Bootstrapping to optimal tail exponents in last passage percolation

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(joint work with Shirshendu Ganguly)

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Junior Integrable Probability Seminar
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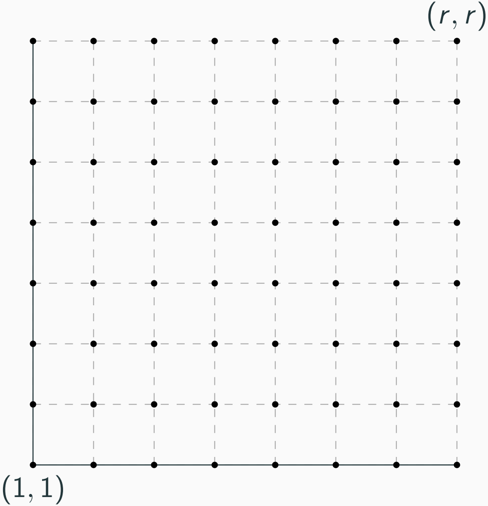
Last passage percolation on $\mathbb{Z}^2$

LPP on $\mathbb{Z}^2$: Paths



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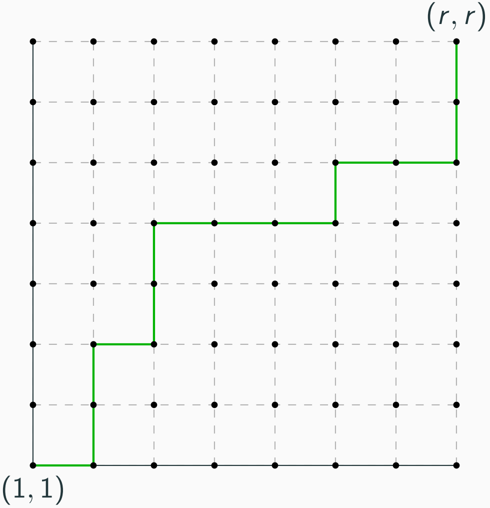

LPP on $\mathbb{Z}^2$: Weights of paths

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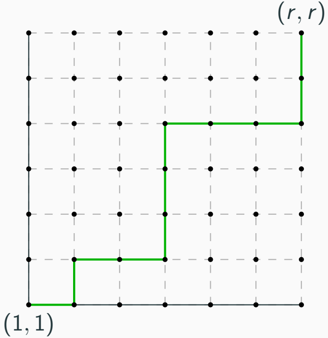
X_r is the maximum weight of paths which go from $(1,1)$ to (r,r).



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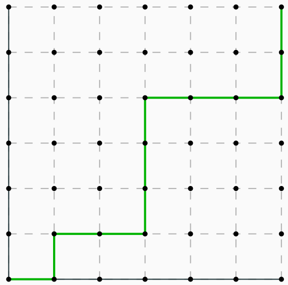

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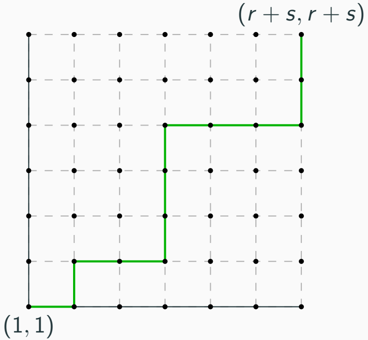
This is because $\{X_r\}_r$ are super-additive:

$$X_{r+s} \overset{d}{\geq} X_r + \widetilde{X}_s.$$



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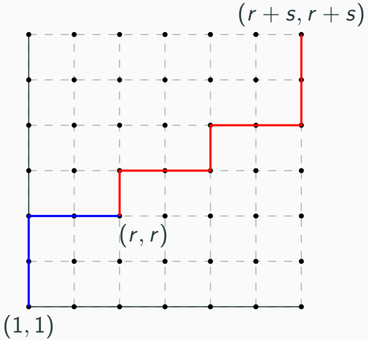


LPP on $\mathbb{Z}^2$: Scalings

X_r's fluctuations around μr should be non-Gaussian of order $r^{1/3}$.

When centred and scaled, the scaling limit should be the GUE Tracy-Widom distribution (known in integrable models).

Theorem (Johansson '00)

Let $\{\xi_v : v \in \mathbb{Z}^2\}$ be i.i.d. exponential rate one random variables. It holds that

$$\frac{X_r - 4r}{2^{4/3} r^{1/3}} \xrightarrow{d} F_{\mathrm{TW}}.$$

GUE Tracy-Widom tail behavior

The GUE Tracy-Widom is also non-Gaussian. It has upper and lower tail exponents 3/2 and 3:

Theorem (eg. Ramirez-Rider-Virág '11)

As $t \to \infty$,

$$F_{\mathrm{TW}}\big([t,\infty)\big) = \exp\left(-\frac{4}{3}t^{3/2}\left(1+o(1)\right)\right) \qquad \textit{and}$$

$$F_{\mathrm{TW}}\big((-\infty,-t]\big) = \exp\left(-\frac{1}{12}t^{3}\left(1+o(1)\right)\right).$$

Tail behaviour in LPP

These tail exponents are also known for LPP in integrable cases!

