained by applying Theorem 1.8 with $s = k^{1/3}$ and $t = 0$ to lower bound the first term and Lemma 4.3 (with r/k in place of r) to lower bound the remaining terms. Theorem 1.8 provides an absolute constant s_0 and its application requires $k^{1/3} > s_0$, a condition that will translate into a lower bound on θ after we set the value of k next; Lemma 4.4 requires that $r/k > r_0$, which will translate to an upper bound on θ.

Take $k = (\min(c_2, C'))^{-3/2}\,\theta^{3/2}$ and recall the value of m from (28). Note that the assumed lower bound of $\ell \geq C\theta^{-1}$ ensures that $m \geq 1$; we additionally impose that $k \geq s_0^3$ to meet the requirement of Theorem 1.8 mentioned above. We also assume without loss of generality that $s_0 \geq 1$ to encode that k must be at least 1. Thus we obtain from (29), for some constant $c_1 > 0$,

$$\mathbb{P}\left(X_r^U \leq \mu r - \theta r^{1/3}\right) \geq \exp\left(-c_1 \min(\ell\theta^{5/2}, \theta^3)\right); \tag{30}$$

this holds for every θ which is consistent with $s_0^3 \leq k \leq r_0^{-1}r$, the latter inequality to ensure that r/k is at least r_0 as obtained from Lemma 4.3. Recalling the value of k, this condition on θ may be written as

$$s_0^2 \cdot \min(c_2, C') \leq \theta \leq \min(c_2, C') r_0^{-2/3} \cdot r^{2/3}.$$

Recall that Theorem 5 must be proven only for $\theta_0 \leq \theta \leq \eta r^{2/3}$. Thus we may meet the condition of the last display by modifying θ_0 to be greater than $s_0^2 \min(c_2, C')$ and η to be less than $\min(c_2, C') r_0^{-2/3}$, if required.

Now we turn to the final statement in Theorem 5 on replacing μr by $\mathbb{E}[X_r^U]$ in (30) when ℓ is bounded below by a constant ε. For this, all we require is that $\mathbb{E}[X_r^U] \geq \mu r - Cr^{1/3}$ for $r > r_0$ for a C and r_0 which may depend on ε. This is because with that bound we may absorb the $Cr^{1/3}$ into $\theta r^{1/3}$ by reducing the constant c_1 (which will then depend on ε). Now the required bound on $\mathbb{E}[X_r^U]$ is provided by (8) of Proposition 1.11.

This completes the proof of Theorem 4 and the lower bound of Theorem 5 □

The remaining task is to prove Lemma 4.3, which lower bounds the lower tail for interval-to-line weights. Recall from Figure 5 our strategy of obtaining bounds on interval-to-interval weights from similar bounds on point-to-point weights by backing up. Notice that we are presently seeking to lower bound the probability that interval-to-line weights are low; in other words, expressing the line as a union of disjoint intervals, we want to lower bound the probability of the intersection of the decreasing events that all the corresponding interval-to-interval weights are low. This will involve an application of the FKG inequality and the following two lemmas, which lower bound the lower tail of interval-to-interval weights. The first one (Lemma 4.4) treats the case when the anti-diagonal displacement between the two intervals is small. The second one (Lemma 4.5) handles intervals at greater anti-diagonal separation, exploiting the natural parabolic loss in the mean which makes the weights unlikely to be high in this case.

Lemma 4.4. *Let I be the interval connecting the points $(-r^{2/3}, r^{2/3})$ and $(r^{2/3}, -r^{2/3})$, J be the interval connecting the points $(r - z - r^{2/3}, r + z + r^{2/3})$ and $(r - z + r^{2/3}, r + z - r^{2/3})$, and ρ be as in Assumption 2. Under Assumptions 3b, 4b, and 2, there exist C'', $\delta'' > 0$, and r_0'' such that, for $r > r_0''$ and $|z| \leq \rho r - 2r^{2/3}$ (this quantifies the closeness of the intervals in the anti-diagonal direction),*

$$\mathbb{P}\left(X_{I,J} < \mu r - C'' r^{1/3}\right) \geq \delta''.$$

Proof. We first prove a similar statement for intervals of size $\varepsilon r^{2/3}$ for an $\varepsilon > 0$ to be fixed later. This is a crucial first step as it is difficult to directly control the possible gain in weight afforded by allowing the endpoints to vary over an interval of size $2r^{2/3}$; initially using the leeway of making the interval sufficiently small (but on the scale $r^{2/3}$) makes this control achievable. This will be done using a backing up strategy similar to the one illustrated in Figure 5 for the interval-to-interval upper tail bound Lemma 3.5.

Let $|w| \le \rho r$, and let I_ε be the interval joining the points $(-\varepsilon r^{2/3}, \varepsilon r^{2/3})$ and $(\varepsilon r^{2/3}, -\varepsilon r^{2/3})$ and J_ε be the interval joining the points $(r - w - \varepsilon r^{2/3}, r + w + \varepsilon r^{2/3})$ and $(r - w + \varepsilon r^{2/3}, r + w - \varepsilon r^{2/3})$. We will prove that there exist positive C'', δ, and ε, independent of w, such that

$$\mathbb{P}\left(X_{I_\varepsilon, J_\varepsilon} < \mu r - C'' r^{1/3}\right) \ge \delta/2. \tag{31}$$

Let $u^* \in I_\varepsilon$ and $v^* \in J_\varepsilon$ be such that $X_{I_\varepsilon, J_\varepsilon} = X_{u^*, v^*}$. Also let $\phi_1 = (-\varepsilon^{3/2} r, -\varepsilon^{3/2} r)$, $\phi_2 = (r - w + \varepsilon^{3/2} r, r + w + \varepsilon^{3/2} r)$ be the backed up points. Then we have the inequality

$$X_{\phi_1, \phi_2} \ge X_{\phi_1, u^* - (1,0)} + X_{I_\varepsilon, J_\varepsilon} + X_{v^* + (1,0), \phi_2}.$$

