The Airy difference profile & Brownian local time

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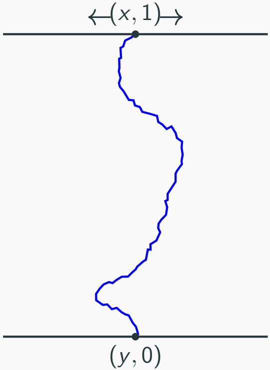
An expository continuum last passage percolation model

Consider noise defined on $\mathbb{R} \times [0,1]$; its distribution is unimportant (for our expository purposes).

Directed paths $\gamma : [0,1] \to \mathbb{R}$ are given by functions: $\gamma(t)$ is the position at height t.

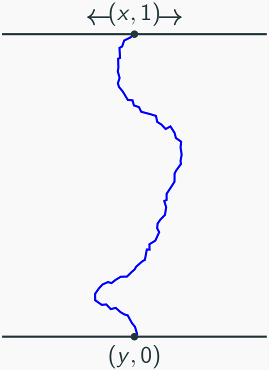


An expository continuum last passage percolation model

Each path is assigned a *weight* based on the environment it traverses.

$\mathcal{S}(y, x)$ is the *maximum* weight over all paths from $(y, 0)$ to $(x, 1)$.

The maximizing path is called a *geodesic*.



An expository continuum last passage percolation model

Fix y, and consider the *weight profile* $x \mapsto \mathcal{P}_1(x) = \mathcal{S}(y, x)$.

For each y, this is a parabolic Airy_2 process.

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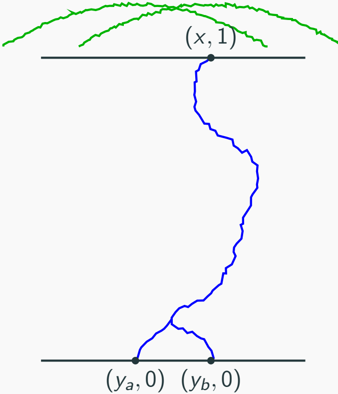
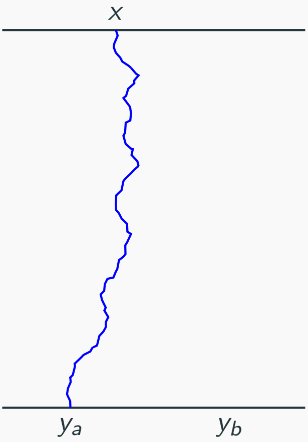
$\mathcal{S}$ provides a coupling of these parabolic Airy_2 processes.



What is the coupling structure?

Fix $y_a < y_b$.



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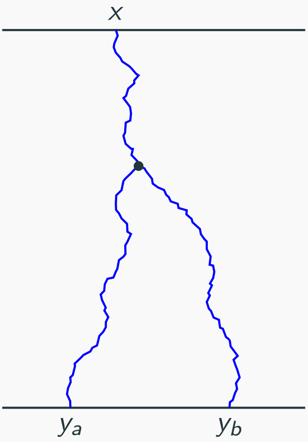

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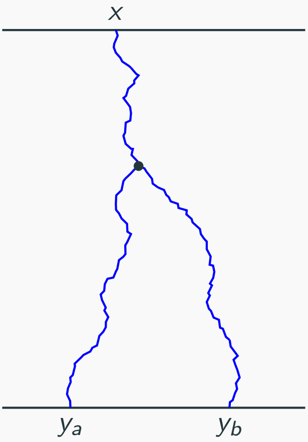

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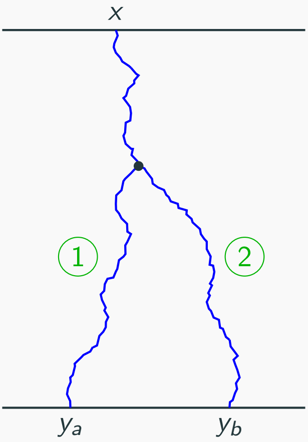

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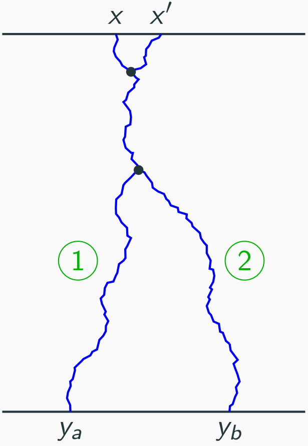
$$= \text{②} - \text{①}$$