Theorem (Joh00, LR10, BGHK19)

Let $\{\xi_v : v \in \mathbb{Z}^2\}$ be i.i.d. exponential rate one random variables. For $t_0 < t < r^{2/3}$,

$$\mathbb{P}\left(X_r > 4r + tr^{1/3}\right) \leq \exp\left(-c_1 t^{3/2}\right) \qquad \text{and}$$

$$\exp\left(-c_2 t^3\right) \leq \mathbb{P}\left(X_r < 4r - tr^{1/3}\right) \leq \exp\left(-c_3 t^3\right).$$

Our main result obtains such inequalities under some natural assumptions.

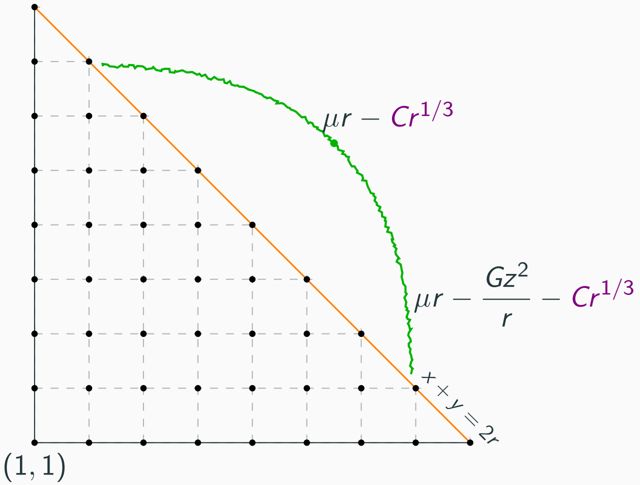
Use of assumptions in the non-integrable model FPP

Progress in the non-integrable model of first passage percolation has been very limited, and has often relied on assumptions comparable to ours.

These include

1. assumptions on fluctuations of the analogue of X_r or of the geodesic (see eg. [Cha13] and [AD14]); and
2. curvature of the limit shape as the endpoint of the geodesic varies (see eg. [NP95]).

LPP on $\mathbb{Z}^2$: Parabolic curvature of weight profile



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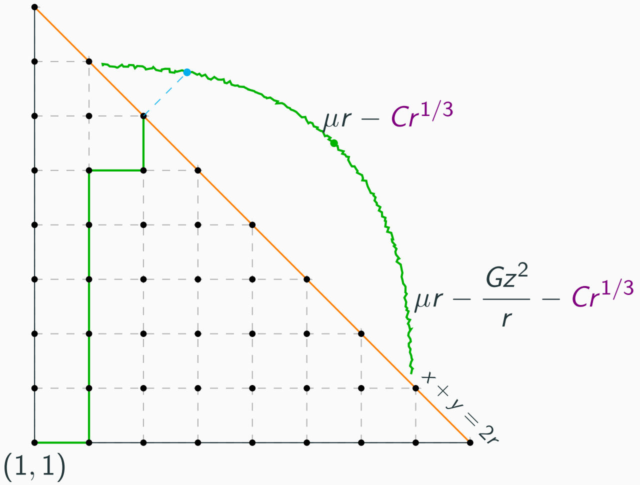

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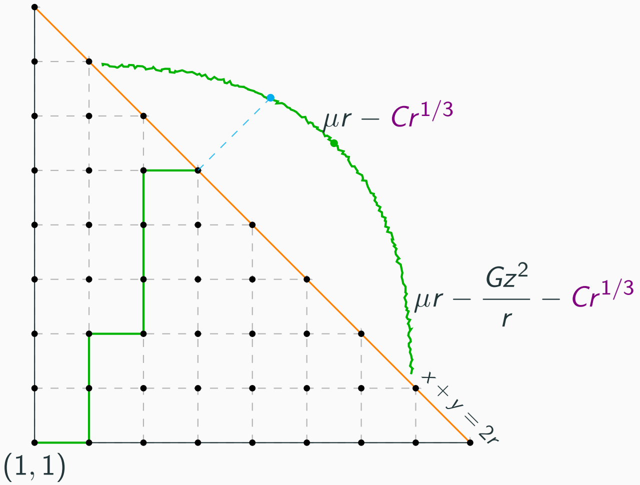

