CHANGES TO "EQUIVARIANT INSTANTON HOMOLOGY" IN SECOND VERSION

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In addition to a number of small edits in the main body, the appendix required two substantial changes.

- As discussed in [Ola22], Section A.8 in the first version stated that a certain completeness property holds which does not appear to. A different approach to the completed complexes, described in [Ola22, Section 5.4], fixes this without changing the statements of any results, and Helle's completion gives the same complexes as my original proposed completion in Section 7, the section devoted to some specific concrete calculations.

  This changes none of the main results in the appendix, nor any of the main results in the main body of the text. Helle's approach is elegant and I was glad to incorporate it.

- As pointed out to me by Helle in private correspondence, the proof of Lemma A.23 in the original revisions is incorrect. It asserts that a certain map is $C_*(G)$-biequivariant, but this is not true, and it seems difficult to modify this into a correct statement. In the new version, Sections A.5 and A.7 have been substantially modified to clarify the proofs and to explain what we can prove under simplifying hypotheses.

  This affects the main results in the appendix and in the main body as follows.

  (1) In the appendix, I obtain for any compact Lie group G and any G-module M an exact triangle of $H^{-*}(BG; R)$-modules,
  $$\cdots \to H^{+,\mathrm{tw}}_G(M) \to H^-_G(M) \to H^\infty_G(M) \to \cdots$$
  Further, we have a natural *additive* isomorphism $H^{+,\mathrm{tw}}_G(M) \cong H^+_G(M)[n]$, a shift of the standard Borel construction. However, this additive isomorphism can only be chosen to be an $H^{-*}(BG; R)$-module isomorphism when we can establish the analogue of Lemma A.23 for $C_*(G; R)$. (Compare the statement of Theorem A.29(7) in the original version to Theorem A.25(7) and A.25(8) in the new revision.)
  I am able to establish this stronger condition for, among other things, $G = SO(3)$ and R such that 2 is either invertible or zero in R.

  (2) In the main body, these results are applied to $G = SO(3)$, so one obtains a natural exact triangle $I^+(Y, \sigma; R)[3] \to I^-(Y, \sigma; R) \to I^\infty(Y, \sigma; R)$ of R-modules for all rational homology spheres Y, and I can prove that this is an exact sequence of $H^{-*}(BSO_3; R)$-modules when $\frac{1}{2} \in R$ or $2 = 0 \in R$.

It's probably true that the analogue of Lemma A.23 holds for all G, but I leave this to an interested reader, as it does not affect the applications to the instanton theory. Notably, all known applications of $SO(3)$-equivariant instanton theory either work with $\frac{1}{2} \in R$ or work with $R = \mathbb{F}_2$. It remains unclear whether one should expect anything useful from the integer-valued theory, even after proving the analogue of Lemma A.23, and in that case other work in Floer theory suggests the thing to do would be to study $CI^-(Y; \mathbb{Z})$ and perform ad hoc algebraic maneuvers to extract some particular information of interest.

In the three sections below, I outline these changes in the opposite order listed above (what I view as the order of importance).

Changes related to A.23

Section A.5. Theorem A.19 [now: Theorem A.15] originally only asked for an equivalence between A and $A^\vee$ as right modules, but the full result uses an equivalence as *bimodules*. One still gets a result under weaker assumptions, which is still useful. This has been clarified in the new revision.

Section A.5 has been changed as follows.

(1) Added Definition A.8 of 'weak Poincaré duality' and 'strong Poincaré duality'.
(2) Added the weaker result Theorem A.14 for algebras satisfying weak Poincaré duality (which will include all $C_*(G;R)$).
(3) Unrelatedly, the assumptions that $H_0(A) = R[G]$ and that A is non-negatively graded are superfluous in Theorem A.19 [now: Theorem A.15], and have been removed.

Section A.7. Section A.7 has been substantially rewritten to clarify the difference between weak Poincaré duality and strong Poincaré duality, and to prove what we can under each hypothesis. Because the structure of the section is fairly different, it seems better to explain the structure of the changes instead of enumerating precise changes.