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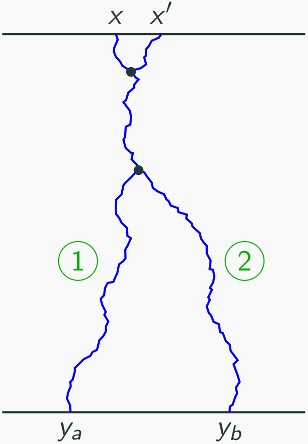

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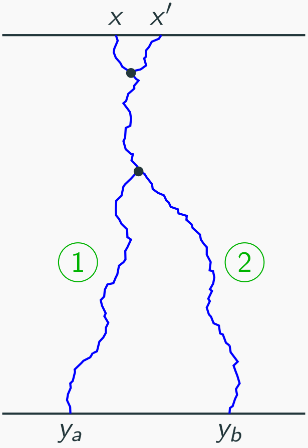
This suggests that

$$\mathcal{D}(x) := \mathcal{S}(y_b, x) - \mathcal{S}(y_a, x)$$

is constant a.e.



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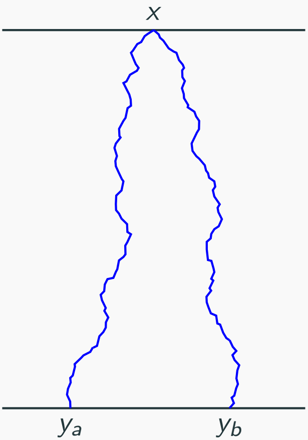
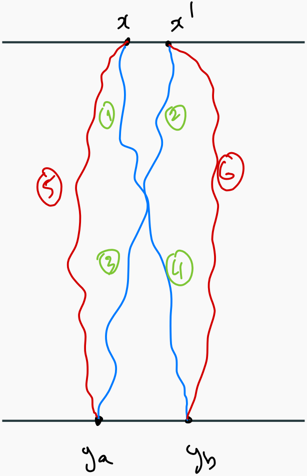


$\mathcal{D}$ is non-decreasing

Lemma

$\mathcal{D}$ is non-decreasing a.s.



Proof

$$S(y_b, x') - S(y_a, x')$$

$$= \textcircled{6} - (\textcircled{3} + \textcircled{2})$$

$$\geq \textcircled{4} + \textcircled{2} - (\textcircled{3} + \textcircled{2})$$

$$= \textcircled{4} - \textcircled{3}$$

$$= \underbrace{\textcircled{4} + \textcircled{1}}_{= S(y_b, x)} - \underbrace{(\textcircled{3} + \textcircled{1})}_{\leq \textcircled{5}}$$

$$= S(y_b, x) - S(y_a, x)$$

The fractal dimension of non-constant points

Let $\mathsf{NC}(\mathcal{D})$ be the set of non-constant points of $\mathcal{D}$.

$\mathsf{NC}(\mathcal{D})$ has Lebesgue measure zero.

So we consider its fractal dimension as a measure of sparsity.

We use Hausdorff dimension: heuristically, $\mathsf{NC}(\mathcal{D})$ has dimension α if it needs $\varepsilon^{-\alpha}$ number of diameter-ε sets to be covered.

The fractal dimension of non-constant points

Theorem (Basu-Ganguly-Hammond)

$\mathrm{NC}(\mathcal{D})$ *has Hausdorff dimension* $\frac{1}{2}$ *a.s.*

Classical fact: The zero set of Brownian motion also has Hausdorff dimension $\frac{1}{2}$ a.s.

There is an associated non-decreasing function $\mathcal{L}$, the *local time*, such that $\mathrm{NC}(\mathcal{L}) = \mathrm{Zero(BM)}$.

Definition of Brownian local time

Definition

Let B be Brownian motion of rate σ^2. Then

$$\mathcal{L}(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|B(s)| \leq \varepsilon} \, \mathrm{d}s.$$

Heuristically, the amount of time B spends at the origin.

Question: Is there a connection between $\mathcal{L}$ and $\mathcal{D}$?