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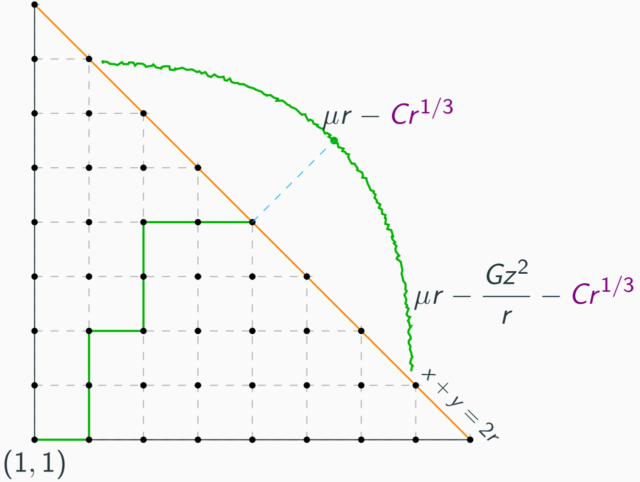

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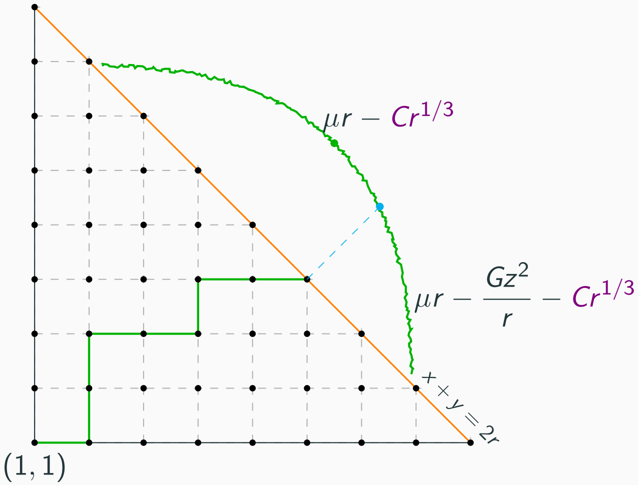

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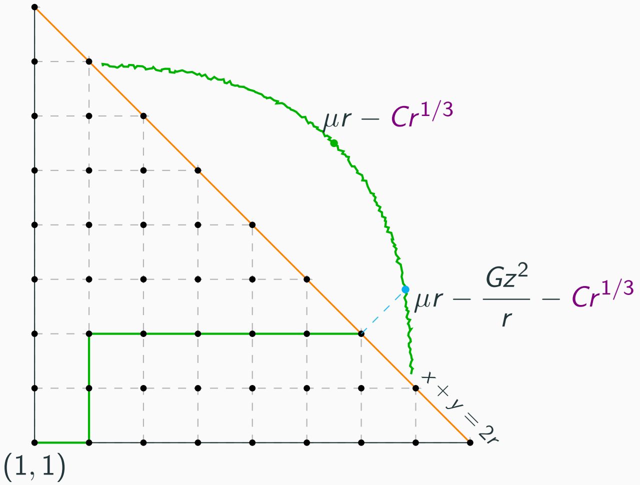
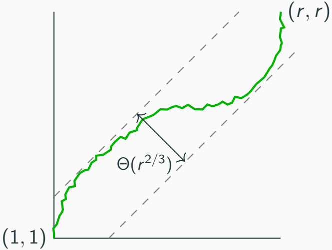


A consequence of the curvature: transversal fluctuations

The transversal fluctuation of a path γ is roughly the maximum distance to the diagonal from γ.

In planar LPP, it is of order $r^{2/3}$: at this value, the weight loss from parabolic curvature is of the order of weight fluctuations, $r^{1/3}$:

$$\frac{G(r^{2/3})^2}{r} = Gr^{1/3}.$$



Assumptions and main results

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2. **Limit shape strong concavity & non-random fluctuation:**
   For $z \in [-\rho r, \rho r]$,

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3. **Upper bounds, uniform in direction:** There exists $\alpha > 0$ s.t.
$$\mathbb{P}\left(|X_r^z - \mathbb{E}[X_r^z]| > tr^{1/3}\right) \leq \exp(-ct^\alpha).$$

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4. **Lower bounds, in diagonal direction:**

$$\min\left\{\mathbb{P}\left(X_r - \mu r > Cr^{1/3}\right), \ \mathbb{P}\left(X_r - \mu r < -Cr^{1/3}\right)\right\} \geq \delta.$$

Main results: Upper tail

Main Theorem (Lower bound on upper tail)

For $t_0 < t < \Theta(r^{2/3})$,

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There exists $\zeta(\alpha) > 0$ *such that, for* $t_0 < t < r^{\zeta}$,