For M to be fixed and C as in Lemma 4.2, we will consider the constant probability events

$$\begin{aligned} E_{p\to p} &:= \left\{X_{\phi_1,\phi_2} \le \mu(1 + 2\varepsilon^{3/2}) r - G\frac{w^2}{(1 + 2\varepsilon^{3/2} r)} - C r^{1/3}\right\}, \\ E_{p\to \mathrm{int}} &:= \left\{X_{\phi_1, u^* - (0,1)} > \mu \varepsilon^{3/2} r - M\varepsilon^{1/2} r^{1/3}\right\}, \qquad \text{and} \\ E_{\mathrm{int}\to p} &:= \left\{X_{v^* + (1,0), \phi_2} > \mu \varepsilon^{3/2} r - M\varepsilon^{1/2} r^{1/3}\right\}. \end{aligned}$$

On the intersection $E_{p\to p} \cap E_{p\to\mathrm{int}} \cap E_{\mathrm{int}\to p}$ we have

$$X_{I_\varepsilon, J_\varepsilon} < \mu r - (C - 2M\varepsilon^{1/2}) r^{1/3}. \tag{32}$$

We must lower bound the probability of this intersection. From Lemma 4.2 we see

$$\mathbb{P}\left(E_{p\to p}\right) \ge \delta,$$

since $C < (1 + \varepsilon^{3/2})^{1/3} C$. Next, recall that u^* is a vertex of I_ε, which lies on the line $x + y = 0$, which is the starting point of a heaviest path from I_ε to J_ε. Thus u^* is independent of the random field below the line $x + y = 0$. Now we see that, for large enough M (depending only on δ),

$$\begin{aligned} \mathbb{P}\left(E^c_{p\to\mathrm{int}}\right) &= \mathbb{P}\left(X_{\phi_1, u^* - (0,1)} \le \mu\varepsilon^{3/2} r - M\varepsilon^{1/2} r^{1/3}\right) \\ &\le \sup_{u \in I_\varepsilon} \mathbb{P}\left(X_{\phi_1, u - (0,1)} \le \mu\varepsilon^{3/2} r - M\varepsilon^{1/2} r^{1/3}\right) \le \frac{\delta}{4}, \end{aligned}$$

with the mentioned independence allowing the uniform bound of the second line. The same bound with the same M holds for $\mathbb{P}(E^c_{\mathrm{int}\to p})$ as well; we fix this M. Now we set $\varepsilon > 0$ such that $C - 2M\varepsilon^{1/2} = C/2$. From (32) the above yields (31) with $C'' = C/2$.

To move from $I_\varepsilon, J_\varepsilon$ to I, J, we let $I_{\varepsilon,i}$ for $i \in [\![1, \varepsilon^{-1}]\!]$ be the intervals of length ε which make up the length one interval I in the obvious way, and similarly for $J_{\varepsilon,j}$ and J. Next observe that

$$\left\{X_{I,J} < \mu r - \tfrac{1}{2} C r^{1/3}\right\} \supseteq \bigcap_{i,j} \left\{X_{I_{\varepsilon,i}, J_{\varepsilon,j}} < \mu r - \tfrac{1}{2} C r^{1/3}\right\}. \tag{33}$$

Now, the bound (31) holds as long as the intervals are of length $\varepsilon r^{2/3}$ and their antidiagonal displacement is at most ρr. The intervals $I_{\varepsilon,i}$ and $J_{\varepsilon,i}$ have this length, and their antidiagonal displacement is at most $|z| + 2r^{2/3}$, where recall z is the antidiagonal displacement between I and J. This occurs, for example, $I_{\varepsilon,i}$ is the left most subinterval of I and $J_{\epsilon,j}$ is the right most sub interval of J. But since we have assumed $|z| + 2r^{2/3} \le \rho r$, the bound (31) applies to $X_{I_{\varepsilon,i}, J_{\varepsilon,j}}$, and so the probability of each event in the intersection of (33) is at least $\delta/2$.

The intersection of (33) is of decreasing events, and so we may invoke the FKG inequality and the just noted probability lower bound to conclude that the probability of the right hand side of (33) is at least $(\delta/2)^{\varepsilon^{-2}}$. This completes the proof of Lemma 4.4 with $C'' = C/2$ and $\delta'' = (\delta/2)^{\varepsilon^{-2}}$. $\square$

While the previous lemma provided control when the destination interval is relatively close to (r, r), i.e., have x- and y-coordinates within ρr of r, the next lemma will be used to treat pairs of intervals which have greater separation. Let $I \subseteq \mathbb{Z}^2$ be the interval connecting $(-r^{2/3}, r^{2/3})$ and $(r^{2/3}, r^{2/3})$, and $J \subseteq \mathbb{Z}^2$ be the interval connecting $(r - w - r^{2/3}, r + w + r^{2/3})$ and $(r - w + r^{2/3}, r + w - r^{2/3})$; thus w represents the intervals' antidiagonal displacement, while z will be used as a variable in the hypothesis.

Lemma 4.5. *Suppose there exist $\alpha \in (0, 1]$, $\lambda > 0$ and constants $c > 0$, t_0, and r_0 such that, for $t > t_0$, $r > r_0$, and $|z| \leq r/2$*

$$\mathbb{P}\left(X_r^z \geq \mu r - \lambda \frac{Gz^2}{r} + tr^{1/3}\right) \leq \exp\left(-ct^\alpha\right).$$

Then there exist $\tilde{c} > 0$, $\tilde{t}_0 = \tilde{t}_0(t_0)$, and $\tilde{r}_0 = \tilde{r}_0(r_0)$ such that, for $r > \tilde{r}_0$, $|w| \leq |r|$, and $t > \tilde{t}_0$,

$$\mathbb{P}\left(X_{I,J} > \mu r - \frac{\lambda}{3} \cdot \frac{Gw^2}{r} + tr^{1/3}\right) \leq \exp\left(-\tilde{c}t^\alpha\right).$$

We impose the condition that $|z| \leq r/2$ because the tail bound hypothesis will be provided by Assumption 3a in our application, and the latter requires $|z|$ to be macroscopically away from 0 and r. But note that we allow $|w|$ to be as large as r, as we do need to allow the destination interval to be placed anywhere between the coordinate axes.

