Understanding the upper tail behaviour of the KPZ equation via the tangent method

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(based on joint work with Shirshendu Ganguly)

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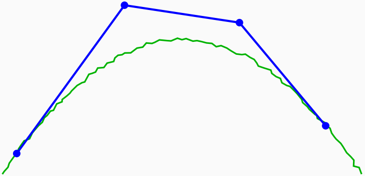
University of Chicago Probability Seminar
April 1, 2022



The Kardar-Parisi-Zhang (KPZ) equation is a non-linear stochastic PDE believed to describe planar random interface growth in a very broad class of models, and is given by

$$\partial_t H = \frac{1}{4}(\partial_x H)^2 + \frac{1}{4}\partial_x^2 H + \xi, \tag{1}$$

where ξ is space-time white noise on $\mathbb{R} \times (0, \infty)$ and $H : \mathbb{R} \times (0, \infty) \to \mathbb{R}$.

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While there are now sophisticated notions of solution available, the one that has underlied most previous studies is known as the Cole-Hopf solution.

Cole-Hopf solution to the KPZ equation

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The Cole-Hopf solution *defines* $H = \log Z$ to solve the KPZ equation, where Z solves the multiplicative stochastic heat equation

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The narrow-wedge solution to the KPZ equation is the solution when $Z(0, \cdot) = \delta_0$, the Dirac mass at the origin.

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So we consider the scaled narrow-wedge solution

$$\mathfrak{h}^t(x) = \frac{H(t, t^{2/3}x) - \frac{t}{12}}{t^{1/3}};$$

this does not grow linearly with t, has unit order fluctuations, and is tight in $t \geq t_0 > 0$.

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It decays *parabolically*; in fact, for fixed t, $x \mapsto \mathfrak{h}^t(x) + x^2$ is a stationary process.

Upper tail behaviour of $\mathfrak{h}^t$

The upper tail behavior has been of significant interest in both the physics and mathematics communities.

In spite of significant recent progress relying on exact formulas available for the narrow wedge solution, the upper tail behavior is not completely understood.

Questions of interest include:

- Asymptotics of one- and multi-point tails, eg. $\mathbb{P}(\mathfrak{h}^t(0) > \theta)$ or $\mathbb{P}(\mathfrak{h}^t(-1) > \theta_-, \mathfrak{h}^t(1) > \theta_+)$.
- The behavior of the profile under the above events.

Existing work has been mainly focused on one-point asymptotics.

A multi-point question: Sharpness of FKG?

Because of connections to statistical mechanics models, it is known that $\mathfrak{h}^t$ enjoys the FKG inequality, so that, for all θ_- and θ_+,

$$\mathbb{P}\left(\mathfrak{h}^t(-1) > \theta_-, \mathfrak{h}^t(1) > \theta_+\right) \geq \mathbb{P}\left(\mathfrak{h}^t(-1) > \theta_-\right) \cdot \mathbb{P}\left(\mathfrak{h}^t(1) > \theta_+\right).$$

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But in many applications the inequality is suboptimal.

So we are led to ask: Is FKG sharp for any values of θ_- and θ_+, and, if so, which ones?

Greater structure in the $t \to \infty$ limit

It has been long expected and recently proven (Quastel-Sarkar and Virág) that the narrow-wedge solution converges to the parabolic Airy_2 process $\mathcal{P}$ as $t \to \infty$.

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$$\mathbb{P}\big(\mathcal{P}(0) > \theta\big) = \exp\left(-\frac{4}{3}\theta^{3/2} + O(\log \theta)\right).$$

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The question of spatial structure has received some more attention however, with Quastel-Tsai proving a large deviation principle for a discrete prelimit (TASEP) of $\mathcal{P}$.

Related work in the finite t case

There has been some work in the case of finite t as well:

- Lamarre-Lin-Tsai investigate the upper tail large deviation limit shape for *short time* ($t \to 0$) using a Feynman-Kac representation. (In this limit the non-linearity disappears and the solution falls into the Gaussian universality class.)