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \geq tr^{1/3}\right) \leq \exp\left(-ct^{3/2}(\log t)^{-1/2}\right).$$

Main results: Lower tail

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For $t > t_0$,

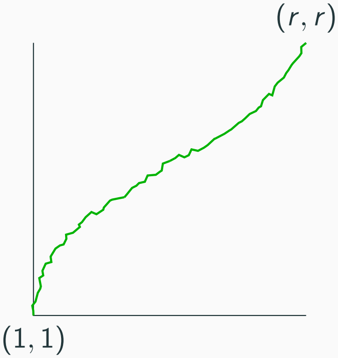
$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] \leq -tr^{1/3}\right) \leq \exp(-ct^3).$$

Proof strategies: Upper tail

An overarching theme for both tails

The basic theme is: look at the geodesic at smaller scales!

But which scale?



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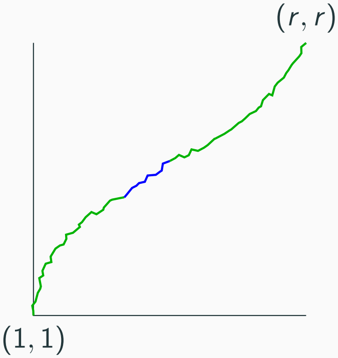
Consider a $1/k$ fraction of it.



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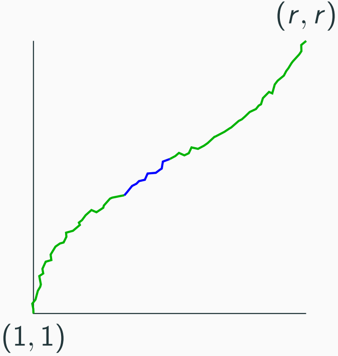

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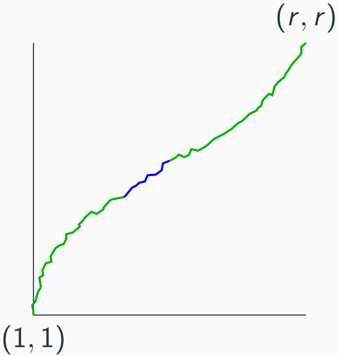

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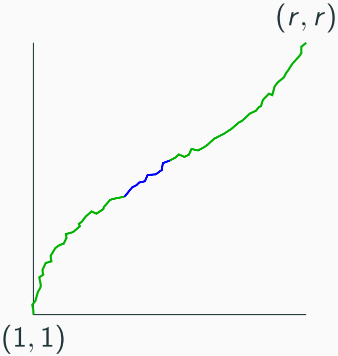
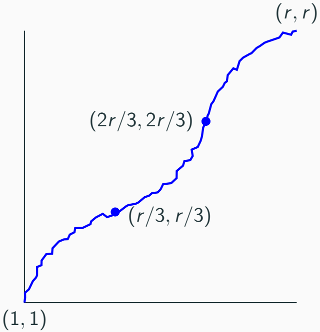
Equating,
$(r/k)^{1/3} \approx t r^{1/3}/k \implies \boxed{k \approx t^{3/2}}$



The easier half of upper tail: lower bound

Let $X^{(i)}_{r/k}$ be the LPP value from $i \cdot (r/k, r/k)$ to $(i+1) \cdot (r/k, r/k)$.
Independent!



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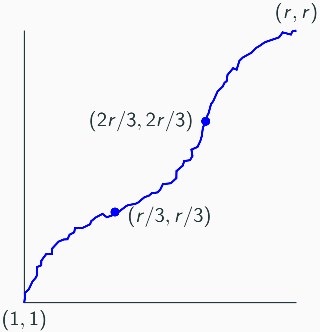

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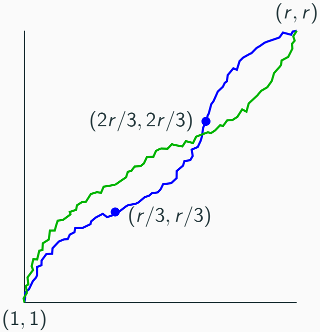

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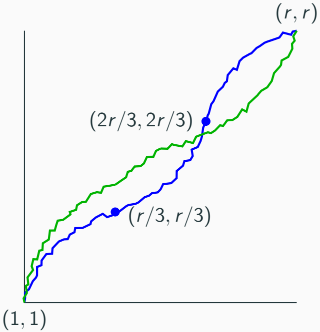

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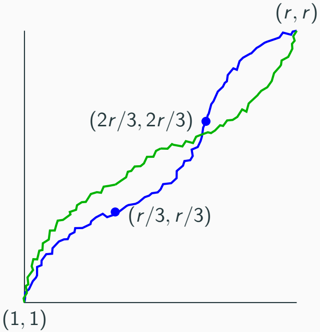

