Brownian structure in universal KPZ objects

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KPZ universality

The Kardar-Parisi-Zhang (KPZ) universality class is a class of stochastic growth models which share certain qualitative features:

- local smoothening
- slope dependent growth rate
- white noise roughening

It is expected that any model with these features exhibits *universal* behaviour independent of the precise details of the model.

Some models in the KPZ class

Totally Asymmetric Simple Exclusion Process

KPZ equation
$$\partial_t \mathcal{H} = \frac{1}{2}\partial_x^2 \mathcal{H} + \frac{1}{2}(\partial_x \mathcal{H})^2 + \xi$$

...

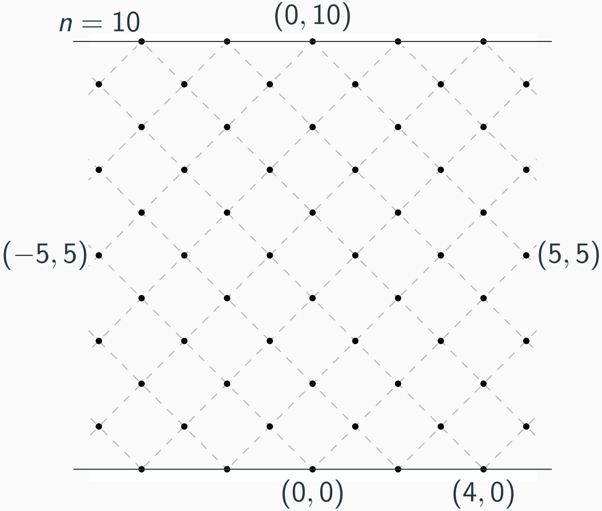
KPZ universality

Continuum Directed Random Polymer

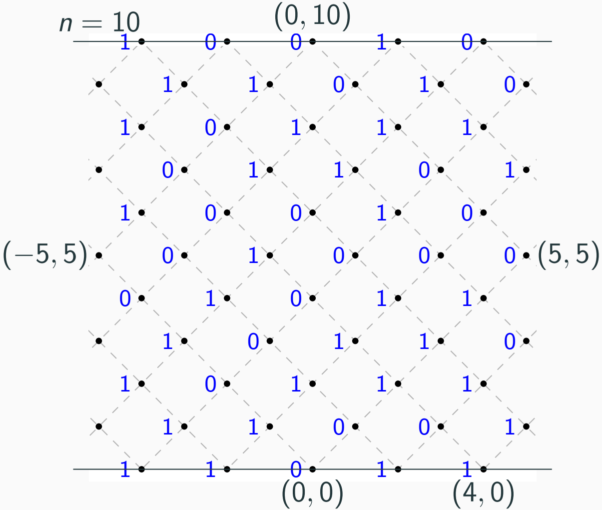
Last Passage Percolation

A simple last passage percolation model: Bernoulli LPP

Bernoulli LPP



Bernoulli LPP



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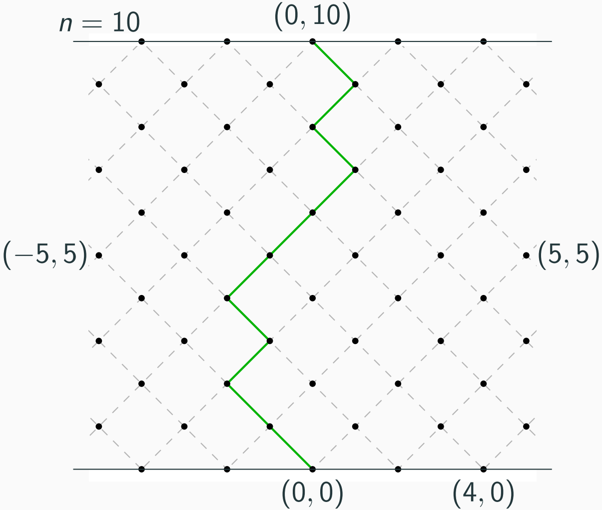
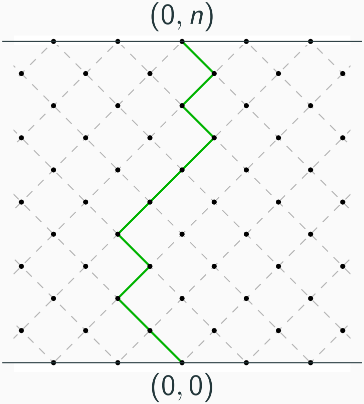


