MONOPOLES, COUPLED MORSE HOMOLOGY, AND HIRSCH ALGEBRAS

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Abstract. Francesco Lin and I recently calculated the groups $\overline{HM}(Y, \mathfrak{s})$ for any 3-manifold, spin^c structure, and twisted coefficient group. Our work builds on Kronheimer and Mrowka's model of $\overline{HM}$ in terms of coupled Morse homology, and then their computation of coupled Morse homology in terms of twisted singular homology, so that $\overline{HM}$ is the twisted singular homology of the torus $\mathbb{T}(Y, \mathfrak{s})$ of flat spin^c connections.

Our computation is essentially algebraic in nature. We study twisted singular homology in a general context; it is controlled by a gadget called a twisting sequence. We determine precisely the necessary structure (Hirsch algebras) to control these twisting sequences when they are transferred along quasi-isomorphisms, and compute this structure explicitly enough on the torus to work with it.

Surprisingly, the answer we obtain is not identical to what is usually called cup homology, but is rather slightly more intricate (stated on the last page below). Computer calculations suggest that they are at least isomorphic as abelian groups.

Context: Floer homology for 3-manifolds

Many moons ago, Oszváth and Szabó defined four invariants of closed oriented 3-manifolds equipped with a spin^c structure $(Y, \mathfrak{s})$. These are the *Heegaard Floer homology groups*: one is the finite-dimensional vector space $\widehat{HF}(Y, \mathfrak{s})$, while the others are the $\mathbb{Z}[U]$-modules HF^+, HF^-, HF^∞. Here U acts as a degree -2 variable; on HF^+, the map U acts nilpotently on any element, while on HF^- it acts as an isomorphism in sufficiently negative degrees, and in HF^∞ it acts invertibly. The simplest examples are

$$HF^-(S^3) \cong \mathbb{Z}[U], \quad HF^+(S^3) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U], \quad HF^\infty(S^3) \cong \mathbb{Z}[U, U^{-1}], \quad \widehat{HF}(S^3) \cong \mathbb{Z}.$$

Of these, the complex describing HF^- contains the most information: from it, one may extract any of the other three complexes. The groups $HF^\infty(Y)$ are by far the simplest in structure. Oszváth and Szabó constructed a spectral sequence with E_2 page $\Lambda^* H^1(Y; \mathbb{Z}) \otimes \mathbb{Z}[U, U^{-1}]$, and offered the following conjecture about its behavior. To state the conjecture below, we should define the *contraction by the triple-cup-product*.

Precisely, consider the map $\iota_\cup : \Lambda^m(H^1 Y) \to \Lambda^{m-3}(H^1 Y)$ given by sending

$$\iota_\cup(\alpha_1 \wedge \cdots \wedge \alpha_m) = \sum_{1 \leqslant i < j < k \leqslant m} (-1)^{i+j+k} (\alpha_i \cup \alpha_j \cup \alpha_k)[Y] \cdot \alpha_1 \wedge \widehat{\alpha}_{i,j,k} \cdots \wedge \alpha_m.$$

Conjecture 1 (Oszváth and Szabó's conjecture)**.** *Let $(Y, \mathfrak{s})$ be a closed oriented 3-manifold equipped with a spin^c structure so that $c_1(\mathfrak{s}) \in H^2(Y; \mathbb{Z})$ is **torsion**. Then the Oszváth-Szabó spectral sequence has only one nontrivial differential, given by* contraction with the triple cup product*:*

$$d_3(\omega \otimes U^k) = \iota_\cup(\omega) \otimes U^{k-1}.$$

This conjecture was verified with $\mathbb{F}_2$ coefficients in Tye Lidman's thesis. Carrying out his argument over the integers seems possible but at the very least extremely difficult: one would need to have a very firm understanding of all of the signs involved in the constructions in Heegaard Floer homology.

We call this conjectural E^3 page the *cup complex* of $(Y, \mathfrak{s})$, and its homology groups the *cup homology* $HC^\infty(Y, \mathfrak{s})$. Notice that this does not depend on the spin^c structure, as stated above. Later we will describe a corresponding version of cup homology with possibly twisted coefficients and possibly torsion $c_1(\mathfrak{s})$; in this case, the resulting complex only depends on the element $[c_1] \in \mathrm{Hom}(H_2 Y, \mathbb{Z})$.

On the other hand, Kronheimer and Mrowka defined a Floer homology called *monopole Floer homology.* It by and large fits into the same structure as outlined above: now instead of $\widehat{HF}, HF^+, HF^-, HF^\infty$, our cast of characters are $\widetilde{HM}, \widecheck{HM}, \widehat{HM}, \overline{HM}$. These behave in roughly the same way as the Heegaard Floer homology groups do, and in fact, the difficult and pioneering work of two different teams has shown that the two collections are *isomorphic*; for instance, for any closed oriented 3-manifold Y, we have

$$HF^{\infty}(Y,\mathfrak{s}) \cong \overline{HM}(Y,\mathfrak{s}).$$

These homology groups are in fact functors, and we do not yet know that HF^{∞} and $\overline{HM}$ are the same up to natural isomorphism; but no matter.

Today's story is really about $\overline{HM}(Y,\mathfrak{s})$. Thanks to Kronheimer and Mrowka's work in their original study of $\overline{HM}$, quite a lot was already known about its structure; we'll explore that next. Using their structure theorems, we will be able to prove the Conjecture above (and more).