Lemma 4.5 will be proved along with Lemma 3.5, as they are both interval-to-interval upper tails, in the appendix via a backing up argument. With the bounds of the previous two lemmas, we can prove the interval-to-line bound in Lemma 4.3 and thus complete the proof of Theorem 5.

As earlier, we will construct an event as an intersection of decreasing events which forces $X_{I,\mathbb{L}_r}$ to be small, and use the FKG inequality to lower bound its probability. So, we need to make any path starting in I and ending on the line $\mathbb{L}_r$ have low weight, which we will do by forcing such paths which end on various intervals on $\mathbb{L}_r$ to separately have low weight. When the destination interval is close to the point (r, r), the probability that all such paths have low weight will be lower bounded by Lemma 4.4. When the destination interval is far from (r, r), Lemma 4.5 says that it is highly likely that the paths will have low weight, and it is a matter of checking that the probabilities approach 1 quickly enough that their product is lower bounded by a positive constant.

Proof of Lemma 4.3. For $j \in [\![-r^{1/3}, r^{1/3}]\!]$, let J_j be the interval connecting the points $(r - jr^{2/3}, r + jr^{2/3})$ and $(r - (j + 1)r^{2/3}, r + (j + 1)r^{2/3})$. We observe that, with C'' as in Lemma 4.4,

$$\left\{X_{I,\mathbb{L}_r} < \mu r - C'' r^{1/3}\right\} \supseteq \bigcap_{|j| \leq r^{1/3}} \left\{X_{I,J_j} < \mu r - C'' r^{1/3}\right\}. \tag{34}$$

Assumption 2 says that, for $|z| \leq \rho r$, $\mathbb{E}[X_r^z] \leq \mu r - Gz^2/r - g_1 r^{1/3}$. But then observe that the concavity of the limit shape implies the existence of a small constant $\lambda \in (0, 1)$ such that

$$\mathbb{E}[X_r^z] \leq \mu r - \lambda \frac{Gz^2}{r} \tag{35}$$

for $|z| \leq r$. For this value of λ, Assumption 3a, with $\varepsilon = 1/2$, implies that there exists $c > 0$ such that, for all $r > r_0$ and $|z| \leq r/2$,

$$\mathbb{P}\left(X_r^z > \mu r - \lambda \frac{Gz^2}{r} + tr^{1/3}\right) \leq \exp(-c\theta^\alpha).$$

With this value of λ and the above bound as input, we will apply Lemma 4.5 with $t = \lambda G j^2/3 - C$. Lemma 4.5 requires $t > \tilde{t}_0$, which implies that $|j|$ must be larger than some j_0. So for $|j| > j_0$, Lemma 4.5 implies that

$$\mathbb{P}\left(X_{I,J_j} < \mu r - C'' r^{1/3}\right) \geq 1 - \exp(-\tilde{c}|j|^{2\alpha}). \tag{36}$$

Applying (36) and the bound from Lemma 4.4 to (34), along with the FKG inequality, gives

$$\begin{aligned}\mathbb{P}\left(X_{I,\mathbb{L}_r} < \mu r - C'' r^{1/3}\right) &\geq \prod_{|j|\leq j_0} \mathbb{P}\left(X_{I,J_j} < \mu r - C'' r^{1/3}\right) \times \prod_{j_0<|j|\leq r^{1/3}} \mathbb{P}\left(X_{I,J_j} < \mu r - C'' r^{1/3}\right)\\ &\geq (\delta'')^{2j_0+1} \times \prod_{j_0\leq|j|\leq r^{1/3}} \left(1-\exp(-\tilde{c}|j|^{\alpha})\right);\end{aligned}$$

we employed Lemma 4.4 for the factors in the first product and (36) for those in the second. The proof of Lemma 4.3 is complete by taking $C' = C''$ and δ' equal to the final line of the last display; note that the second product in the last display is bounded below by a positive constant independent of r since $\exp(-\tilde{c}|j|^{\alpha})$ is summable over j for any $\alpha > 0$, and so $\delta' > 0$. ☐

Appendix A. Transversal fluctuation, interval-to-interval, and crude lower tail bounds

In this appendix we explain how to obtain the first, second, and fourth tools of Section 1.8, i.e., Theorem 1.8 and Propositions 1.9 and 1.11, and provide the outstanding proofs of Lemmas 3.5, 4.5, and 4.2 from the main text. The third tool was already explained in Section 4.

The proofs of the first and fourth tools follow verbatim from corresponding results in [BGHH20] by replacing the upper and lower tails used there with Assumption 3; the parabolic curvature assumption there is provided by our Assumption 2. In particular, Theorem 1.8 follows from [BGHH20, Theorem 3.3] and Proposition 1.11 from [BGHH20, Proposition 3.7].

The proof of the second tool, Proposition 1.9, will be addressed in Section A.1, after we next provide the outstanding proofs of Lemmas 3.5, 4.5, and 4.2 from the main text.

We start with the proofs of Lemmas 3.5 and 4.5 on an upper tail bound of the interval-to-interval weight; this largely follows the proof of [BGHH20, Proposition 3.5]. The strategy is to back up from the intervals appropriately and consider a point-to-point weight for which we have tail bounds by hypothesis; this strategy was illustrated in Figure 5 and in the proof of Lemma 4.4.