(1) The beginning of the section is now devoted to establishing what we can about Poincaré duality for groups $C_*(G)$.
    - First, we establish that $C_*(G;R)$ always satisfies weak Poincaré duality.
    - We then establish that $C_*(SO_n;\mathbb{F}_2)$ satisfies strong Poincaré duality, as does $C_*(G;R)$ for finite G, as does $C_*(G;R)$ if $G = S^1$ or S^3 (and therefore if $G = SO_3$ and $\frac{1}{2} \in R$).
    - Added Remark A.5 explaining one strategy one might take to establishing strong Poincaré duality in general, and Remark A.6 explaining why another strategy doesn't work.
(2) Like before, the end of the section is devoted to computing $H^\bullet_G(G/H)$. These results do not change at all, and the proofs are unchanged. However, the proof of Theorem A.26 [now Theorem A.23] has been expanded to clarify some points; what was once part of its proof has now been separated as Lemma A.22; what was Lemma A.27 has been absorbed into the proof of Theorem A.26 [now A.23] itself.

Elsewhere. Here are the places these changes affected the rest of the document.

(1) Theorem 2 in the introduction has been modified to say the exact triangle is one of $H^-(BSO_3;R)$-modules so long as either $2 = 0$ or $\frac{1}{2} \in R$, and point the reader to the appendix for consideration of the general case.
(2) Section 6.4 points briefly to the subtleties involved in defining the norm map as a map of H^-_G-modules.
(3) The proof of Theorem 6.21 is expanded to explain the relevance of strong duality (and the condition that $2 = 0$ or 2 is invertible in R). The statement of Theorem 6.21 is changed in the same way the statement of Theorem 2 in the introduction is.
(4) Theorem A.29 in the final section (now: Theorem A.25) has had item (7) split into two statements: (7) the existence of a long exact triangle between three theories (one the twisted Borel homology), and (8) the existence of an isomorphism between the twisted theory and the standard Borel theory — as R-modules with weak Poincaré duality, as H^- modules with strong Poincaré duality.

Changes related to completion

While these changes mainly affect Section A.8 in content, they also suggest slightly modifying the way previous discussions were written. The changes are enumerated here.

(1) Following a suggestion of Helle, the discussion of convergent spectral sequence is much simplified by referring only to spectral sequences arising from filtered complexes. References to Boardman have been replaced with references to Cartan–Eilenberg and Weibel. This primarily affects Section A.2, where the discussion of spectral sequences has been revised accordingly; references in the rest of the document have been changed accordingly.

(2) Section A.6 on spectral sequences has now been rephrased in terms of the filtrations that give rise to these spectral sequences, as opposed to the spectral sequences themselves.

(3) The statement of Proposition A.20 [now: Proposition A.16] has been modified to phrase in terms of filtrations instead of spectral sequences. Also, the statement originally included an assumption that $H_*(M)$ is bounded below to guarantee strong convergence, but the correct assumption is that it is bounded above. (This does not affect the rest of the work, as in reality the complexes of interest to us are unbounded in both directions.)

(4) Assumptions on the algebra A and module M were progressively added in Theorem A.25 [previously: A.29] instead of assumed at the beginning of the statement, because Helle's construction allows us to assume less for the main results.

(5) The proof of Theorem A.25 [previously: A.29] is simplified, thanks to Helle's construction. Previous references to completed complexes in other sections (and invariance results for them) have been removed.

Elsewhere. Here are the places these changes affected the rest of the document.

(1) The proof of Theorem 6.19 is slightly simplified with the new phrasing of A.25.

(2) In Section 7.2, when introducing the complex DCI^+ a discussion about the full completion is given to explain where the power series comes from. A similar but more brief discussion appears when discussing DCI^-.

Other changes

First, I list the changes to the main body of the text.

(1) Changed name from "S. Michael Miller" to "Mike Miller Eismeier".
(2) Changed title of second section from "Analysis of configuration spaces" to "Manifold models of configuration spaces".
(3) Acknowledgements updated to thank Roberto Ladu, Mariano Eccheveria, and Gard Olav Helle
(4) Added Remark 1 to the introduction, pointing out that Conjecture 1 has been proved since the preprint first appeared.