Bernoulli LPP

The energy of a path is the sum of the values of the covered vertices.

M_n is the maximum energy of paths which go from $(0,0)$ to $(0,n)$.



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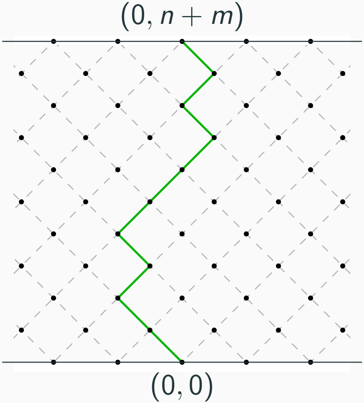
To first order, M_n is *linear* in n: $\lim_{n\to\infty} M_n/n = a$ almost surely.

This is because the subadditive ergodic theorem applies to $\{-M_n\}_n$:

$$M_{n+m} \overset{d}{\geq} M_n + \widetilde{M}_m.$$



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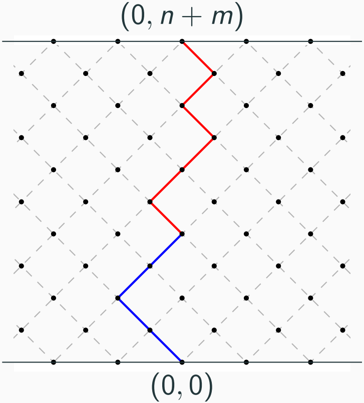


Bernoulli LPP

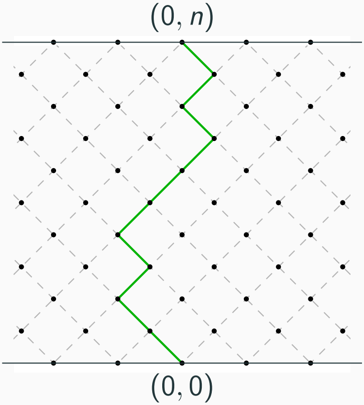
The scale of the second order term is $n^{1/3}$. That is,

$$n^{-1/3}(M_n - an) \text{ is tight in } n.$$

Define the weight

$$\text{Wgt}_n := b^{-1}n^{-1/3}(M_n - an).$$

For appropriate choice of b, Wgt_n is believed to converge to the GUE Tracy-Widom distribution.



Bernoulli LPP: Maximum energy profile

So far our endpoints were fixed at the unscaled coordinates $(0, 0)$ and $(0, n)$.

Denote the maximum energy with other endpoints $(x, 0)$ and (y, n) by $M_n\big[(x, 0) \to (y, n)\big]$.

If we scale $M_n[(0, 0) \to (y, n)]$ appropriately, it will be a tight sequence of functions.