Proofs of Lemmas 3.5 and 4.5. We prove Lemma 3.5 and indicate at the end the modifications for Lemma 4.5. We set $\lambda = \lambda_j$ and $\lambda' = \lambda_{j+1}$ to avoid confusion later when we describe the modifications for Lemma 4.5. Note that $\lambda' < \lambda$.

By considering the event that Z is large and two events defined in terms of the environment outside of U, we find a point-to-point path which has large length. To define these events, first define points ϕ_{low} and ϕ_{up} on either side of the lower and upper intervals as follows, where $\delta = \frac{1}{2}\left(\frac{\lambda}{\lambda'} - 1\right) > 0$:

$$\begin{aligned}\phi_{\text{low}} &:= (-\delta r, -\delta r)\\ \phi_{\text{up}} &:= ((1+\delta)r - w, (1+\delta)r + w)\,.\end{aligned}$$

Let u^* and v^* be the points on A and B where the suprema in the definition of Z are attained, and let the events E_{low} and E_{up} be defined as

$$E_{\text{low}} = \left\{X_{\phi_{\text{low}},u^*-(1,0)} > \mu\delta r - \frac{t}{3} r^{1/3}\right\} \quad \text{and} \quad E_{\text{up}} = \left\{X_{v^*+(1,0),\phi_{\text{up}}} > \mu\delta r - \frac{t}{3} r^{1/3}\right\}.$$

Let $\tilde{r} = \frac{\lambda}{\lambda'} r = (1+2\delta)r$, and observe that the diagonal distance of ϕ_{low} and ϕ_{up} is $\tilde{r}$. Also note

$$X_{\phi_{\text{low}},\phi_{\text{up}}} \geq X_{\phi_{\text{low}},u*-(1,0)} + Z + X_{v^*+(1,0),\phi_{\text{up}}}.$$

Then we have the following:

$$\begin{aligned}\mathbb{P}\left(Z > \mu r - \lambda'\frac{Gw^2}{r} + tr^{1/3}, E_{\text{low}}, E_{\text{up}}\right) &\leq \mathbb{P}\left(X_{\phi_{\text{low}},\phi_{\text{up}}} \geq \mu(1+2\delta)r - \lambda'\frac{Gw^2}{r} + \frac{t}{3}r^{1/3}\right) \qquad (37)\\ &= \mathbb{P}\left(X_{\phi_{\text{low}},\phi_{\text{up}}} \geq \mu\tilde{r} - \lambda\frac{Gw^2}{\tilde{r}} + \frac{t}{3}\cdot\left(\frac{\lambda'}{\lambda}\right)^{1/3}\cdot(\tilde{r})^{1/3}\right)\\ &\leq \begin{cases}\exp(-\tilde{c}t^{\beta}) & t_0 < t < r^{\zeta}\\ \exp(-\tilde{c}t^{\alpha}) & t \geq r^{\zeta}.\end{cases}\end{aligned}$$

The final inequality uses the hypothesis (15) on the point-to-point tail, which is applicable since the antidiagonal separation of ϕ_{low} and ϕ_{up} is w while the diagonal separation is $(1+2\delta)r$, and clearly $|w| \leq r^{5/6}$ implies $|w| \leq (1+2\delta)^{5/6}r^{5/6}$. We applied (15) with $\theta = t(\lambda'/\lambda)^{1/3}/3$, which is required to be greater than θ_0. This translates to $t \geq t_0$ for a t_0 depending on θ_0 and λ'/λ. Similarly, we absorbed the λ'/λ dependency in the tail into the value of $\tilde{c}$, which thus depends on the original tail coefficient c in (15) and λ'/λ.

Let us denote conditioning on the environment U by the notation $\mathbb{P}(\cdot \mid U)$. By this we mean we condition on the weights of vertices interior to U as well as those on the lower side A, but not those on the upper side B. Then we see

$$\begin{aligned}&\mathbb{P}\left(Z > \mu r - \lambda'\frac{Gw^2}{r} + tr^{1/3}, E_{\text{low}}, E_{\text{up}} \mid U\right)\\ &\qquad = \mathbb{P}\left(Z > \mu r - \lambda'\frac{Gw^2}{r} + tr^{1/3} \mid U\right)\cdot\mathbb{P}\left(E_{\text{low}} \mid U\right)\cdot\mathbb{P}\left(E_{\text{up}} \mid U\right).\end{aligned}$$

So with (37), all we need is a lower bound on $\mathbb{P}(E_{\text{low}} \mid U)$ and $\mathbb{P}(E_{\text{up}} \mid U)$. This is straightforward using independence of the environment below and above U from U:

$$\mathbb{P}\left(E^c_{\text{lower}} \mid U\right) \leq \sup_{u\in A}\mathbb{P}\left(X_{\phi_{\text{low}},u} \leq \mu\delta r - \frac{t}{3}r^{1/3}\right) \leq \frac{1}{2} \qquad (38)$$

for large enough t (independent of δ) and r (depending on δ), using Assumption 3b. A similar upper bound holds for $\mathbb{P}\left(E^c_{\text{upper}} \mid U\right)$. Together this gives

$$\mathbb{P}\left(Z > \mu r - \lambda'\frac{Gw^2}{r} + tr^{1/3}, E_{\text{low}}, E_{\text{up}} \mid U\right) \geq \frac{1}{4}\cdot\mathbb{P}\left(Z > \mu r - \lambda'\frac{Gw^2}{r} + tr^{1/3}\right),$$

and taking expectation on both sides, combined with (37), gives Lemma 3.5. The fact that λ'/λ depends only on j and the previously mentioned dependencies gives the claimed dependencies of $\tilde{t}_0, \tilde{r}_0$, and $\tilde{c}$.