A global comparison for $\mathcal{D}$

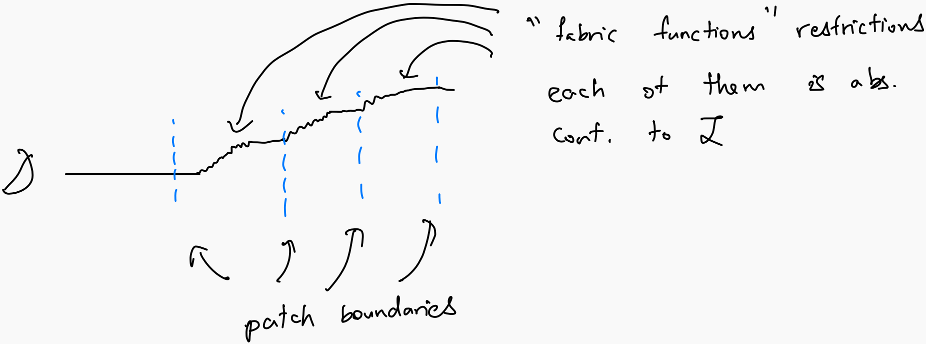
A first form of global comparison might be the absolute continuity of $\mathcal{D}$ to $\mathcal{L}$.

Unfortunately, this appears to be difficult.

Instead, we give ourselves a little flexibility.

Main Theorem

$\mathcal{D}$ is a Brownian local time patchwork quilt of rate four.



$\mathcal{D}$ is these fabric functions "sewn" together to form a patchwork quilt

A global comparison for $\mathcal{D}$

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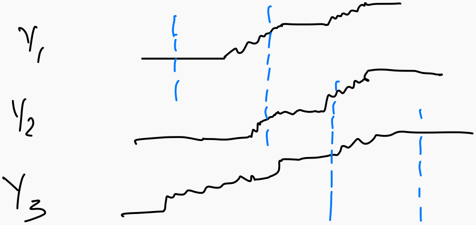

$Y_i : [0, \infty) \to \mathbb{R}$ and abs. cont. to $\mathcal{L}$ on any interval $[\delta, \delta^{-1}]$

vertically shift them to be continuous

A local limit for $\mathcal{D}$

Main Theorem

Let $\lambda \in \mathbb{R}$ and $\tau_\lambda = \inf\{t > \lambda : t \in \mathsf{NC}(\mathcal{D})\}$. Then,

$$\varepsilon^{-1/2} \left(\mathcal{D}(\tau_\lambda + \varepsilon t) - \mathcal{D}(\tau_\lambda) \right) \xrightarrow{d} \mathcal{L}(t)$$

in the topology of uniform convergence on compact sets.

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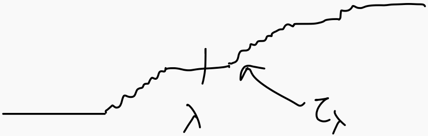
in the topology of uniform convergence on compact sets.

Observe that τ_λ is in some sense a size-biased choice: larger preceding flat portions are preferred.



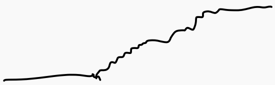
Local limits with a uniform modes of selection

Main Theorem

Additionally, with uniform convergence on compact sets,

$$\varepsilon^{-1/2}\left(\mathcal{D}(\xi+\varepsilon t)-\mathcal{D}(\xi)\right)\xrightarrow{d}\mathcal{L}(t),$$

where $\xi = \xi_{[a,b]}$ is an independent sample from the probability measure on $[a,b]$ with distribution function $\mathcal{D}$ (normalized).



Brownianity of the parabolic Airy$_2$ process

Recall the parabolic Airy$_2$ process $\mathcal{P}_1$.

It possesses a form of global Brownianity: it is absolutely continuous to Brownian motion on compact intervals.

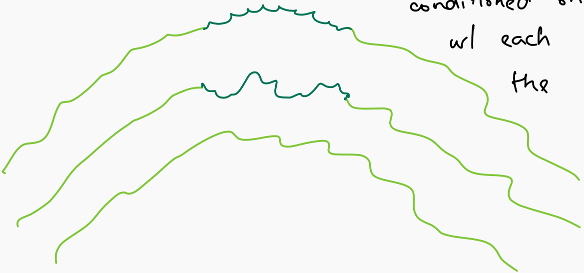
Proved by Corwin-Hammond by construction of the parabolic Airy line ensemble $\mathcal{P}$.

