Fractal dimension of multiple maximizers in the KPZ fixed point

Milind Hegde
(joint work with Ivan Corwin, Alan Hammond, and Konstantin Matetski)

Berkeley Probability Seminar
January 27, 2021

KPZ universality

The Kardar-Parisi-Zhang (KPZ) universality class is a class of stochastic growth models which share certain qualitative features:

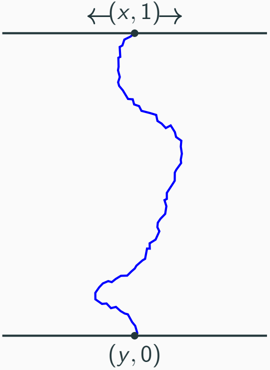
- local smoothening
- slope dependent growth rate
- white noise roughening

It is expected that any model with these features exhibits *universal* behaviour—such as the KPZ fixed point as a certain scaling limit—independent of the precise details of the model.

An expository continuum LPP model

Consider noise defined on $\mathbb{R} \times [0,1]$; its distribution is unimportant (for our expository purposes).

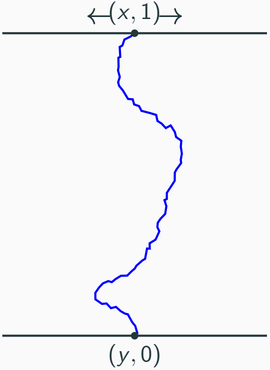
Directed paths $\gamma : [0,1] \to \mathbb{R}$ are given by functions: $\gamma(t)$ is the position at height t.



An expository continuum LPP model

Each path is assigned a *weight* based on the environment it traverses.

$\mathcal{L}(y,0;x,1)$ is the *maximum* weight over all paths from $(y,0)$ to $(x,1)$.

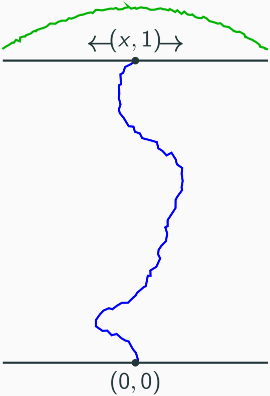


An expository continuum LPP model

Fix $y = 0$, and consider the *weight profile* $x \mapsto \mathfrak{h}_1(x) = \mathcal{L}(0,0;x,1)$.

This is the parabolic Airy$_2$ process, the KPZ fixed point from narrow-wedge initial condition at time one.

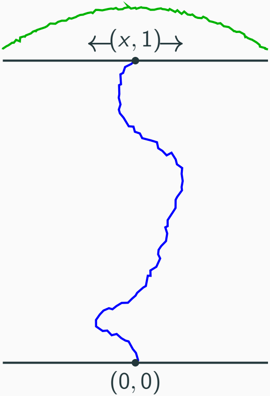
Its maximizer is where a polymer with *unconstrained* endpoint concludes.



An expository continuum LPP model

Johansson conjectured in 2003 that $\mathfrak{h}_1$ has a unique maximizer a.s.; the unconstrained polymer endpoint is unique.

Proved in 2014 by Corwin-Hammond, and subsequently by Moreno Flores-Quastel--Remenik and Pimentel.



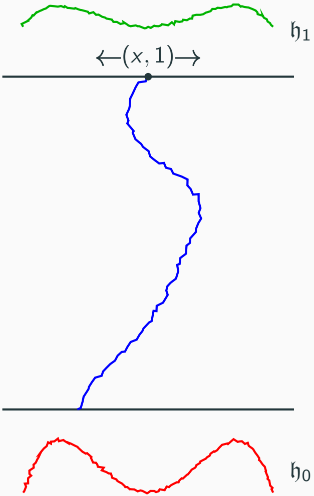
An expository continuum LPP model

We can consider other initial conditions with the starting point not fixed.

Let $\mathfrak{h}_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ specify an auxiliary weight added to paths based on their starting point.

The maximization is done with the full weight:

$$\mathfrak{h}_1(x) = \sup_{y \in \mathbb{R}} \Big\{ \mathfrak{h}_0(y) + \mathcal{L}(y, 0; x, 1) \Big\}$$



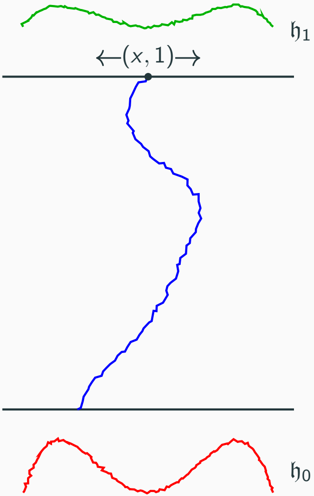
An expository continuum LPP model

$$\mathfrak{h}_1(x) = \sup_{y\in\mathbb{R}}\Big\{\mathfrak{h}_0(y) + \mathcal{L}(y,0;x,1)\Big\}$$

$\mathfrak{h}_1$ is the KPZ fixed point (at time one).