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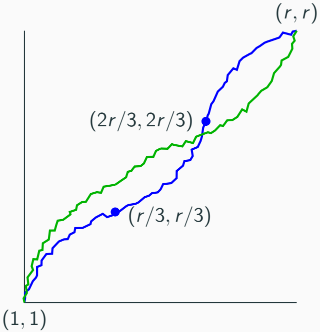

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Since $X_{r/k}^{(i)}$ are independent, by the assumed lower bound,

$$\mathbb{P}\left(X_r \geq \mu r + t r^{1/3}\right) \geq \prod_{i=0}^{k-1} \mathbb{P}\left(X_{r/k}^{(i)} \geq \mu r/k + C(r/k)^{1/3}\right)$$

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This is only for $t < \Theta(r^{2/3})$: $r/k = r t^{-3/2}$ has to be at least 1.

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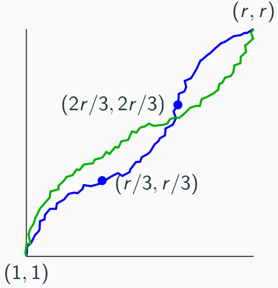
For illustration, suppose X_r were *sub*-additive: $X_r \leq \sum_{i=0}^{k-1} X_{r/k}^{(i)}$.



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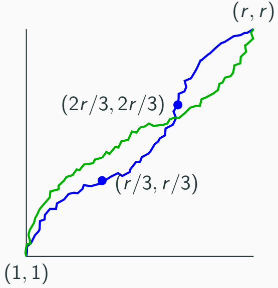

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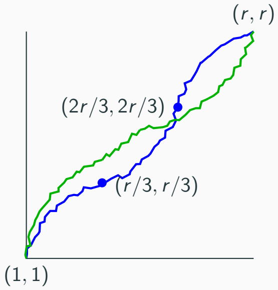

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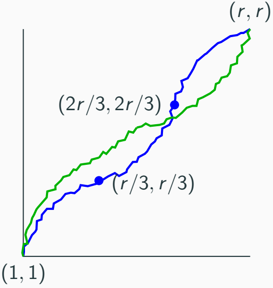

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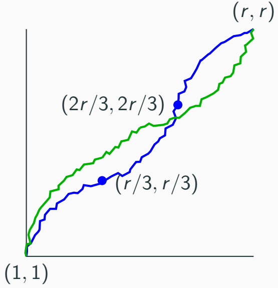
$$- k \cdot \left(\mu r/k - C(r/k)^{1/3} \right)$$

$$= \Theta(k^{2/3} r^{1/3}).$$



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$$\leq \mathbb{P}\left(\sum_{i=0}^{k-1} X_{r/k}^{(i)} \geq \mathbb{E}[X_r] + t r^{1/3}\right)$$
$$\leq \mathbb{P}\left(\sum_{i=0}^{k-1} \left(X_{r/k}^{(i)} - \mathbb{E}[X_{r/k}^{(i)}]\right) \geq t r^{1/3} - k^{2/3} r^{1/3}\right)$$

The hard half of upper tail: upper bound

Set $k = \Theta(t^{3/2})$ such that $k^{2/3}r^{1/3}$ is less than $\frac{1}{2}tr^{1/3}$.

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The is a sum of independent mean zero α-stretched exponentials.

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(Remember again $k = \Theta(t^{3/2})$.)

The hard half of upper tail: upper bound

So $\mathbb{P}\left(X_r > \mathbb{E}[X_r] + t r^{1/3}\right) \le \exp(-c t^{3\alpha/2})$.

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This is one round of the bootstrap: from tail exponent α to $3\alpha/2$.

Iterate till we get an exponent bigger than 1. One last round gives the exponent $3/2 \cdot 1 = 3/2$.

$$\implies \mathbb{P}\left(X_r > \mathbb{E}[X_r] + t r^{1/3}\right) \leq \exp(-c t^{3/2}).$$

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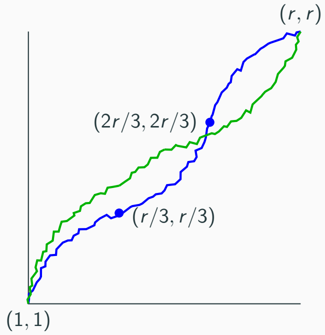
But $k = \Theta(t^{3/2})$, so this is $\exp(-ct^{3/2})$. Somewhat mysterious!

Dealing with the simplifications

We assumed that $X_r \leq \sum_{i=0}^{k-1} X_{r/k}^{(i)}$, which is not true.