The parabolic Airy line ensemble and Brownian Gibbs

$\mathcal{P}$ is an infinite collection of random non-intersecting continuous curves, with $\mathcal{P}_1$ as its top curve.

$\mathcal{P}$ possesses the *Brownian Gibbs property.*

The cond. dist. is given by Brownian bridge conditioned on non-intersection w/ each other or the below curve



Proof ideas for patchwork quilt

Motivation: discrete RSK correspondence

The discrete Robinson-Schensted-Knuth correspondence provides a transformed environment that preserves LPP values.

1	0	2
3	1	3
1	2	0

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iid LPP
environment

1	0	2
3	1	3
1	2	0

$\xrightarrow[\text{across rows}]{\text{increments}}$

1	-1	2
3	-2	2
1	1	-2

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increments $\longrightarrow$ across rows

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best wgt = 6

best collection of 2 paths has weight = 7

RSK

5	4	6
	-1	1
		-2

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The numbers in the same column are *ordered* after RSK.

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1	-1	2
3	-2	2
1	1	-2

weight = 3

RSK

5	4	6
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increments $\longrightarrow$ across rows

5	-1	2
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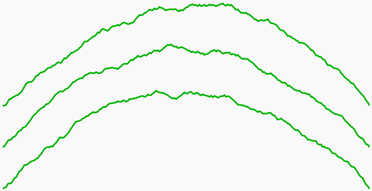
Semi-discrete LPP

A similar preservation of geodesic weights will hold for $\mathcal{S}$, via the parabolic Airy line ensemble $\mathcal{P}$.



$\mathcal{P}_1$

$\mathcal{P}_2$

$\mathcal{P}_3$

Semi-discrete LPP

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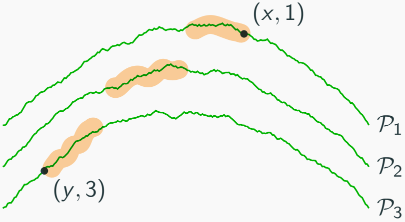
Weight of an up-right path in $\mathcal{P}$ = sum of increments across $\mathcal{P}_i$.

LPP value from (y, k) to $(x, 1)$ is denoted $\mathcal{P}[(y, k) \to (x, 1)]$.



A continuous RSK correspondence for $\mathcal{S}$

Here is the limiting relation between LPP values in the original and transformed environments, between $\mathcal{S}$ and $\mathcal{P}$.

Theorem (Dauvergne-Ortmann-Virág)

$\mathcal{S}$ exists and has a coupling with $\mathcal{P}$ such that

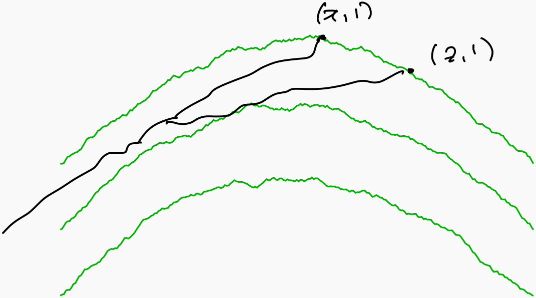
$$\mathcal{S}(y,x) - \mathcal{S}(y,z) = \lim_{k\to\infty}\Big(\mathcal{P}[(-y_k,k)\to(x,1)] - \mathcal{P}[(-y_k,k)\to(z,1)]\Big).$$

Here $\{y_k\}$ is a sequence of points defined by y which goes to ∞.

Coalescence intuition and difficulties of relation

$\mathcal{S}(y, x) - \mathcal{S}(y, z)$ has common starting and differing ending points.

This is important for the result because of coalescence.



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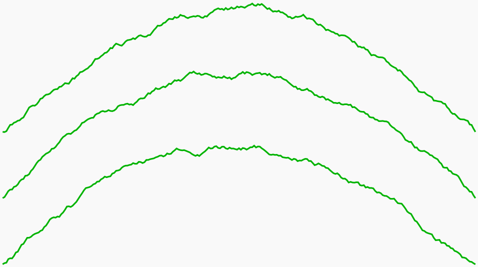

But $\mathcal{D}(x) = \mathcal{S}(y_b, x) - \mathcal{S}(y_a, x)$ is opposite!