(5) The statement of Theorem 3 in the introduction should only have asserted that the spectral sequences compute homology for $\tilde{I}$ and I^-, but detect isomorphisms for all four flavors.
(6) Definition 1.2 of the action $\tau_t(s) = t - s$ has been changed to the inverse action $\tau_t(s) = t + s$ to be consistent with other literature. Added a remark following this to help conceptualize the action.

(7) An important clause of Proposition 4.4 was missing (the first in the new enumeration), and has been added.
(8) Corrected a typo in the ODE in the proof of Lemma 4.21 (a missing \\ in align mode made the display nonsense), and added references for two earlier facts about extending perturbations to lower-order Sobolev spaces, including the clause added to Prop 4.4 mentioned above.

(9) Made micro-edits to the proofs of Lemmas 4.35-4.36 for the sake of clarity.

(10) The proof of what was Proposition 6.2 [now Proposition 6.1] was incorrect, as it asserted a certain fiber product was a topological manifold with corners, but the surrounding text gives a counter-example. However, the argument it has been replaced by is in fact simpler, and requires one fewer definition, and Lemma 6.1 has been removed.

(11) The statement of Theorem 6.8(2) [now: 6.7(2)] should read "with $c_1(E)$ divisible by 2", and has been corrected.
(12) The final paragraph of Theorem 6.10(6) [now 6.9(6)] was superfluous and incorrect. The relevant statement appears when needed later.
(13) Some superscript W's were erroneously dropped in the statement of Theorem 6.10 (now 6.9) and have been reintroduced.

(14) A number of inconsequential typos in Section 7 have been corrected.

Signs. A small number of sign errors in the main body have also been corrected.

(1) Some signs in Propositions 5.7 and 5.12 have been corrected, and the proof of Proposition 5.7 expanded in the hope of making it more clear. The sign in the formula preceding Proposition 5.7 has also been changed to be consistent with work in Section 6 later.
(2) The signs in Theorem 6.9(6) [now 6.8(6)] and Theorem 6.10(6) [now 6.9(6)] were incorrect as written, and have been corrected.
(3) With the correct signs, the definition of the chain map in the proof of Lemma 6.12 [now 6.11] is incorrect: there should not be a $(-1)^{|\sigma|}$ factor. This has been removed. (This sign is still appropriate in the definition of the differential and chain homotopy.) Note that all calculations in this paper are at the level of homology groups, and not at the level of cobordism maps, so this is a self-contained correction.

Remaining appendix changes. Here I list some less significant changes to the appendix.

(1) The formula for ϵ_i in Definition A.1 is incorrect: the $+i$ should not be there, and the conventions of Gugenheim and May do not include it. This has been corrected. Fortunately,

this affects none of the signs in Section 7: for the relevant algebras, the terms with incorrect signs are automatically zero.

(2) In Section A.3 an action of C^- on the twisted Borel complex is introduced. In the revised version, a formula for this action has been added, and it is made more explicit that this action is only defined because the C^- action on D_A commutes with the A action.
(3) The proof of Lemma A.10 [now: Lemma A.8] made an incorrect assertion ($D_A = \cdots$) which uses finiteness assumptions. This assertion was not relevant to the proof, so has been removed.
(4) The statement of Theorem A.11(2) [now: Theorem A.9(2)] was too general: one must either assume X is finite-dimensional or that A is free in each degree. I chose the former: it is all that is used in the rest of the paper, and the proof in the latter case requires additional argument.
(5) Added Remark A.12 about the independence of the choice of dga, slightly extending the discussion in the preceding Proposition.
(6) There were a handful of sign errors in the proof of Lemma A.17 [now: Lemma A.13] which have been corrected; these do not enter into the statement of the Lemma and do not affect work in the rest of the document. There was an unnecessary assumption that n was odd, which has been removed (this assumption was suggested by the erroneous signs). A remark has been added to explain how the parity of n is relevant.
(7) Theorem A.29(6) [now: Theorem A.25(6)] has been modified to add the finiteness hypothesis on $X \otimes A$ mentioned above.

References

[Ola22] Gard Olav Helle. Equivariant instanton Floer homology and calculations for the binary polyhedral spaces. *arXiv e-prints*, page arXiv:2203.09471, March 2022.