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The above rely on techniques such as the Feynman-Kac representation, PDE methods, and the exact formulas available for the narrow-wedge solution.

In contrast, our approach is more geometric and will yield near-optimal results.

Related work in the finite t case

Theorem (Corwin-Ghosal)

The following holds for a wide class of initial data as well as the narrow-wedge case: there exist $c_1, c_2 > 0$ and θ_0 such that, for $\theta > \theta_0$ and $t \geq 1$,

$$\exp(-c_1\theta^{3/2}) \leq \mathbb{P}(\mathfrak{h}^t(0) > \theta) \leq \exp(-c_2\theta^{3/2}).$$

The constants c_1 and c_2 are explicit but non-optimal for general initial data (predicted to be 4/3 in the physics literature).

For narrow wedge the methods did obtain the optimal constant of $\frac{4}{3} + o(1)$, but only in certain regimes of θ.

Main results

Main results: one-point asymptotics

Theorem

There exist $C < \infty$ and θ_0 such that, for all $t \geq 1$ and $\theta > \theta_0$,

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{9/8}\right) \leq \frac{1}{\mathrm{d}\theta}\mathbb{P}\left(\mathfrak{h}^t(0) \in \mathrm{d}\theta\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{9/8}\right).$$

As an immediate consequence, the same bounds also hold for $\mathbb{P}(\mathfrak{h}^t(0) > \theta)$.

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- This gives a sharp bound with the optimal $\frac{4}{3}$ for the *density*. To our knowledge, bounds on the density were not previously available in the literature.
- The bound holds for all large values of θ with the optimal coefficient $\frac{4}{3}$, and the error is uniform in t.
- The bound also holds for $\mathcal{P}$ (as mentioned a sharper version for the *upper tail* of $\mathcal{P}$ was already known, but the density bound is new).
- We will give similar tail bounds for general initial data shortly.

Main results: the one-point limit shape

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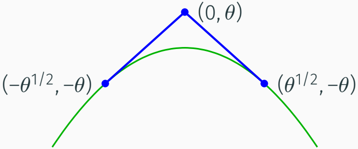

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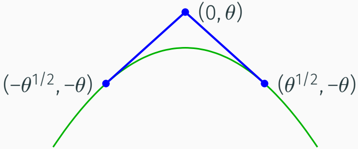

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Theorem

There exist θ_0 and $c > 0$ such that, for all $t \geq 1$, $\theta > \theta_0$, and $M > 0$,

$$\mathbb{P}\left(\sup_{x\in[-\theta^{1/2},\theta^{1/2}]} |\mathfrak{h}^t(x) - \mathrm{Triangle}_\theta(x)| > M\theta^{1/4} \;\middle|\; \mathfrak{h}^t(0) > \theta\right) \leq \exp(-cM^2).$$

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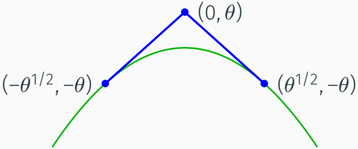

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- The bound also holds with the conditioning $\mathfrak{h}^t(0) = \theta$, and for $\mathcal{P}$.
- $\theta^{1/4}$ is the Brownian fluctuation scale on an interval of size $\theta^{1/2}$ and is optimal.