First constructed in terms of Fredholm determinantal formulas via TASEP by Matetski-Quastel-Remenik.

The variational formula uses the directed landscape $\mathcal{L}$, constructed by Dauvergne-Ortmann-Virág, and related to TASEP by Nica-Quastel-Remenik.



Uniqueness of maximizer for the KPZ fixed point

Johansson's conjecture is true for a much wider class of initial conditions:

Theorem (Corwin-Hammond-H.-Matetski)

Under mild conditions on $\mathfrak{h}_0$, $\mathfrak{h}_1$ *has a unique maximizer almost surely.*

The full KPZ fixed point

We previously set the height at 1; but we can allow any height in $(0, \infty)$.

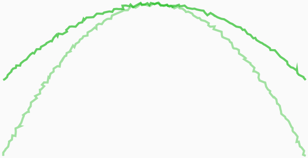
Doing so we can define

$$\mathfrak{h}_t(x) = \sup_{y \in \mathbb{R}} \Big\{ \mathfrak{h}_0(y) + \mathcal{L}(y, 0; x, t) \Big\}.$$

The KPZ fixed point is $t \mapsto \mathfrak{h}_t$.

The full KPZ fixed point

We previously set the height at 1; but we can allow any height in $(0, \infty)$.



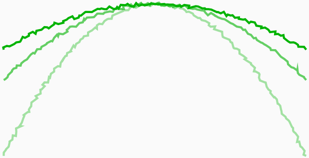
Doing so we can define

$$\mathfrak{h}_t(x) = \sup_{y \in \mathbb{R}} \Big\{ \mathfrak{h}_0(y) + \mathcal{L}(y, 0; x, t) \Big\}.$$

The KPZ fixed point is $t \mapsto \mathfrak{h}_t$.

The full KPZ fixed point

We previously set the height at 1; but we can allow any height in $(0, \infty)$.



Doing so we can define

$$\mathfrak{h}_t(x) = \sup_{y \in \mathbb{R}} \Big\{ \mathfrak{h}_0(y) + \mathcal{L}(y, 0; x, t) \Big\}.$$

The KPZ fixed point is $t \mapsto \mathfrak{h}_t$.

Exceptional times

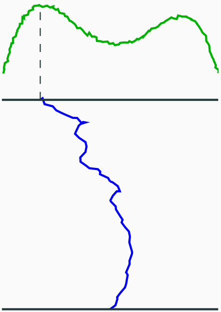
With a time evolution, we can ask whether there exist random times when $x \mapsto \mathfrak{h}_t(x)$ has multiple maximizers.

As a rough analogue, Brownian motion has probability zero of being at the origin at fixed time, but still equals zero at random times. The Hausdorff dimension of the zero set is $\frac{1}{2}$.

Or, dynamical critical percolation has an infinite cluster at a fixed time with probability zero, but still a.s. has exceptional times when an infinite cluster appears. The Hausdorff dimension of such times is $\frac{31}{36}$ on the honeycomb lattice.

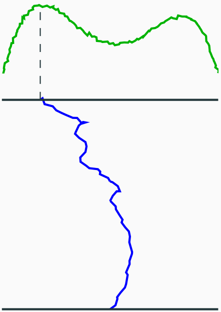
Polymer instability at exceptional times

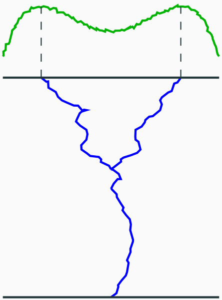
Exceptional times of multiple maximizers are times of polymer instability.



Polymer instability at exceptional times

Exceptional times of multiple maximizers are times of polymer instability.