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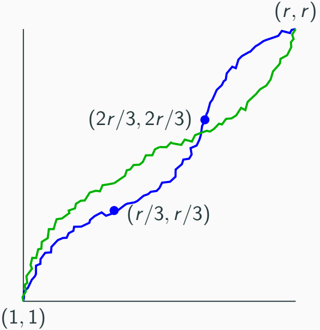

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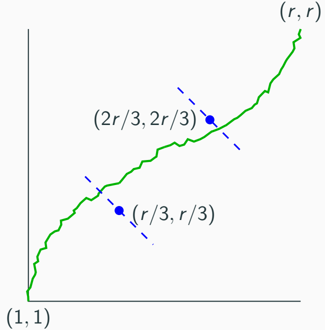

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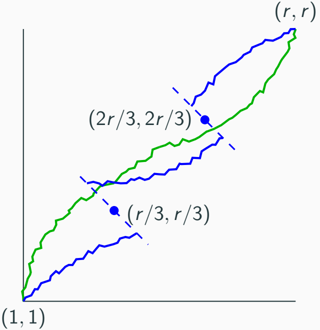
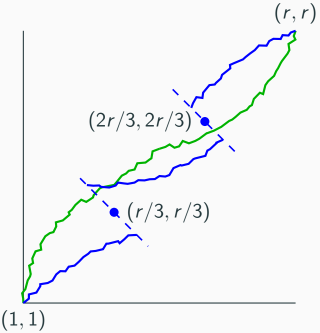


Interval-to-interval weights

Interval-to-interval weights have a sub-additive relation with X_r.

But we don't know which intervals the geodesic will pass through!



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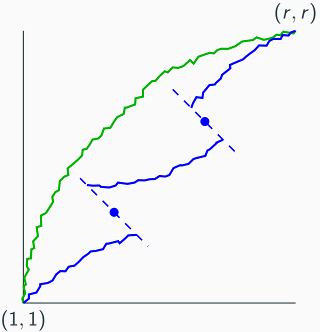
(With probability $\exp(-ct^{3/2})$, it can fluctuate $\text{poly}(t)r^{2/3}$.)

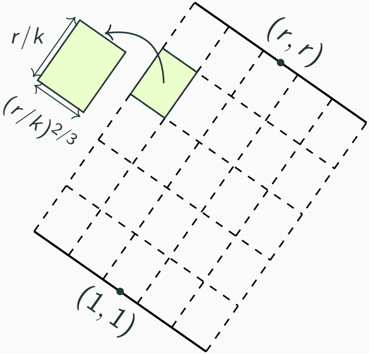


Interval-to-interval weights

So we do a grid-based discretization.

Its width is such that the geodesic exits the grid with probability at most $\exp(-ct^{3/2})$.

For any fixed choice of intervals, bootstrapping upgrades the tail.



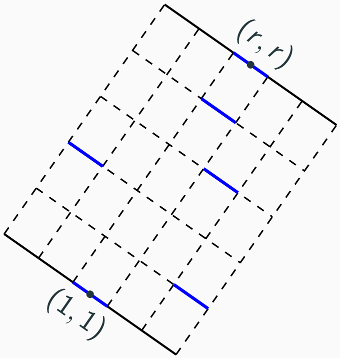
Interval-to-interval weights

So we also have to handle very zig-zaggy paths!

We need better tail bounds in *many* directions in each bootstrap round.

We also have to do a union bound over all possible choices of intervals.

The union bound entropy introduces the $(\log t)^{-1/2}$ factor.



Proof strategies: Lower tail

The exponent 3 vs 3/2

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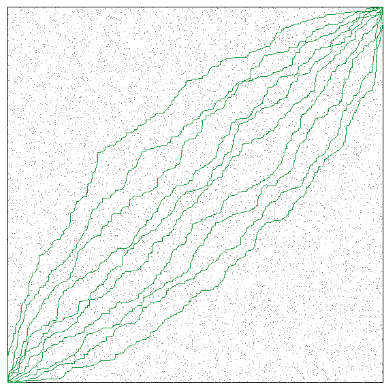
The lower tail makes *all* paths have low weight.

For the upper bound, we will find $t^{3/2}$ disjoint paths, each with weight at most $\mu r - tr^{1/3}$. The probability will be

$$\exp(-ct^{3/2} \cdot t^{3/2}) = \exp(-ct^3).$$

The k-geodesic watermelon

More precisely, consider the maximal weight collection of k disjoint paths between $(1, 1)$ and (r, r): the *k-geodesic watermelon*.



Constructing k good paths

The weight X_r^k of the k-melon was recently lower bounded by an explicit construction (under stronger assumptions, $\alpha = 3/2$).