To prove Lemma 4.5, we take $\delta = 1$, which is equivalent to $\lambda' = \lambda/3$. Then in (37) the final bound is done with the hypothesized bound on $X_{\phi_{\text{low}},\phi_{\text{up}}}$, i.e.,

$$\mathbb{P}\left(X_{\phi_{\text{low}},\phi_{\text{up}}} > \mu\tilde{r} - \lambda\frac{Gw^2}{\tilde{r}} + tr^{1/3}\right) \leq \exp(-\tilde{c}t^{\alpha}).$$

Applying this bound requires $|w| \leq \tilde{r}/2$. Since $|w| \leq r$ and $\tilde{r} = \lambda r/\lambda' = 3r$, this is valid. □

Next we prove Lemma 4.2, on a constant probability lower bound on the lower tail, based on Assumptions 3b and 2.

Proof of Lemma 4.2. Let $\tilde{X}^z_r = X^z_r - \mu r + Gz^2/r$. We know from Assumption 2 that $\mathbb{E}[\tilde{X}^z_r] \leq -g_2 r^{1/3}$. Let E be the event

$$E = E(\theta) = \left\{X^z_r < \mu r - \frac{Gz^2}{r} - \theta r^{1/3}\right\},$$

so that $\mathbb{P}(E) \leq \exp(-c\theta^\alpha)$ for $\theta > \theta_0$, by Assumption 3b.

Observe that $-\tilde{X}_r^z \mathbb{1}_E$ is a positive random variable and so, by Assumption 3b,

$$\begin{aligned}
\mathbb{E}[-\tilde{X}_r^z \mathbb{1}_E] &= r^{1/3} \int_0^\infty \mathbb{P}\left(\tilde{X}_r^z \mathbb{1}_E < -tr^{1/3}\right) \mathrm{d}t \\
&= r^{1/3}\left[\theta \cdot \mathbb{P}\left(X_r^z < \mu r - \frac{Gz^2}{r} - \theta r^{1/3}\right) + \int_\theta^\infty \mathbb{P}\left(X_r^z < \mu r - \frac{Gz^2}{r} - tr^{1/3}\right) \mathrm{d}t\right] \\
&\leq r^{1/3}\left[\theta \exp(-c\theta^\alpha) + \int_\theta^\infty \exp(-ct^\alpha)\,\mathrm{d}t\right];
\end{aligned}$$

this may be made smaller than $0.5 g_2 r^{1/3}$ by taking θ large enough. We now set θ to such a value. We also have $\mathbb{E}[\tilde{X}_r^z] = \mathbb{E}[\tilde{X}_r^z(\mathbb{1}_E + \mathbb{1}_{E^c})]$. Combining this, the above lower bound on $\mathbb{E}[\tilde{X}_r^z \mathbb{1}_E]$, and the upper bound on $\mathbb{E}[X_r^z]$, gives that

$$\mathbb{E}[\tilde{X}_r^z \mathbb{1}_{E^c}] \leq -\frac{1}{2} g_2 r^{1/3}. \tag{39}$$

The fact that $\tilde{X}_r^z \mathbb{1}_E^c$ is supported on $[-\theta r^{1/3}, \infty)$ implies that

$$\mathbb{P}\left(X_r^z \mathbb{1}_{E^c} < \mu r - \frac{Gz^2}{r} - \frac{1}{4} g_2 r^{1/3}\right) \geq \frac{g_2}{4\theta}; \tag{40}$$

this follows from (39) and since

$$\begin{aligned}
\mathbb{E}[\tilde{X}_r^z \mathbb{1}_E^c] &\geq -\theta \cdot \mathbb{P}\left(\tilde{X}_r^z \mathbb{1}_E^c < -\frac{1}{4} g_2 r^{1/3}\right) - \frac{1}{4} g_2 r^{1/3} \mathbb{P}\left(\tilde{X}_r^z \mathbb{1}_E^c \geq -\frac{1}{4} g_2 r^{1/3}\right) \\
&\geq -\theta \cdot \mathbb{P}\left(\tilde{X}_r^z \mathbb{1}_E^c < -\frac{1}{4} g_2 r^{1/3}\right) - \frac{1}{4} g_2 r^{1/3}.
\end{aligned}$$

Since $\tilde{X}_r^z \mathbb{1}_E < 0$, it follows that $\tilde{X}_r^z \leq \tilde{X}_r^z \mathbb{1}_{E^c}$, so (40) gives a lower bound on the lower tail of X_r^z, as desired, with $C = \frac{1}{4} g_2$ and $\delta = g_2/4\theta$. ☐

A.1. **Proof of transversal fluctuation bound, Proposition 1.9.** In this section we prove Proposition 1.9 on the tail (with exponent 2α) of the transversal fluctuation of the geodesic path on scale $r^{2/3}$; we closely follow the proof of Theorem 11.1 of the preprint [BSS14], but adapted to our setting and assumptions. We give the argument for the left-most geodesic Γ_r^z from $(1,1)$ to $(r-z, r+z)$; the argument is symmetric for the right-most geodesic. (Note that these are well-defined by the planarity and the weight-maximizing properties of all geodesics).

We start with a similar bound at the midpoint of the geodesic, which needs some notation. For $x \in [\![1, r]\!]$, let $\Gamma_r^z(x)$ be the unique point y such that $(x-y, x+y) \in \Gamma_r^z$.

Proposition A.1. *Under Assumption 2 and 3, there exist $c = c(\alpha) > 0$, r_0, and s_0 such that, for $r > r_0$, $s > s_0$, and $|z| \leq r^{5/6}$,*

$$\mathbb{P}\left(|\Gamma_r^z(r/2)| > z/2 + sr^{2/3}\right) \leq 2\exp(-cs^{2\alpha}).$$

To prove this we will need a bound on the maximum, $\widetilde{Z}$, of fluctuations of the point-to-point weight as the endpoint varies over an interval, i.e.,

$$\widetilde{Z} = \sup_{v \in \mathbb{L}_{\mathrm{up}}} \Big(X_v - \mathbb{E}[X_v]\Big),$$

where $\mathbb{L}_{\mathrm{up}}$ is the interval of width $2r^{2/3}$ around $(r-w, r+w)$. Note that this is not the same as the point-to-interval weight.