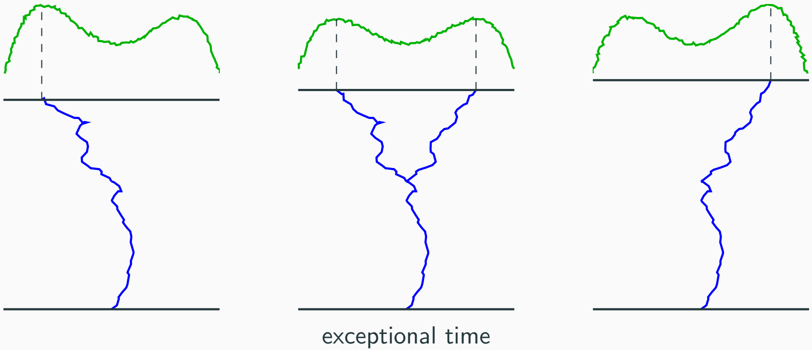




exceptional time

Polymer instability at exceptional times

Exceptional times of multiple maximizers are times of polymer instability.



Our result

Assumption: There exist $\gamma > 0$ and $\lambda \in \mathbb{R}$ such that
(i) $\mathfrak{h}_0(x) \leq -\gamma x^2$ for all $x \in \mathbb{R}$ and (ii) $\mathfrak{h}_0(x) = -\infty$ for $x \leq -\lambda$.

For $A \geq 0$ and $T > 0$, define

$$\mathcal{T}_A = \Big\{ t \in [0, T] : \mathfrak{h}_t \text{ has two maximizers at distance } > A \Big\}.$$

Theorem (Corwin-Hammond-H.-Matetski)

Under this assumption, $\mathcal{T}_A \neq \emptyset$ with positive probability.
Conditional on this, its Hausdorff dimension is almost surely $\frac{2}{3}$.

Discussion of the result

Theorem (Corwin-Hammond-H.-Matetski)

Under this assumption, $\mathcal{T}_A \neq \emptyset$ with positive probability. Conditional on this, its Hausdorff dimension is almost surely $\frac{2}{3}$.

Corollary

For narrow-wedge initial condition, $\mathcal{T}_{A=0} \neq \emptyset$ almost surely, and $\dim(\mathcal{T}_{A=0}) = \frac{2}{3}$ almost surely.

Conjecture: $\mathcal{T}_{A=0}$ is almost surely dense for all initial conditions satisfying the assumption.

Discussion of the result

For $A > 0$, it is not the case that $\mathcal{T}_A \neq \emptyset$ almost surely.

This is analogous to dynamical critical percolation: with probability one there are exceptional times with an infinite cluster somewhere (and these times are dense).

But it is only with positive probability that there is an exceptional time which has an infinite cluster containing a *given* face.

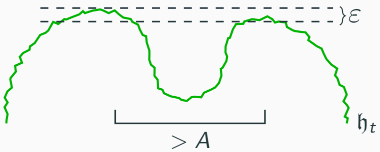
A heuristic for $\frac{2}{3}$

A Hausdorff dimension of $\frac{2}{3}$ roughly means that we need ε^{-2} sets of diameter ε^3 to cover $\mathcal{T}_A$: $\varepsilon^{3\alpha-2} \to 0$ as $\varepsilon \to 0$ for any $\alpha > 2/3$.

A heuristic for $\frac{2}{3}$

A Hausdorff dimension of $\frac{2}{3}$ roughly means that we need ε^{-2} sets of diameter ε^3 to cover $\mathcal{T}_A$: $\varepsilon^{3\alpha-2} \to 0$ as $\varepsilon \to 0$ for any $\alpha > 2/3$.

Consider a proxy set $\mathcal{T}_A^\varepsilon$ of times when ε-*twin peaks* occurs:
$t \in \mathcal{T}_A^\varepsilon$ if $\mathfrak{h}_t \in \mathrm{TP}_A^\varepsilon$, i.e.,



A heuristic for $\frac{2}{3}$

(i) $\mathfrak{h}_t(x)$ is Hölder-$\frac{1}{3}^-$ in t

(i) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at least order ε^3.

A heuristic for $\frac{2}{3}$

(i) $\mathfrak{h}_t(x)$ is Hölder-$\frac{1}{3}^{-}$ in t

(i) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at least order ε^3.

(ii) $\mathbb{P}(\mathfrak{h}_t \in \mathrm{TP}_A^\varepsilon) \leq ct^{-1/3}\varepsilon$

(ii) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at most order ε^3.

A heuristic for $\frac{2}{3}$

(i) $\mathfrak{h}_t(x)$ is Hölder-$\frac{1}{3}^-$ in t

(i) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at least order ε^3.