Theorem (Basu-Ganguly-Hammond-H. '20)

For some $C < \infty$,

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Why $k^{5/3} r^{1/3}$? There are k paths, each has k subparts on scale r/k, which each lose $(r/k)^{1/3}$:

$$k \cdot k \cdot (r/k)^{1/3} = k^{5/3} r^{1/3}.$$

The k-melon to the lower tail's upper bound

Theorem (stated again)

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With this, the lower tail is easy. When $k = \Theta(t^{3/2})$, $k^{5/3} \approx kt$, so

$$\mathbb{P}\left(X_r \leq \mu r - tr^{1/3}\right) \leq \mathbb{P}\left(X_r^k \leq \mu rk - ktr^{1/3}\right)$$

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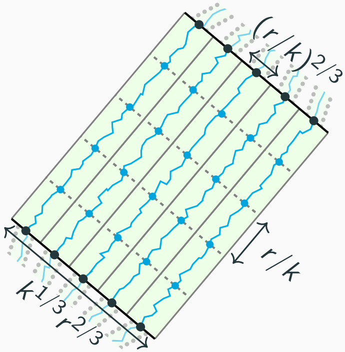
The construction, briefly



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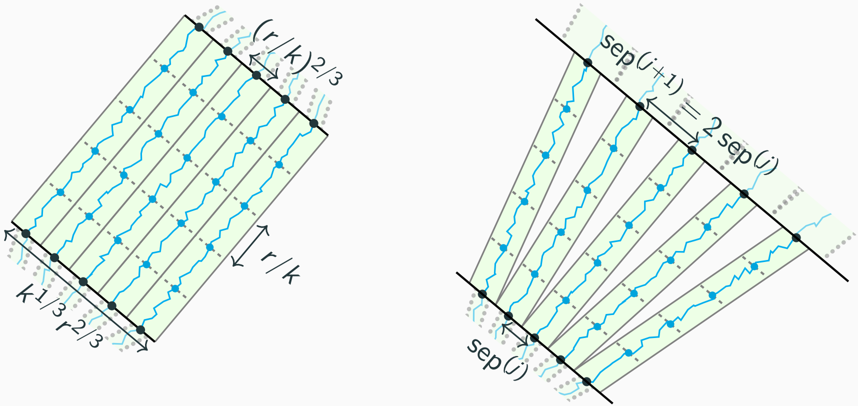

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1. Parabolic curvature (which we have).
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3. Lower bound on the constrained weight mean: follows from previous.

Lower bound on the lower tail

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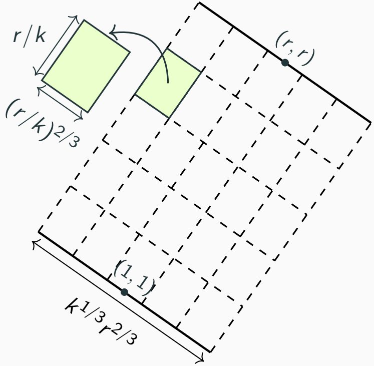
So only need to make the geodesic weight low when it's inside $\mathcal{R}$.

Another grid

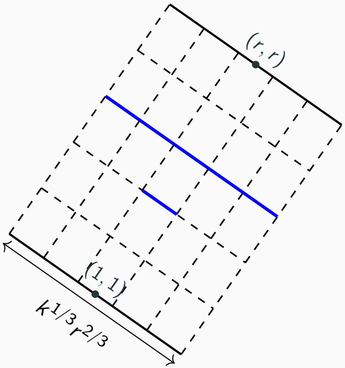
We divide $\mathcal{R}$ into a grid of intervals.

There are k rows, each with k intervals: k^2 cells total.



Another grid

Consider the event that the best weight from one interval to the next *row* is less than $\mu r/k - C(r/k)^{1/3}$.



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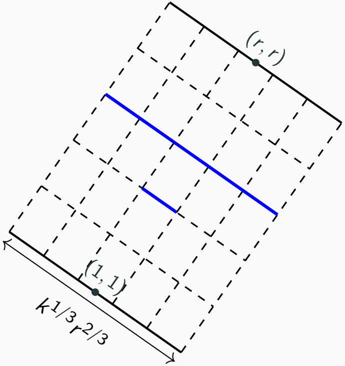

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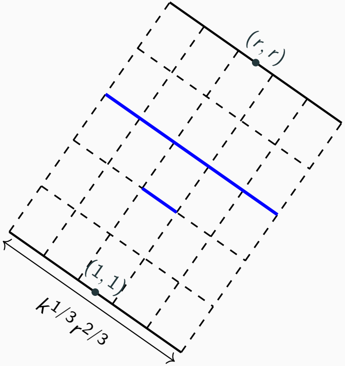