Bernoulli LPP: Heuristic for spatial scaling

Assume M_n is "locally Brownian". Then we expect

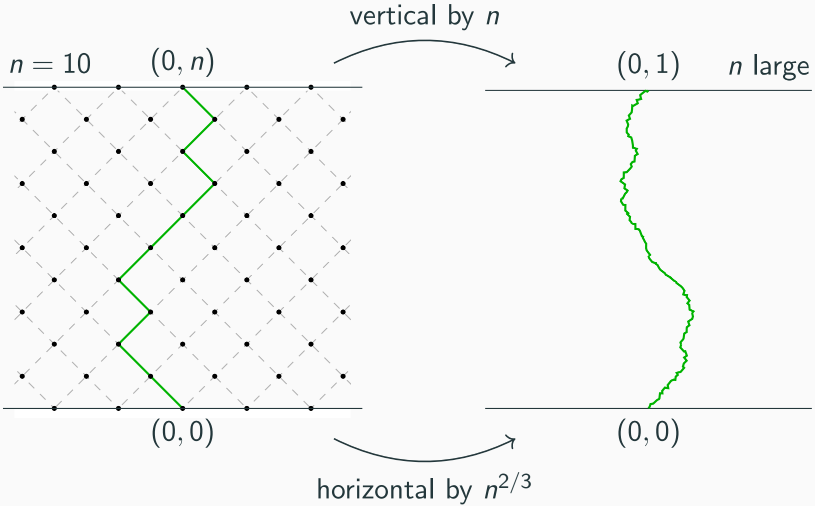
$$M_n[(0,0) \to (x,n)] - M_n[(0,0) \to (0,n)] \approx x^{1/2}.$$

Since the fluctuations of M_n are of order $n^{1/3}$, we expect

$$x^{1/2} \approx n^{1/3} \implies x \approx n^{2/3}.$$

This suggests the correct spatial scale is $n^{2/3}$.

Bernoulli LPP: Scalings



Bernoulli LPP: Weight profile

We define the weight profile respecting these scalings:

$$\mathrm{Wgt}_n\big[(x,0) \to (y,1)\big] = b^{-1} n^{-1/3} \Big(M_n\big[(xn^{2/3},0) \to (yn^{2/3},n)\big] - an \Big) .$$

This sequence of functions is expected to be tight in n.

The narrow-wedge weight profile and parabolic Airy_2

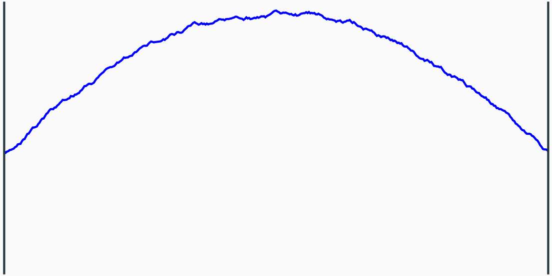
Fix $x = 0$. This is like a Dirac mass initial condition, and is called the narrow-wedge profile.

The sequence $\text{Wgt}_n[(0,0) \to (y,1)]$ is believed to be tight in n as a function of y.

The limit should be $\mathcal{A}(y) - y^2$, where $\mathcal{A}$ is the Airy_2 process. (This is known in Brownian LPP.)

What does the limiting weight profile look like?

This is a depiction of $\mathcal{A}(y) - y^2$:



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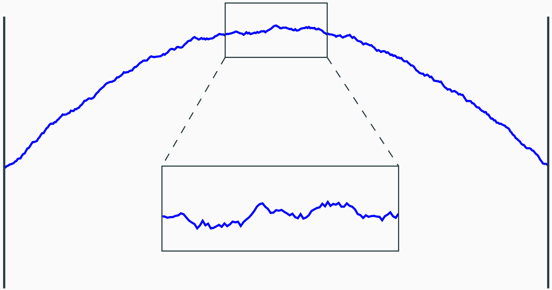

The limiting weight profile is globally parabolic, but locally Brownian.

What does locally Brownian mean?

Define $\mathcal{L}(y) = 2^{-1/2}\left(\mathcal{A}(y) - y^2\right)$. Fix $d > 0$ and let

- $\mathcal{C} = \left\{f : [-d, d] \to \mathbb{R}, \text{continuous}, f(-d) = 0\right\}$
- $\mathcal{B}$ be the law of standard (rate one) Brownian motion on $[-d, d]$.

Theorem (Corwin-Hammond)

Let $d > 0$. The law of $\mathcal{L}(\cdot) - \mathcal{L}(-d)$ is absolutely continuous to $\mathcal{B}$.