Main results: two-point limit shape

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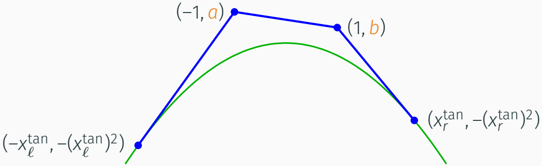

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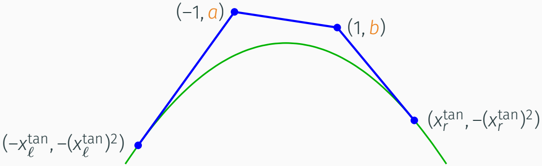

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Theorem

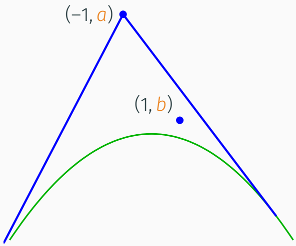
Assuming some non-degeneracy conditions on a and b, there exists $c > 0$ such that, for all $t \geq 1$, $M > 0$, and large enough a, b,

$$\mathbb{P}\left(\sup_{x \in [-x_\ell^{\tan}, x_r^{\tan}]} |\mathfrak{h}^t(x) - \mathrm{Quad}_{a,b}(x)| > M(a^{1/4} + b^{1/4}) \,\middle|\, \mathfrak{h}^t(-1) > a, \mathfrak{h}^t(1) > b \right) \leq \exp(-cM^2).$$

- The bound again also holds for $\mathcal{P}$.

Main results: two-point limit shape

The non-degeneracy conditions are to ensure that both $(-1, a)$ and $(1, b)$ are extreme points of the convex hull, unlike below.



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For $t \geq 1$ and if a, b are large enough and satisfy the non-degeneracy condition, then

$$\mathbb{P}\left(\mathfrak{h}^t(-1) \geq a, \mathfrak{h}^t(1) \geq b\right)$$
$$= \exp\left(-\frac{1}{24}\left[16\left((1+a)^{3/2} + (1+b)^{3/2}\right) + 3(a-b)^2 + 24(a+b) + 32\right] + \text{error}\right).$$

The error term has explicit upper and lower bounds, uniformly in t. The asymptotic also holds for $\mathcal{P}$.

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- This is the first sharp asymptotic for the two-point distribution we know of, and is also new for $\mathcal{P}$.
- The non-degeneracy condition on a, b implies that $(b-a)^2 \ll a^{3/2}, b^{3/2}$. Without it, the one-point asymptotic dominates.
- A similar bound also holds at $\pm K$ in place of ± 1.

Corollary: Geometric condition for sharpness of FKG

Recall that the FKG inequality implies

$$\mathbb{P}\left(\mathfrak{h}^t(-K^{1/2}) > a, \mathfrak{h}^t(K^{1/2}) > b\right) \geq \mathbb{P}\left(\mathfrak{h}^t(-K^{1/2}) > a\right) \cdot \mathbb{P}\left(\mathfrak{h}^t(K^{1/2}) > b\right),$$

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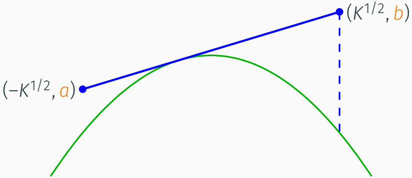

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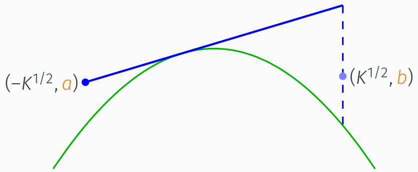

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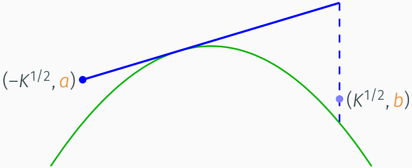

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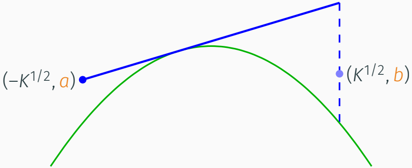

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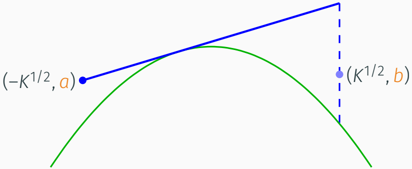