Lemma A.2. *Let $K > 0$ and $|w| \leq Kr^{5/6}$. Under Assumptions 2 and 3, there exist $c > 0$, $\theta_0 = \theta_0(K)$, and r_0, such that, for $\theta > \theta_0$ and $r > r_0$,*

$$\mathbb{P}\left(\widetilde{Z} > \theta r^{1/3}\right) \leq \exp(-c\theta^\alpha).$$

Proof. The proof is very similar to that of Lemma 3.5 above.

We take $\phi_{\mathrm{up}} = (2(r-w), 2(r+w))$ to be the backed up point. Let $v^* \in \mathbb{L}_{\mathrm{up}}$ be the maximizing point in the definition of $\widetilde{Z}$. For clarity, define the lower and upper mean weight functions M_{low} and M_{up} by $M_{\mathrm{low}}(v) = \mathbb{E}[X_v]$ and $M_{\mathrm{up}}(v) = \mathbb{E}[X_{v,\phi_{\mathrm{up}}}]$; this is to use the unambiguous notation $M_{\mathrm{low}}(v^*)$ (which is a function of v^*) instead of $\mathbb{E}[X_{v^*}]$. We also define

$$E_{\mathrm{up}} = \left\{X_{v^*+(1,0),\phi_{\mathrm{up}}} - M_{\mathrm{up}}(v^* + (1,0)) > -\frac{\theta}{2}r^{1/3}\right\}.$$

Now observe that

$$\begin{aligned} X_{v^*} - M_{\mathrm{low}}(v^*) + X_{v^*+(1,0),\phi_{\mathrm{up}}} - M_{\mathrm{up}}(v^* + (1,0)) \\ \leq X_{\phi_{\mathrm{up}}} - \inf_{v\in\mathbb{L}_{\mathrm{up}}} \left(M_{\mathrm{low}}(v) + M_{\mathrm{up}}(v + (1,0))\right). \end{aligned} \tag{41}$$

We want to replace the infimum on the right hand side by $\mathbb{E}[X_{\phi_{\mathrm{up}}}]$. The latter is at most $2\mu r - 2Gw^2/r$. We need to show that the infimum term is at least something which is within $O(r^{1/3})$ of this expression. For this we do the following calculation using Assumption 2. Parametrize $v \in \mathbb{L}_{\mathrm{up}}$ as $(r - w - tr^{2/3}, r + w + tr^{2/3})$ for $t \in [-1, 1]$. Then, for all $t \in [-1, 1]$,

$$\begin{aligned} M_{\mathrm{low}}(v) + M_{\mathrm{up}}(v + (1,0)) &\geq \left[\mu r - \frac{G(w + tr^{2/3})^2}{r} - H\frac{(w + tr^{2/3})^4}{r^3}\right] \\ &\quad + \left[\mu r - \frac{G(w - tr^{2/3})^2}{r} - H\frac{(w - tr^{2/3})^4}{r^3}\right] \\ &\geq 2\mu r - \frac{2Gw^2}{r} - 2Gt^2r^{1/3} - 32HK^4r^{1/3}, \end{aligned}$$

the last inequality since $|w \pm tr^{2/3}| \leq 2Kr^{5/6}$. Since $t \in [-1, 1]$, $2Gt^2r^{1/3} \leq 2Gr^{1/3}$, and so the right hand side of (41) is at most $X_{\phi_{\mathrm{up}}} - \mathbb{E}[X_{\phi_{\mathrm{up}}}] + \frac{\theta}{4}r^{1/3}$ for all large enough θ (depending on K). Thus, recalling the definition of E_{up},

$$\mathbb{P}\left(\widetilde{Z} > \theta r^{1/3}, E_{\mathrm{up}}\right) \leq \mathbb{P}\left(X_{\phi_{\mathrm{up}}} - \mathbb{E}[X_{\phi_{\mathrm{up}}}] > \frac{\theta}{4}r^{1/3}\right) \leq \exp(-c\theta^\alpha).$$

We now claim that E_{up} has probability at least $1/2$; since E_{up} is independent of $\widetilde{Z}$, this will imply that $\mathbb{P}(\widetilde{Z} > \theta r^{1/3}) \leq 2\exp(-c\theta^\alpha)$. The proof of the claim is straightforward using the independence of u^* with the environment above $\mathbb{L}_{\mathrm{up}}$ and Assumption 3b, for

$$\mathbb{P}\left(E_{\mathrm{up}}^c\right) \leq \sup_{v\in\mathbb{L}_{\mathrm{up}}} \mathbb{P}\left(X_{v+(1,0)} - M_{\mathrm{up}}(v + (1,0)) \leq -\frac{\theta}{2}r^{1/3}\right) \leq 1/2,$$

for all θ larger than an absolute constant. □

Proof of Proposition A.1. We will prove the bound for the event that $\Gamma_r^z(r/2) > z/2 + sr^{2/3}$, as the event that it is less than $-z/2 - sr^{2/3}$ is symmetric.

For $j \in [\![0, r^{1/3}]\!]$, let I_j be the interval

$$\left(\frac{r}{2} - \frac{z}{2} - sr^{2/3}, \frac{r}{2} + \frac{z}{2} + sr^{2/3}\right) - [j, j+1] \cdot (r^{2/3}, -r^{2/3}).$$

Let A_j be the event that Γ_r^z passes through I_j, for $j \in [\![0, r^{1/3}]\!]$. Observe that

$$\left\{\Gamma_r^z(r/2) > z/2 + sr^{2/3}\right\} \subseteq \bigcup_{j=0}^{r^{1/3}} A_j. \tag{42}$$

We claim that $\mathbb{P}(A_j) \leq \exp(-c(s+j)^{2\alpha})$ for each such j; this will imply Proposition A.1 by a union bound which we perform at the end.