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Theorem (Corwin-Hammond)

Let $d > 0$. The law of $\mathcal{L}(\cdot) - \mathcal{L}(-d)$ is absolutely continuous to $\mathcal{B}$.

So if an event has probability *zero* under $\mathcal{B}$, it has probability zero under $\mathcal{L}(\cdot) - \mathcal{L}(-d)$ as well.

We would like to know more: how do low probability events compare quantitatively?

Main result

Main Theorem

Let $d \geq 1$*, let* A *be a Borel measurable subset of* $\mathcal{C}$*, and let* $\varepsilon = \mathcal{B}(A)$*. Then for all* $\varepsilon \in [0,1]$*,*

$$\mathbb{P}\Big(\mathcal{L}(\,\cdot\,) - \mathcal{L}(-d) \in A\Big) \leq \varepsilon \cdot (\textit{subpolynomial-in-}\varepsilon \textit{ error}).$$

- There is no condition (except measurability) on the set A.
- The subpolynomial error we prove is $\exp\big(Gd(\log \varepsilon^{-1})^{5/6}\big)$.
- Essentially the same bound applies to the *Brownian* LPP weight profile $\mathsf{Wgt}_n[(0,0) \to (y,1)]$.

Applications

A simple illustrative consequence

Corollary

Let $d \geq 1$. Then there exist $G < \infty$ and $x_0 > 0$ such that, for $x > x_0$,

$$\mathbb{P}\left(\sup_{s\in[-d,d]} |\mathcal{L}(s) - \mathcal{L}(-d)| \geq x\right) \leq e^{-x^2/4d + Gd^{1/6}x^{5/3}}.$$

A simple illustrative consequence

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Proof.

Enough to show $\mathbb{P}\left(\sup_{s \in [-d,d]} |B(s)| \geq x\right) \leq C \cdot e^{-x^2/4d}$.
This follows from the reflection principle for Brownian motion; the denominator of $4d$ is because the interval is of length $2d$. □

A quantified local version of Johansson's conjecture

Suppose we consider paths with starting point 0 and ending point *horizontally free*.

Where will the endpoint of maximum energy paths fall? Will it be unique?

The uniqueness of the endpoint is equivalent to the uniqueness of the maximiser of the weight profile.

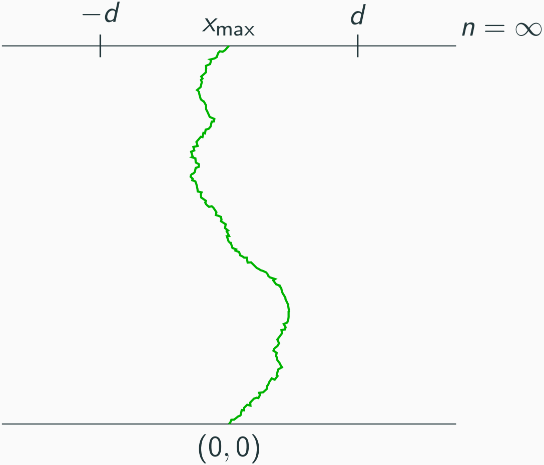
A quantified local version of Johansson's conjecture

Johansson conjectured that in the limiting narrow-wedge weight profile the maximiser is unique.

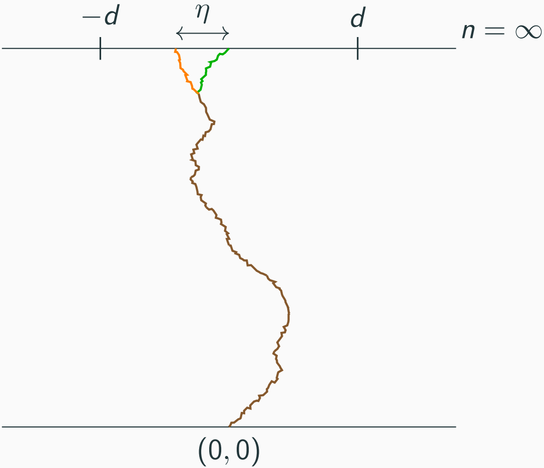
This fact now has a number of proofs, including by Corwin-Hammond, Flores-Quastel-Remenik, and Pimentel.