Corollary

Let K be fixed. If the line joining $(-K^{1/2}, a)$ and $(K^{1/2}, b)$ is tangent to or intersects $-x^2$ inside $[-K^{1/2}, K^{1/2}]$, then

$$\mathbb{P}\left(\mathfrak{h}^t(-K^{1/2}) > a, \mathfrak{h}^t(K^{1/2}) > b\right) = \exp\left(-\frac{4}{3}[(K+a)^{3/2} + (K+b)^{3/2}] + \text{error}\right)$$
$$\approx \mathbb{P}(\mathfrak{h}^t(-K^{1/2}) > a) \cdot \mathbb{P}(\mathfrak{h}^t(K^{1/2}) > b).$$

- The second line is via the one-point asymptotics and since $\mathfrak{h}^t(\pm K^{1/2}) + K \stackrel{d}{=} \mathfrak{h}^t(0)$ by stationarity of $\mathfrak{h}^t(x) + x^2$.

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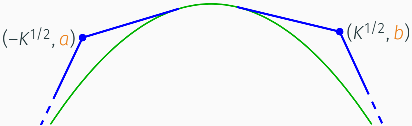

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- In essence, the parabola acts as a barrier to the interaction of the events $\{\mathfrak{h}^t(-K^{1/2}) > a)\}$ and $\{\mathfrak{h}^t(K^{1/2}) > b\}$.

Main results: One-point asymptotics for general initial condition

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- **At most linear growth**: $\limsup_{|x|\to\infty} x^{-1}\mathfrak{h}^0(x) < \infty$.
- **Not $-\infty$ everywhere**: There is a positive measure set where $\mathfrak{h}^0 \neq -\infty$.

(The hypotheses essentially ensure $\mathfrak{h}^t$ exists for all $t \geq 1$ and is non-trivial.)

Recall that the solution $\mathfrak{h}^t_{\text{gen}}$ is given by $\log Z^t$, where $Z^t(x)$ solves the stochastic heat equation with $Z^0(x) = \exp(\mathfrak{h}^0(x))$.

Main results: One-point asymptotics for general initial condition

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(The hypotheses essentially ensure $\mathfrak{h}^t$ exists for all $t \geq 1$ and is non-trivial.)

Recall that the solution $\mathfrak{h}^t_{\mathrm{gen}}$ is given by $\log Z^t$, where $Z^t(x)$ solves the stochastic heat equation with $Z^0(x) = \exp(\mathfrak{h}^0(x))$.

Theorem

There exist C, θ_0 (depending on $\mathfrak{h}^0$) such that, for $t \geq 1$, and $\theta > \theta_0$,

$$\exp\left(-\frac{4}{3}\theta^{3/2} - C\theta^{9/8}\right) \leq \mathbb{P}\left(\mathfrak{h}^t_{\mathrm{gen}}(0) \geq \theta\right) \leq \exp\left(-\frac{4}{3}\theta^{3/2} + C\theta^{9/8}\right).$$

The constants C and θ_0 can be made uniform over a class of initial data by quantifying the hypotheses on $\mathfrak{h}^0$.

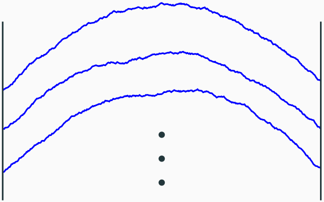
Summary of results

- The limit shapes for large one- and two-point values are given in terms of tangent lines to the parabola through the high points, and the fluctuations around the shape are Brownian.
- The information about the limit shapes can be combined with Brownian estimates to give sharp asymptotics for the one- and two-point probabilities.
- These asymptotics give a geometric understanding of asymptotic independence of upper-tail events, i.e., the sharpness of FKG.
- This can be extended to give sharp one-point asymptotics for general initial data.

The Brownian Gibbs property

The resampling property

Both $\mathfrak{h}^t$ and $\mathcal{P}$ can be embedded as the top, or lowest-indexed, curve in a $\mathbb{N}$-indexed ensemble of random continuous curves.