(ii) $\mathbb{P}(\mathfrak{h}_t \in \mathsf{TP}_A^\varepsilon) \leq ct^{-1/3}\varepsilon$

(ii) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at most order ε^3.

(iii) $\mathbb{P}(\mathfrak{h}_t \in \mathsf{TP}_A^\varepsilon) \geq c(t)\varepsilon$.

(iii) $\implies$ $\mathcal{T}_A^\varepsilon$ has size order ε (eg. in expectation by Fubini).

A heuristic for $\frac{2}{3}$

(i) $\mathfrak{h}_t(x)$ is Hölder-$\frac{1}{3}^-$ in t

(i) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at least order ε^3.

(ii) $\mathbb{P}(\mathfrak{h}_t \in \mathrm{TP}_A^\varepsilon) \leq ct^{-1/3}\varepsilon$

(ii) $\implies$ intervals in $\mathcal{T}_A^\varepsilon$ have size at most order ε^3.

(iii) $\mathbb{P}(\mathfrak{h}_t \in \mathrm{TP}_A^\varepsilon) \geq c(t)\varepsilon$.

(iii) $\implies$ $\mathcal{T}_A^\varepsilon$ has size order ε (eg. in expectation by Fubini).

Thus we need $\varepsilon/\varepsilon^3 = \varepsilon^{-2}$ intervals of diameter ε^3 to cover $\mathcal{T}_A^\varepsilon$.

General strategy for proving Hausdorff dimension

Upper bound on HD is easier: identify an efficient covering.
Accomplished for us with steps (i) and (ii).

General strategy for proving Hausdorff dimension

Upper bound on HD is easier: identify an efficient covering. Accomplished for us with steps (i) and (ii).

Lower bounds are harder. Usually have to find a measure supported on the fractal set which is well-spread out.

General strategy for proving Hausdorff dimension

Upper bound on HD is easier: identify an efficient covering. Accomplished for us with steps (i) and (ii).

Lower bounds are harder. Usually have to find a measure supported on the fractal set which is well-spread out.

To prove the “spread-out”-ness, a precise estimate of decorrelation across times needed: captured here in step (ii) upper bound’s $t^{-1/3}$ factor.

Step (iii) lower bound proves the measure has non-trivial mass.

Difficulty of the steps

Step (i) on Hölder continuity of $t \mapsto \mathfrak{h}_t(x)$ is straightforward, via the variational formula and known properties of $\mathcal{L}$.

Steps (ii) and (iii) on upper & lower bounds on $\mathbb{P}(\mathfrak{h}_t \in \mathsf{TP}^{\varepsilon}_{A})$ are much more difficult.

Difficulty of the steps

Step (i) on Hölder continuity of $t \mapsto \mathfrak{h}_t(x)$ is straightforward, via the variational formula and known properties of $\mathcal{L}$.

Steps (ii) and (iii) on upper & lower bounds on $\mathbb{P}(\mathfrak{h}_t \in \mathsf{TP}^{\varepsilon}_{A})$ are much more difficult.

The upper bound uses exact transition probability formulas of the KPZ fixed point $t \mapsto \mathfrak{h}_t$ given in terms of Fredholm determinants.

Difficulty of the steps

Step (i) on Hölder continuity of $t \mapsto \mathfrak{h}_t(x)$ is straightforward, via the variational formula and known properties of $\mathcal{L}$.

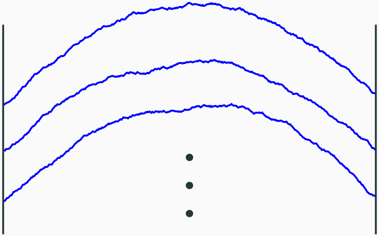
Steps (ii) and (iii) on upper & lower bounds on $\mathbb{P}(\mathfrak{h}_t \in \mathsf{TP}_A^{\varepsilon})$ are much more difficult.

The upper bound uses exact transition probability formulas of the KPZ fixed point $t \mapsto \mathfrak{h}_t$ given in terms of Fredholm determinants.

The lower bound relies on a Brownian Gibbs resampling argument.

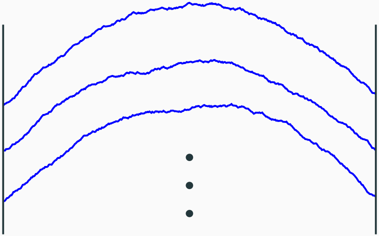
Outline of lower bound for narrow-wedge

The narrow-wedge case is special and easier: here $\mathfrak{h}_1$ can be embedded as the top curve of the *parabolic Airy line ensemble* $\mathcal{P}$.