Theorem

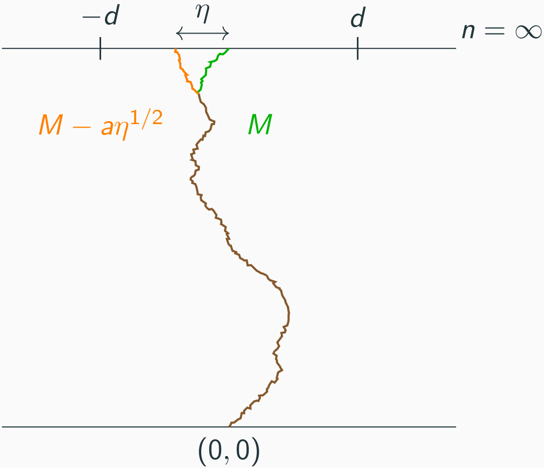
The process $y \mapsto \mathcal{A}(y) - y^2$ almost surely has a unique maximiser.



$-d$
$x_{\max}$
d
$n = \infty$
$(0, 0)$


$-d$
η
d
$n = \infty$
$(0, 0)$

$$M = \sup_{y\in[-d,d]} \mathcal{L}(y) = 2^{-1/2} \sup_{y\in[-d,d]} \left(\mathcal{A}(y) - y^2\right)$$



A quantified local version of Johansson's conjecture

We call this event the *near-touch* event. Precisely,

$$\mathsf{NT}(\mathcal{L},\eta,a) := \left\{ \sup_{|z|\geq 1} \mathcal{L}(x_{\max} + z\eta) \geq M - a\eta^{1/2} \right\}.$$

Theorem

Let $d \geq 1$ and $\eta \in (0,1)$. There exists $G < \infty$ such that, for all $a \in (0,1)$,

$$\mathbb{P}\big(\mathsf{NT}(\mathcal{L},\eta,a)\big) \leq a \cdot \exp\left(Gd(\log a^{-1})^{5/6}\right).$$

The proof is via the Brownian meander decomposition of Brownian motion around its maximiser to estimate the Brownian probability.

General initial conditions

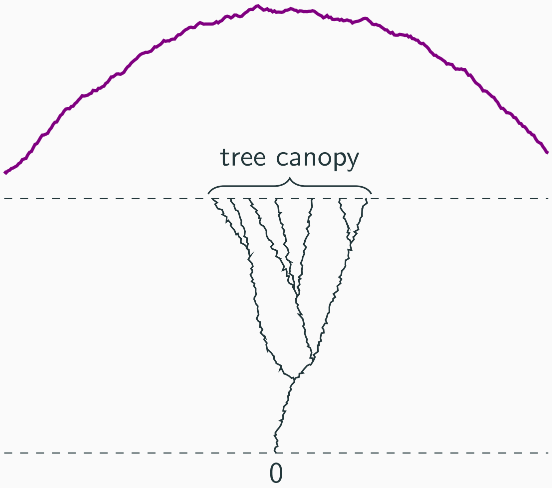
General initial conditions

Till now we discussed the specific initial condition of narrow-wedge and its limiting weight profile.

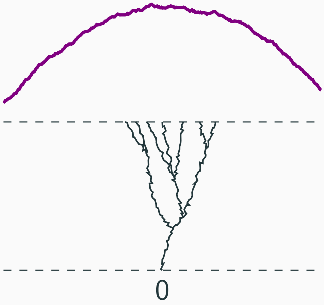
We would like to study the limiting weight profile which arises with other initial conditions as well.