Let $Z_j^{(1)} = X_{(1,1),I_j}$ and $Z_j^{(2)} = X_{I_j,(r-z,r+z)}$. Also, let $\widetilde{Z}_j^{(1)} = \sup_{v\in I_j}(X_v - \mathbb{E}[X_v])$, and define $\widetilde{Z}_j^{(2)}$ analogously.

We have to bound the probability of A_j. The basic idea is to show that any path from $(1,1)$ to $(r-z, r+z)$ which passes through I_j suffers a weight loss greater than that which X_r^z typically suffers (which is of order Gz^2/r), and so such paths are not competitive. When j is very large, it is possible to show this even if we do not have the sharp coefficient of G for the parabolic loss; but for smaller values of j, we will need to be very tight with the coefficient of the parabolic loss. So we divide into two cases, depending on the size of j, and first address the case when j is large (in a sense to be specified more precisely shortly). Observe that, for a $c_2 > 0$ to be fixed,

$$\mathbb{P}(A_j) \leq \mathbb{P}\left(X_r^z < \mathbb{E}[X_r^z] - c_2(s+j)^2 r^{1/3}\right) + \mathbb{P}\left(Z_j^{(1)} + Z_j^{(2)} > \mathbb{E}[X_r^z] - c_2(s+j)^2 r^{1/3}\right);$$

the first term is bounded by $\exp(-c(s+j)^{2\alpha})$ by Assumption 3b for a c depending on c_2, and we must show a similar bound for the second. Note that the second term is bounded by

$$\mathbb{P}\left(Z_j^{(1)} + Z_j^{(2)} > \mu r - \frac{Gz^2}{r} - Hr^{1/3} - c_2(s+j)^2 r^{1/3}\right), \tag{43}$$

using Assumption 2 and since $|z| \leq r^{5/6}$.

Recall from (35) and Lemma 4.5 that there exists a $\lambda \in (0,1)$ such that, for $|z/2 + (s+j)r^{2/3}| \leq r$, and $i = 1$ and 2,

$$\mathbb{P}\left(Z_j^{(i)} > \nu_{i,j} + \theta r^{1/3}\right) \leq \exp(-c\theta^\alpha), \tag{44}$$

where $\nu_{i,j} = \frac{1}{2}\mu r - \lambda \cdot \frac{G}{r/2} \cdot (\frac{1}{2}z \pm (s+j)r^{2/3})^2$ with the $+$ for $i=1$ and $-$ for $i=2$; $\nu_{i,j}$ captures the typical weight of these paths. Note that we are very crude with the parabolic coefficient, but the bound (44) holds for all j; and also that we measure the deviation from the same expression $\nu_{i,j}$ (which is obtained by evaluating (35) at one endpoint) for all points in the interval. As we will see, comparing the full interval to a single point will not work for the second case of small j.

We want to show that the typical weight $\nu_{1,j} + \nu_{2,j}$ is much lower than $\mu r - Gz^2/r$. Simple algebraic manipulations show that, if $(s+j)r^{2/3} > (\lambda^{-1} - 1)^{1/2} r^{5/6}$ (which is the largeness condition on j defining the first case),

$$\sum_{i=1}^{2} \nu_{i,j} < \mu r - \lambda\frac{Gz^2}{r} - (1-\lambda)Gr^{2/3} - 3\lambda G(s+j)^2 r^{1/3} < \mu r - \frac{Gz^2}{r} - 3\lambda G(s+j)^2 r^{1/3},$$

the final inequality since $|z| \leq r^{5/6}$. We have to bound (43) with some value of c_2, and we take it to be $2\lambda G$; note that any bound we prove on (43) will still be true if we later further lower c_2. The previous displayed bound shows that, for $(s+j)r^{2/3} > (\lambda^{-1} - 1)^{1/2} r^{5/6}$,

$$\begin{aligned}\mathbb{P}&\left(Z_j^{(1)} + Z_j^{(2)} > \mu r - \frac{Gz^2}{r} - Hr^{1/3} - c_2(s+j)^2 r^{1/3}\right) \\ &\qquad\qquad \leq \mathbb{P}\left(Z_j^{(1)} + Z_j^{(2)} > \nu_{1,j} + \nu_{2,j} + \tfrac{1}{2}\lambda G(s+j)^2 r^{1/3}\right).\end{aligned}$$

In the inequality we absorbed $-Hr^{1/3}$ into the last term by imposing that s is large enough, depending on λ, G, and H. Now by a union bound and (44), the last display, and hence (43), is bounded by $2\exp(-c(s+j)^{2\alpha})$.

Now we address the other case that $(s+j)r^{2/3} \leq (\lambda^{-1}-1)^{1/2}r^{5/6}$. Thus I_j is close to the interpolating line, and we need a bound on the interval-to-interval weight with a much sharper parabolic term than in the previous case. Here above approach of the first case faces an issue. Since the gradient of Gz^2/r at z is $2Gz/r$, the weight difference across an interval of length $r^{2/3}$ at antidiagonal displacement z is of order $z/r^{1/3}$, which is much larger than the bearable error of $O(r^{1/3})$ when z is, say, $r^{5/6}$; so the crude approach of using the same expression (which we need to be less than $\mu r - Gz^2/r$) for the typical weight of all points in the interval, as we did in the first case, is insufficient—to have a single expression for which a tail bound exists for all points in the interval, we must necessarily include the linear gain of moving across the interval in the expression, and this will force it above $\mu r - Gz^2/r$. So, for this case, we will use Lemma A.2, which avoids the problem by taking the supremum after centering by the point-specific expectation.