Outline of lower bound for narrow-wedge

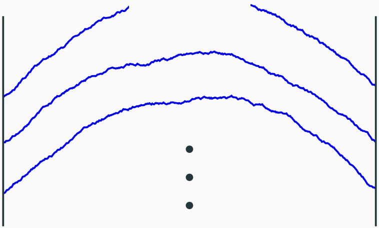
The narrow-wedge case is special and easier: here $\mathfrak{h}_1$ can be embedded as the top curve of the *parabolic Airy line ensemble* $\mathcal{P}$.



This ensemble has the Brownian Gibbs property.

Outline of lower bound for narrow-wedge

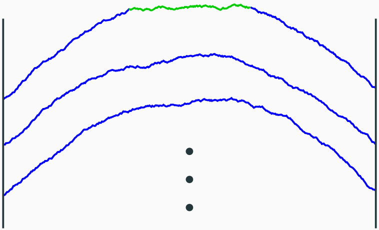
The narrow-wedge case is special and easier: here $\mathfrak{h}_1$ can be embedded as the top curve of the *parabolic Airy line ensemble* $\mathcal{P}$.



This ensemble has the Brownian Gibbs property.

Outline of lower bound for narrow-wedge

The narrow-wedge case is special and easier: here $\mathfrak{h}_1$ can be embedded as the top curve of the *parabolic Airy line ensemble* $\mathcal{P}$.



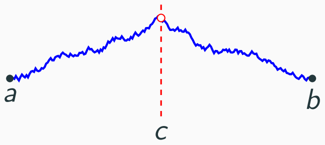
This ensemble has the Brownian Gibbs property.

Brownian bridge recap

Suppose you have a Brownian bridge B on $[a, b]$. You can condition on the "side bridges" on $[a, c]$ and $[c, b]$.

The only remaining information, $B(c)$, is an *independent Gaussian*.

The value of $B(c)$ allows you to reconstruct B on $[a, b]$ by affine shifts of side bridges.

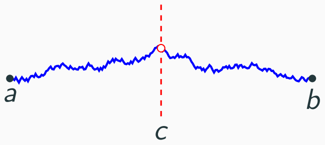


Brownian bridge recap

Suppose you have a Brownian bridge B on $[a, b]$. You can condition on the "side bridges" on $[a, c]$ and $[c, b]$.

The only remaining information, $B(c)$, is an *independent Gaussian*.

The value of $B(c)$ allows you to reconstruct B on $[a, b]$ by affine shifts of side bridges.

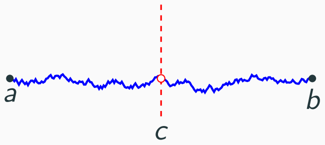


Brownian bridge recap

Suppose you have a Brownian bridge B on $[a, b]$. You can condition on the "side bridges" on $[a, c]$ and $[c, b]$.

The only remaining information, $B(c)$, is an *independent Gaussian*.

The value of $B(c)$ allows you to reconstruct B on $[a, b]$ by affine shifts of side bridges.

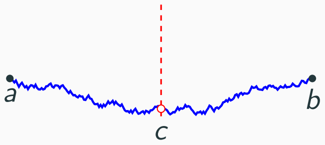


Brownian bridge recap

Suppose you have a Brownian bridge B on $[a, b]$. You can condition on the "side bridges" on $[a, c]$ and $[c, b]$.

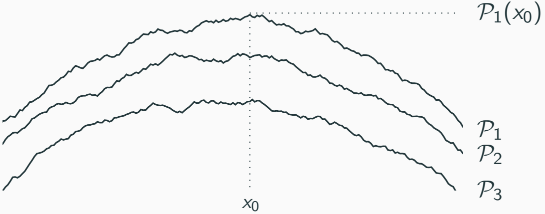
The only remaining information, $B(c)$, is an *independent Gaussian*.

The value of $B(c)$ allows you to reconstruct B on $[a, b]$ by affine shifts of side bridges.



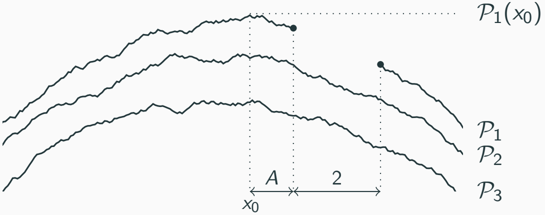
The resampling argument

Let x_0 be the maximizer of $\mathfrak{h}_1 = \mathcal{P}_1$. We want to show that, with probability at least order ε, $\mathcal{P}_1$ has ε-twin peaks.