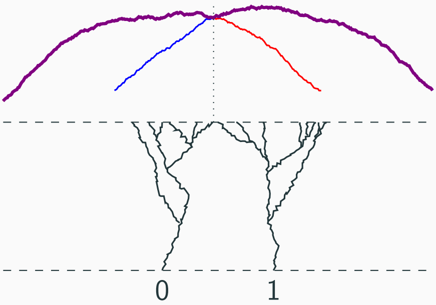
Proof of its existence makes use of the recent advance of Dauvergne, Ortmann, and Virág, which constructs the full scaling limit of Brownian LPP, the *directed landscape*.

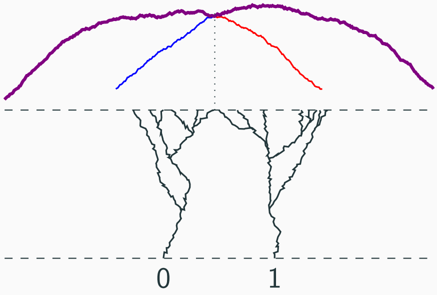
Tree structure in narrow-wedge



Finding a polymer forest







Idea: Can break up the general weight profile into (a random number of) patches whose boundaries correspond to boundaries of polymer tree canopies.

In each patch, the restriction of the profile (a fabric piece) looks like the narrow-wedge profile.

We get a patchwork quilt of Brownian fabrics.

Result for general initial conditions

Let $f : \mathbb{R} \to \mathbb{R}$ be a given function which is the initial condition. The *f-rewarded weight* of a given path started at x under f will be

$$f(x) + \text{weight collected by path.}$$

$\mathrm{Wgt}^f_\infty[(*, 0) \to (y, 1)]$ is the limiting maximum f-rewarded weight of all paths ending at y.

Theorem (Informal extension of [Ham19]'s result)

The function $y \mapsto \mathrm{Wgt}^f_\infty[(*, 0) \to (y, 1)]$ is Brownian motion patchwork quiltable; the Brownian comparison in each patch may be made in L^{3-}, and the random number of patches has polynomial tail with exponent $2-$.

A conjecture of greater Brownian regularity

Conjecture (Informal statement from [Ham19])

Let $A \subseteq \mathcal{C}$ and $\varepsilon = \mathcal{B}(A)$. For a wide class of f, we have

$$\mathbb{P}\Big(\mathrm{Wgt}^f_\infty[(*,0) \to (\,\cdot, 1)] - \mathrm{Wgt}^f_\infty[(*,0) \to (-1,1)] \in A\Big) \\ \leq \varepsilon \cdot (\textit{subpolynomial-in-}\varepsilon \textit{ error}).$$

This would essentially be our main theorem, but for general initial conditions.

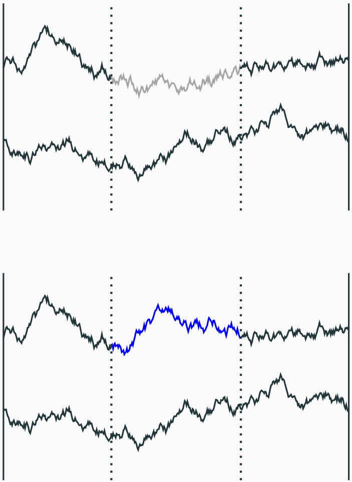
In terms of the patchwork quilt, this means proving that *one* patch is sufficient and has Radon-Nikodym derivative in $L^{\infty-}$.

Proof flavours: The Brownian Gibbs property

The Brownian Gibbs Property

Embed $\mathcal{L}$ in a system of infinite non-intersecting curves with the Brownian Gibbs property:

The conditional distribution of the **gray**, given the **black**, is a **Brownian bridge** with the given end points, conditioned to not intersect the lower curve.



High level proof outline

1. Consider a “half-way house” process J in between $\mathcal{L}$ and Brownian motion.
2. Prove the estimate for J using its Brownian Gibbs property.
3. Transfer the estimate back to $\mathcal{L}$ using results from [Ham19].

Thank you!

Selected References

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