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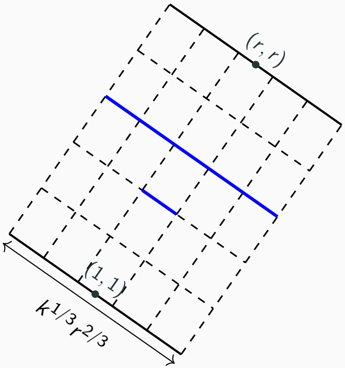

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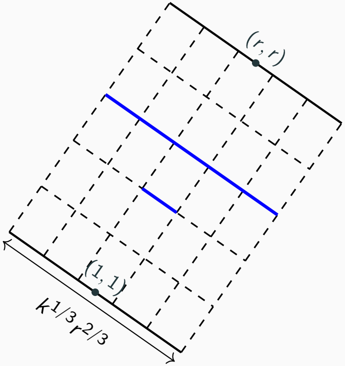

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If we ignore the unlikely event that the geodesic exits the rectangle $\mathcal{R}$, then

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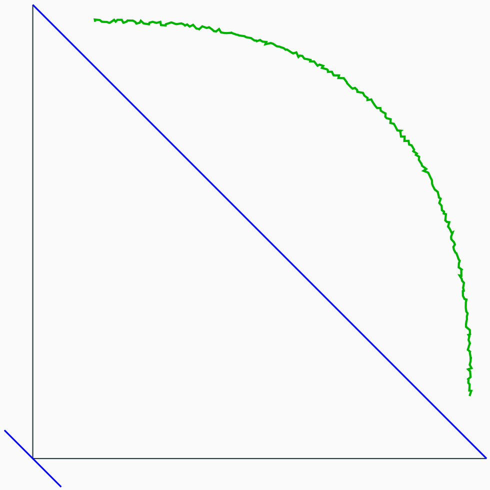
The interval-to-row lower bound



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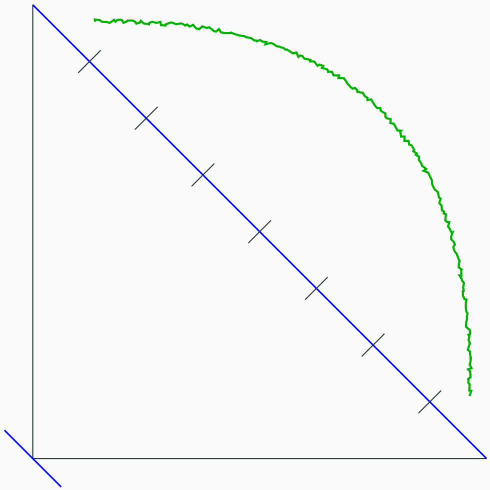

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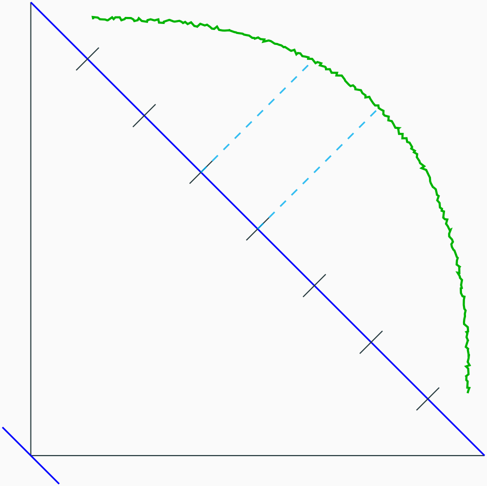

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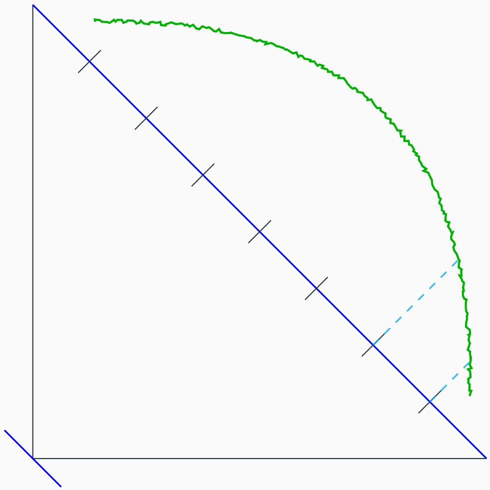

Conclusion

- Surprisingly, the upper and lower tail exponents of $3/2$ and 3 can be explained under natural assumptions by closely studying weight maximizing paths on appropriate scales.
- There is an unexpected connection to concentration of measure that plays an important role.
- The techniques are robust and should be applicable to other non-integrable contexts.

Thank you!

Selected References

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