A direct relation between $\mathcal{S}$ and $\mathcal{P}$?

Ideally, we would like a RSK description for $\mathcal{S}$ *directly*:

$$\mathcal{S}(y, x) = \lim_{k \to \infty} \mathcal{P}[(-y_k, k) \to (x, 1)].$$

But this is difficult: an open problem.

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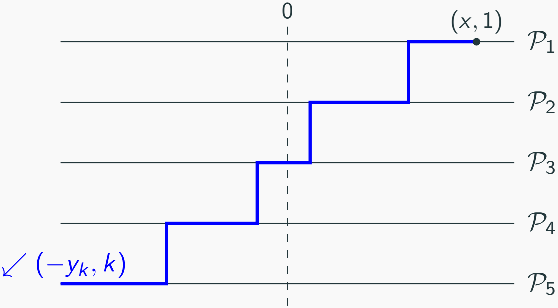
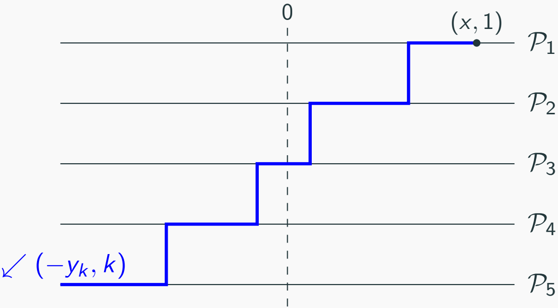
To get around this, we use a notion of boundary data.

A notion of boundary data



A notion of boundary data

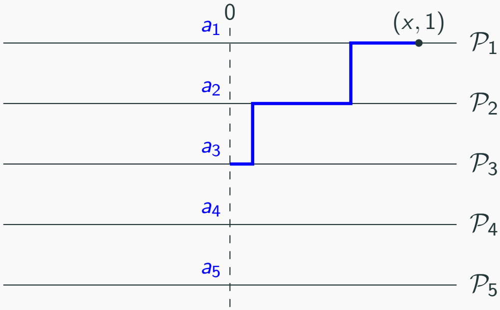


Lemma (Sarkar-Virág)

There exist $\{a_i\}_{i \in \mathbb{N}}$ *so that*

$$\mathcal{S}(y_a, x) = \sup_{i \in \mathbb{N}} \Big\{ a_i + \mathcal{P}[(0,i) \to (x,1)] \Big\}.$$

A notion of boundary data



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The utility of boundary data

By varying the starting points, there exist $\{a_i\}_{i\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}}$ so that

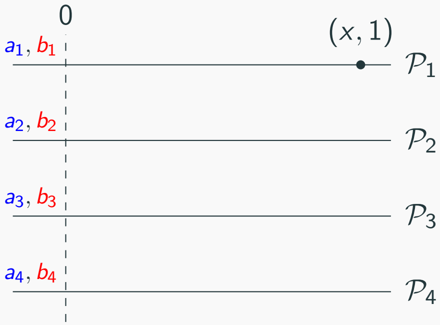
$$\mathcal{S}(y_a, x) = \sup_{i\in\mathbb{N}}\Big\{a_i + \mathcal{P}[(0,i) \to (x,1)]\Big\}$$
$$\mathcal{S}(y_b, x) = \sup_{j\in\mathbb{N}}\Big\{b_j + \mathcal{P}[(0,j) \to (x,1)]\Big\}.$$

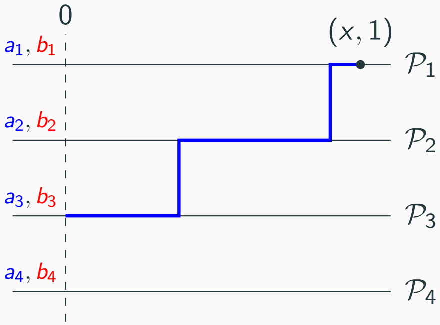
We have the *same* LPP problems $\mathcal{P}[(0,i) \to (x,1)]$ for y_a and y_b!