Let $X'_v = X_{v,(r-z,r+z)}$. Now we observe

$$\mathbb{P}\left(A_j\right) \leq \mathbb{P}\left(X_r^z < \mathbb{E}[X_r^z] - c_2(s+j)^2r^{1/3}\right) + \mathbb{P}\left(\sup_{v\in I_j}(X_v + X'_v) > \mathbb{E}[X_r^z] - c_2(s+j)^2r^{1/3}\right);$$

note that $X_v + X'_v$ counts the weight of v twice, but this is acceptable as this sum dominates the weight of the best path through v. The first term is at most $\exp(-c(s+j)^{2\alpha})$ for a $c>0$ depending on c_2. We bound the second term as follows. First we note that $\mathbb{E}[X_r^z] \geq \mu r - Gz^2/r - Hr^{1/3}$ and that $\sup_{v\in I_j}\left(\mathbb{E}[X_v + X'_v]\right) \leq \mu r - Gz^2/r - G(s+j)^2r^{1/3}$ by a simple calculation with Assumption 2, and so

$$\begin{aligned}\mathbb{P}&\left(\sup_{v\in I_j}(X_v + X'_v) > \mathbb{E}[X_r^z] - c_2(s+j)^2r^{1/3}\right)\\ &\leq \mathbb{P}\left(\sup_{v\in I_j}(X_v - \mathbb{E}[X_v] + X'_v - \mathbb{E}[X'_v]) > -Hr^{1/3} + (G-c_2)(s+j)^2r^{1/3}\right).\end{aligned}$$

We lower c_2 (if required) from its earlier value to be less than $G/2$. Now, we need to absorb the $-Hr^{1/3}$ term above into the $(s+j)^2r^{1/3}$ term, which we can do for $s > s_0$ by setting s_0 large enough depending on G and H. So for such s, by a union bound we see that the previous display is at most

$$\mathbb{P}\left(\sup_{v\in I_j}(X_v - \mathbb{E}[X_v]) > \tfrac{1}{6}G(s+j)^2r^{1/3}\right) + \mathbb{P}\left(\sup_{v\in I_j}(X'_v - \mathbb{E}[X'_v]) > \tfrac{1}{6}G(s+j)^2r^{1/3}\right).$$

We bound this by applying Lemma A.2, with $K = (\lambda^{-1}-1)^{1/2}$ and $\theta = \frac{1}{6}G(s+j)^2$. Recall that the bound of Lemma A.2 holds for $\theta > \theta_0(K)$. Thus we raise s_0 further if necessary so that $(s+j)^2 > \theta_0(K)$ for all $s > s_0$ and $j \geq 0$. Then we see that, for s and j such that $s > s_0$ and $(s+j)r^{2/3} \leq (\lambda^{-1}-1)r^{5/6}$, the last display is at most $2\exp(-c(s+j)^{2\alpha})$.

Returning to the inclusion (42) and the bound of $\exp(-c(s+j)^{2\alpha})$ of $\mathbb{P}(A_j)$ for the two cases, we see that

$$\mathbb{P}\left(\Gamma_r^z(r/2) > z/2 + sr^{2/3}\right) \leq \sum_{j=1}^{r^{1/3}} \exp(-c(s+j)^{2\alpha}) \leq C\exp(-cs^{2\alpha})$$

for some absolute constant $C < \infty$ and $c > 0$ depending on α. Here we used that, if $\alpha \in (0,1/2)$, then $(s+j)^{2\alpha} \geq 2^{2\alpha-1}(s^{2\alpha}+j^{2\alpha})$, while if $\alpha \geq 1/2$, then $(s+j)^{2\alpha} \geq s^{2\alpha}+j^{2\alpha}$; and finally $\exp(-cj^{2\alpha})$ is summable over j. This completes the proof of Proposition A.1. ∎

To extend the transversal fluctuation bound from the midpoint (as in Proposition A.1) to anywhere along the geodesic (as in Proposition 1.9), we follow very closely a multiscale argument previously employed in [BSS14, Theorem 11.1] and [BGHH20, Theorem 3.3] for similar purposes. For this reason, we will not write a detailed proof but only outline the idea.

Proof sketch of Proposition 1.9. First, the interpolating line is divided up into dyadic scales, indexed by j. The j^{th} scale consists of 2^j+1 anti-diagonal intervals, placed at separation $2^{-j}r$, of length of order $s_j r^{2/3} := \prod_{i=1}^{j}(1+2^{-i/3})sr^{2/3}$. By choosing the maximum j for which this is done large enough, it can be shown that, on the event that $\mathrm{TF}(\Gamma_r^z) > sr^{2/3}$, there must be a j such that there is a pair (I_1, I_3) of consecutive intervals on the j^{th} scale, and the interval I_2 of the $(j+1)^{\text{th}}$ scale in between such that the following holds: the geodesic passes through I_1 and I_3, but fluctuates enough that it avoids I_2, say by passing to its left.

Planarity and that the geodesic is a weight-maximising path then implies that the geodesic from the left endpoint of I_1 to that of I_3 is to the left of the geodesic Γ_r^z (this observation is often called geodesic or polymer ordering), and so must have midpoint transversal fluctuation at least of order $(s_{j+1}-s_j)r^{2/3} = 2^{-(j+1)/3}sr^{2/3}$. But since this transversal fluctuation happens across a scale of length $r' = 2^{-j}r$, in scaled coordinates it is of order $2^{j/3}s(r')^{2/3}$. Applying Proposition A.1 says that this probability is at most $\exp(-c2^{2\alpha j/3}s^{2\alpha})$. Now it remains to take a union bound over all the scales and the intervals within each scale. Since the number of intervals in the j^{th} scale is 2^j, and since $2^j \exp(-c2^{2\alpha j/3}s^{2\alpha})$ is summable in j, we obtain the overall probability bound of $\exp(-cs^{2\alpha})$ of Proposition 1.9. □

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