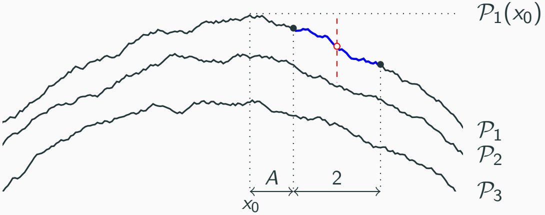
The resampling argument

Let x_0 be the maximizer of $\mathfrak{h}_1 = \mathcal{P}_1$. We want to show that, with probability at least order ε, $\mathcal{P}_1$ has ε-twin peaks.



The resampling argument

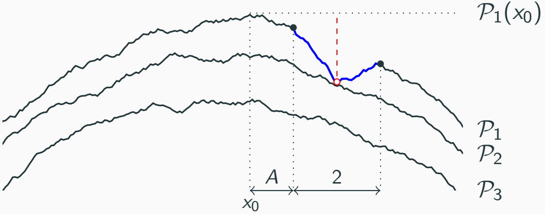
Let x_0 be the maximizer of $\mathfrak{h}_1 = \mathcal{P}_1$. We want to show that, with probability at least order ε, $\mathcal{P}_1$ has ε-twin peaks.



The location of the bead controls $\mathcal{P}_1$ on $[x_0 + A + 1, x_0 + A + 2]$.

The resampling argument

Let x_0 be the maximizer of $\mathfrak{h}_1 = \mathcal{P}_1$. We want to show that, with probability at least order ε, $\mathcal{P}_1$ has ε-twin peaks.

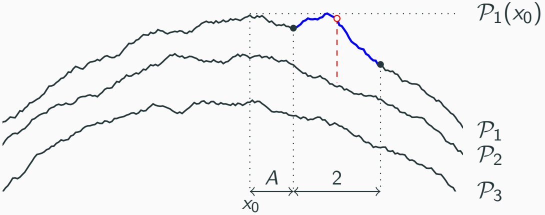


The location of the bead controls $\mathcal{P}_1$ on $[x_0 + A + 1, x_0 + A + 2]$.

Its distribution is Gaussian, *conditioned on lying in an interval* (dashed red line).

The resampling argument

Let x_0 be the maximizer of $\mathfrak{h}_1 = \mathcal{P}_1$. We want to show that, with probability at least order ε, $\mathcal{P}_1$ has ε-twin peaks.

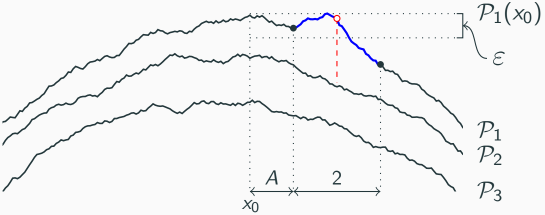


The location of the bead controls $\mathcal{P}_1$ on $[x_0 + A + 1, x_0 + A + 2]$.

Its distribution is Gaussian, *conditioned on lying in an interval* (dashed red line).

The resampling argument

Let x_0 be the maximizer of $\mathfrak{h}_1 = \mathcal{P}_1$. We want to show that, with probability at least order ε, $\mathcal{P}_1$ has ε-twin peaks.



The location of the bead controls $\mathcal{P}_1$ on $[x_0 + A + 1, x_0 + A + 2]$.

Its distribution is Gaussian, *conditioned on lying in an interval* (dashed red line).

Difficulties of the general case

The main difficulty is that there is no corresponding ensemble with the Brownian Gibbs property for general initial condition.

But an important recent discovery of Dauvergne-Ortmann-Virág allows the expression of $\mathfrak{h}_1$ as a complicated functional of $\mathcal{P}$.

The general case proof comes down to understanding the effect of the above resampling on the functional.

Thank you!

Selected References

Ivan Corwin, Alan Hammond, Milind Hegde, Konstantin Matetski (2021)
Exceptional times when the KPZ fixed point violates Johansson's conjecture on maximizer uniqueness.
arXiv preprint 2101.04205.

Duncan Dauvergne, Janosch Ortmann, Bálint Virág (2018)
The directed landscape.
arXiv preprint 1812.00309.

Konstantin Matetski, Jeremy Quastel, Daniel Remenik (2017)
The KPZ fixed point.
To appear in Acta Mathematica.