Recall our goal

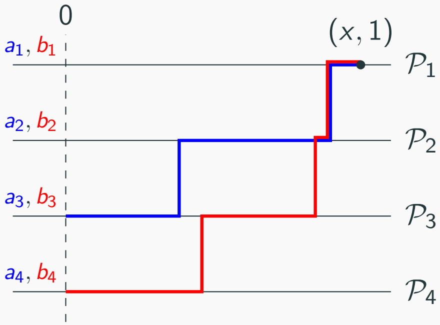
We want to show that $\mathcal{D}$ is a Brownian local time patchwork quilt.

So we have to find random *fabric* functions that $\mathcal{D}$ agrees with on certain (random) intervals and are absolutely continuous to $\mathcal{L}$.


0
(x, 1)
a1, b1
a2, b2
a3, b3
a4, b4
P1
P2
P3
P4



0
$(x, 1)$
a_1, b_1
a_2, b_2
a_3, b_3
a_4, b_4
$\mathcal{P}_1$
$\mathcal{P}_2$
$\mathcal{P}_3$
$\mathcal{P}_4$


0
$(x, 1)$
a_1, b_1
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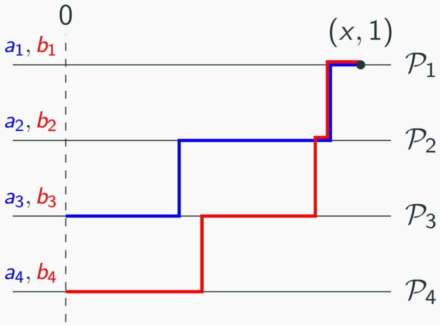
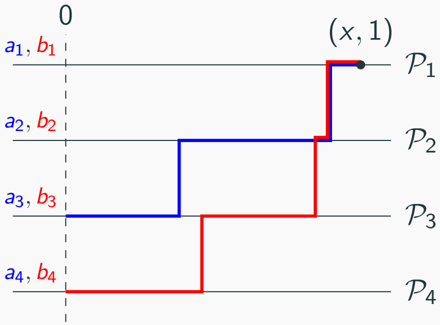


$$\mathcal{S}(y_b, x) - \mathcal{S}(y_a, x)$$
$$= b_j + \mathcal{P}[(0, j) \to (x, 1)]$$
$$- (a_i + \mathcal{P}[(0, i) \to (x, 1)]).$$



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$$= b_j + \mathcal{P}[(0, j) \to (x, 1)]$$
$$- (a_i + \mathcal{P}[(0, i) \to (x, 1)]).$$

Our fabric functions should be

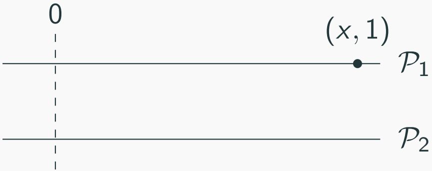
$$\mathcal{P}_{j \to 1}(x) - \mathcal{P}_{i \to 1}(x),$$

where

$$\mathcal{P}_{k \to 1}(x) = \mathcal{P}[(0, k) \to (x, 1)].$$

Why should $\mathcal{P}_{j\to 1}(x) - \mathcal{P}_{i\to 1}(x)$ look like local time?

The simplest case of $j = 2$ and $i = 1$:



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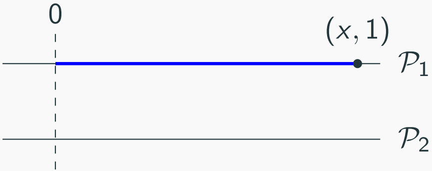

$$\mathcal{P}_{1\to 1}(x) = \mathcal{P}_1(x) - \mathcal{P}_1(0)$$

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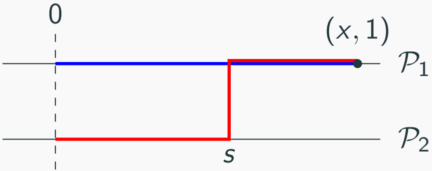

$$\mathcal{P}_{1\to 1}(x) = \mathcal{P}_1(x) - \mathcal{P}_1(0)$$

$$\mathcal{P}_{2\to 1}(x) = \mathcal{P}_1(x) + \max_{0\le s\le x} (\mathcal{P}_2(s) - \mathcal{P}_1(s)) - \mathcal{P}_2(0)$$