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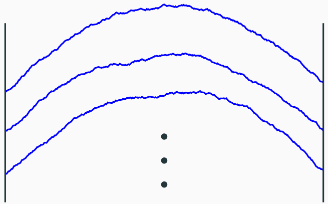

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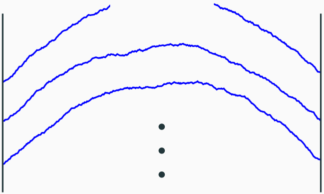

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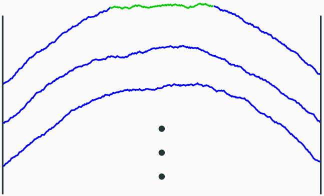

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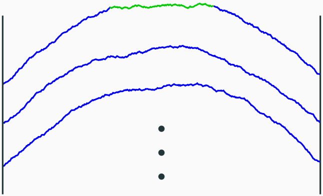


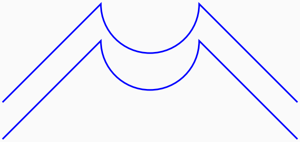
A useful heuristic to keep in mind:

$\mathfrak{h}^t$ and $\mathcal{P}$ are like Brownian bridges conditioned to stay above a parabola $-x^2$ with which they share endpoints.

A convex consequence

A key idea is that the resampling in terms of Brownian bridges implies that the limit shapes should be convex.

Indeed, suppose the limit shape of the top curve is not convex in some neighbourhood. This pushes the second curve down on the interval.



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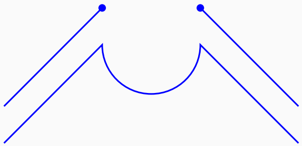

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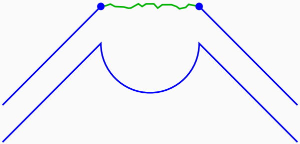

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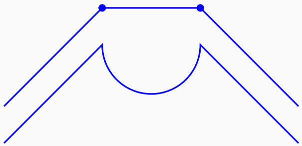

A convex consequence

In the six-vertex model, a similar idea was termed the tangent method by Colomo-Sportiello and was rigorously implemented by Aggarwal.

The six-vertex model is an example of a "zero temperature" model (similar to $\mathcal{P}$), while the KPZ equation at finite t is a "positive temperature" model.

Proof ideas

Getting the one-point limit shape (upper bound): pinning

To use the linear trajectories of Brownian bridges, we first need to "pin" $\mathfrak{h}^t(x)$ to the tangent line at some x; then we will know that Triangle_θ is followed. Take $x = \theta^{1/2} z$.

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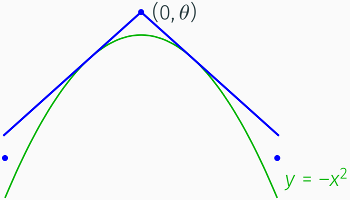

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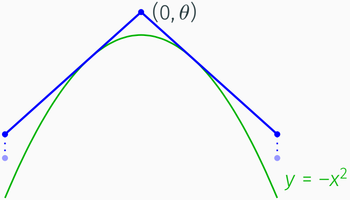

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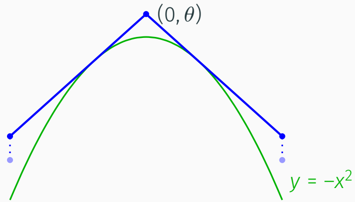

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We need $\mathfrak{h}^t$ to be below the tangent with high probability *conditional on* $\mathfrak{h}^t(0) > \theta$. But by stationarity + parabolic curvature,

$$\mathbb{P}\left(\mathfrak{h}^t(\theta^{1/2}z) > \text{Tangent}(\theta^{1/2}z) \mid \mathfrak{h}^t(0) > \theta\right) \leq \frac{\mathbb{P}(\mathfrak{h}^t(0) > \text{Tangent}(\theta^{1/2}z) + \theta z^2)}{\mathbb{P}(\mathfrak{h}^t(0) > \theta)},$$

which is small for large enough z (but still $O(1)$) by the Corwin-Ghosal tail bounds.