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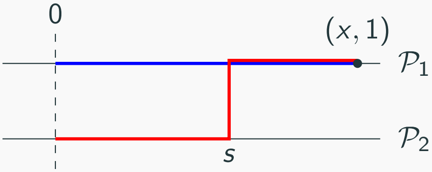
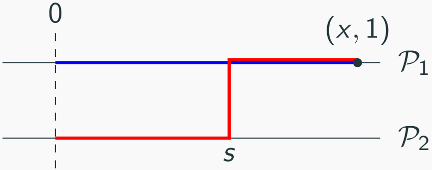


$$\mathcal{P}_{1\to 1}(x) = \mathcal{P}_1(x) - \mathcal{P}_1(0)$$

$$\mathcal{P}_{2\to 1}(x) = \mathcal{P}_1(x) + \max_{0\le s\le x} (\mathcal{P}_2(s) - \mathcal{P}_1(s)) - \mathcal{P}_2(0)$$

$$(\mathcal{P}_{2\to 1} - \mathcal{P}_{1\to 1})(x) = \max_{0\le s\le x} \Big\{ \big[\mathcal{P}_2(s) - \mathcal{P}_2(0)\big] - \big[\mathcal{P}_1(s) - \mathcal{P}_1(0)\big] \Big\}.$$



$$(\mathcal{P}_{2\to 1} - \mathcal{P}_{1\to 1})(x) = \max_{0\le s\le x}\Big\{\big[\mathcal{P}_2(s) - \mathcal{P}_2(0)\big] - \big[\mathcal{P}_1(s) - \mathcal{P}_1(0)\big]\Big\}.$$

By the Brownian Gibbs property, $\mathcal{P}_2(\cdot) - \mathcal{P}_2(0)$ and $\mathcal{P}_1(\cdot) - \mathcal{P}_1(0)$ are jointly absolutely continuous to independent rate two BMs!

So their difference is absolutely continuous to rate *four* BM.

So $\mathcal{P}_{2\to 1} - \mathcal{P}_{1\to 1}$ is absolutely continuous to $\max_{0\leq s\leq t} B(s)$, where B is rate four Brownian motion.

The latter is equal in law to rate four Brownian local time:

Theorem (Lévy's identity)

Let B be rate σ^2 Brownian motion, $\mathcal{L}$ its local time at zero, and M its running maximum. M and $\mathcal{L}$ are equal in law as processes.

The general case

For the general case ($j > i$), we work with *sequences* of transformations like the single one in the $j = 2, i = 1$ case.

These are called Pitman transforms PT, defined for $f_1, f_2 : [0, \infty) \to \mathbb{R}$ by

$$\big(\mathrm{PT}(f_1, f_2)\big)_1(t) = f_{2\to 1}(t)$$
$$\big(\mathrm{PT}(f_1, f_2)\big)_2(t) = f_1(t) + f_2(t) - f_{2\to 1}(t).$$

Certain sequences of these transforms are known to yield LPP values like $\mathcal{P}_{j\to 1}(x)$ (work of Biane-Bougerol-O'Connell, and DOV).

A complication at the origin

$\mathcal{P}_{2\to1} - \mathcal{P}_{1\to1}$ was absolutely continuous to $\mathcal{L}$ on $[0, t]$.

$$= \max (P_2(s) - P_1(s)) - (P_2(0) - P_1(0))$$

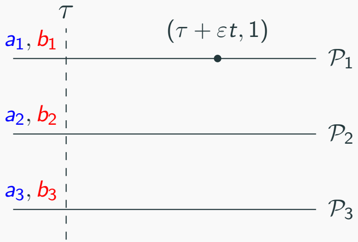
But with multiple Pitman transforms, we expect $\mathcal{P}_{j\to1} - \mathcal{P}_{i\to1}$ to be absolutely continuous on only $[\varepsilon, t]$, for any $\varepsilon > 0$.

For example, $\mathrm{PT}(B_1, B_2)$ is 2-Dyson Brownian motion, which is only comparable to Brownian motion away from 0.

The local limit

Here we consider the geodesic to the *random* location $(\tau, 1)$.

To avoid the singularity at the origin, we need to know we are in the $j = 2$ and $i = 1$ case, and this is the heart of the argument.



Finally reduces to non-point-recurrence of planar Brownian motion.

Thank you!

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