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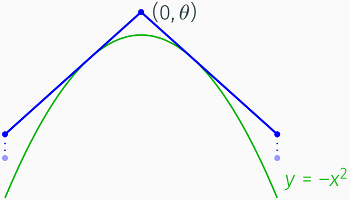

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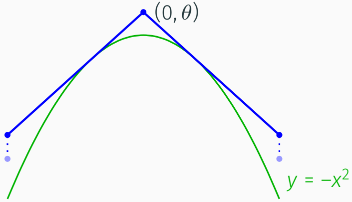

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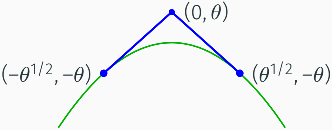
This yields the $\exp(-cM^2)$ bound for a deviation of size M on the Brownian scale $\theta^{1/4}$.

Getting the one point tail asymptotics



Since $\mathfrak{h}^t$ looks like Triangle$_\theta$ on $[-\theta^{1/2},\theta^{1/2}]$ when $\mathfrak{h}^t(0) > \theta$, $\mathbb{P}(\mathfrak{h}^t(0) > \theta)$ becomes a Brownian calculation: with B a (rate 2) Brownian bridge from $(-\theta^{1/2},-\theta)$ to $(\theta^{1/2},-\theta)$,

$$\mathbb{P}\left(\mathfrak{h}^t(0) > \theta\right) \approx \mathbb{P}\left(B(0) > \theta \mid B(x) > -x^2\ \forall x \in [-\theta^{1/2},\theta^{1/2}]\right)$$

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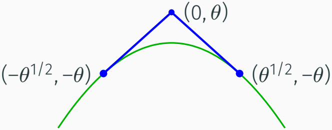

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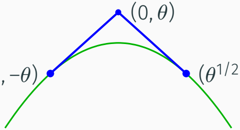

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The numerator is $\exp\left(-\frac{(\theta+\theta)^2}{2\theta^{1/2}}\right) = \exp(-2\theta^{3/2})$.

The denominator D is $\geq \exp\left(-\frac{2}{3}\theta^{3/2}\right)$.

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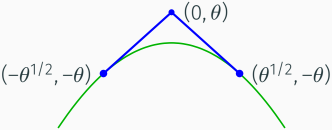

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$$D \leq \mathbb{P}\left(\int_{-\theta^{1/2}}^{\theta^{1/2}} B(x) + x^2 > 0\right) = \mathbb{P}\left(N\left(-\frac{4}{3}\theta^{3/2}, \frac{4}{3}\theta^{3/2}\right) > 0\right) \approx \exp\left(-\frac{2}{3}\theta^{3/2}\right),$$

since $\exp(-y^2/2y) = \exp(-y/2)$.

The argument for general initial data

A distributional convolution formula allows one to get one-point information for general initial data from spatial information for narrow-wedge:

$$\mathfrak{h}^t_{\text{gen}}(0) \stackrel{d}{=} t^{-1/3} \log \int_{\mathbb{R}} \exp\left[t^{1/3}\left\{\mathfrak{h}^t_{\text{nw}}(x) + \mathfrak{h}^0(x)\right\}\right] \,\mathrm{d}x.$$

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This can be done with further resampling arguments using the Brownian Gibbs property.

Summary

- Using geometric methods combined with the Brownian Gibbs properties, we can obtain sharp asymptotics for one- and two-point upper tails for narrow-wedge solutions, as well as for one-point asymptotics for general initial data.
- As a first step for this, and also for independent interest, we need to understand the shape of the profile under these asymptotic events, which we do using ideas similar to the tangent method.
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Thank you!