Coupled Morse homology

Before explaining the connection, let's discuss a variation of the usual Morse theory which arises naturally in the monopole theory.

Definition 1. *Let Q be a compact manifold (of arbitrary dimension), equipped with a bundle $\mathcal{H} \to Q$ of separable, infinite-dimensional Hilbert spaces, and a family $L : \mathcal{H} \to \mathcal{H}$ of self-adjoint Fredholm operators on each fiber.*[1] *We write this data as (Q, L), suppressing the bundle of Hilbert spaces from notation.*

If Q is given a Riemannian metric and if $f : Q \to \mathbb{R}$ is a Morse-Smale function, then we define the coupled morse complex *$CMC_*(Q,L)$ to be generated by pairs (x,λ), where $x \in Crit(f)$ and λ is an eigenvalue of $L_x : \mathcal{H}_x \to \mathcal{H}_x$. The component of the differential running from (x,λ) to (y,λ') counts isolated curves $\widetilde{\gamma} : \mathbb{R} \to \mathcal{H}\backslash 0$ satisfying the following equations, where $\gamma = p\widetilde{\gamma} : \mathbb{R} \to Q$ is the curve projected to the base:*

$$\gamma'(t) + (\nabla f)(\gamma(t)) = 0$$
$$\widetilde{\gamma}'(t)_{\text{vert}} + L_{\gamma(t)}\widetilde{\gamma}(t) = 0.$$

The first equation says that γ is a Morse flowline on P, while the second equation says that its lift is a flowline of the (time-dependent) operator $L_{\gamma(t)}$. The path γ is required to limit to x and y, respectively, while $\widetilde{\gamma}$ is asked to asymptotically take the form $e^{t\lambda}v$ for some $v \in \mathcal{H}_\lambda$.

We say the homology groups are the coupled Morse homology $CMH_*(Q,L)$.

Remark 1. Like the Heegaard Floer CF^{∞}, this complex carries a degree (-2) endomorphism — which we'll call U — essentially given by flowing "in each fiber". If $CMC_*(Q,L)$ is filtered by the Morse index of x, then the associated graded map of U is given by $U(x,\lambda_i) = (x,\lambda_{i-1})$, where the notation indicates that λ_{i-1} is the immediate predecessor of λ_i. (There may be other terms in lower filtration.)

Remark 2. The data of a pair $(\mathcal{H}, L)$ up to 'homotopy' is the same as the data of a map $L : Q \to U(\infty)$ up to homotopy. The coupled Morse homology then depends only on an element of $K^1(Q)$. Kronheimer and Mrowka ask whether it depends on less: there is a cohomology class $c_2' \in H^3U(\infty)$, and they ask if $L^*[c_2']$ determines the coupled Morse homology. One of our results is that this is not true for general Q, but that it is true for the case of relevance to monopole Floer theory.

One of Kronheimer and Mrowka's results is that *the flavor $\overline{HM}$ can be described as a coupled Morse homology group.* In fact, there is a certain torus $\mathbb{T}(Y,\mathfrak{s})$ (the torus of flat connections) and a certain family of operators L_A over $\mathbb{T}(Y,\mathfrak{s})$ (the Chern-Simons-Dirac operators of these flat connections). Precisely,

$$\overline{HM}_*(Y,\mathfrak{s}) \cong CMH_*(\mathbb{T}(Y,\mathfrak{s}); D).$$

This isomorphism respects the $\mathbb{F}[U]$-module structure, as well as the additional structure of a $\mathbb{Z}[H_1(Y;\mathbb{Z})/\text{Tors}]$-module (which we will otherwise not discuss).

This coupled Morse homology is lots of fun if you like to play with Morse theory itself. It is still in want for non-Floer theoretic applications. (For one: does this have any connection to twisted K-theory?)

The above theorem is nice, but what really gets the machine moving is their calculation of coupled Morse homology in a special case.

[1] I am suppressing a lot of technical details in this definition. Actually, we want these to be densely defined Fredholm operators with a well-behaved spectral theory. The details can be found in Chapter 33 of 'Monopoles and Three-Manifolds'.

Theorem 1 (Monopoles and Three-Manifolds, Theorem 34.2.1)**.** *Suppose $L : Q \to U(\infty)$ factors through $SU(2)$, and furthermore that $L : Q \to SU(2)$ is a* simplicial map *for an appropriate triangulation of $SU(2)$. Then there exists a simplicial cocycle $\xi_3 \in C^3_\Delta(Q;\mathbb{Z})$* ***which squares to zero*** *so that*

$$CMH_*(Q,L) \cong H\left(C^\Delta_*(Q) \otimes \mathbb{Z}[U,U^{-1}], d_{tw}(x \otimes U^k) = (dx) \otimes U^k + (\xi_3 \cap x) \otimes U^{k+1}.\right)$$

This expression on the right is the homology of $C^\Delta_*(Q)[U,U^{-1}]$ with a *perturbed* differential. We use the normal simplicial differential, but there is an extra term — which takes us up one power of U — which is given by cap product against ξ_3.

To intuit this, one should imagine the eigenvalues are globally labeled λ_i with $\lambda_i \leqslant \lambda_{i+1}$, so that our complex CMC has generators (x,i), where i labels λ_i and $U(x,i) = (x,i-1)$ (up to lower filtration). That second term in the differential above thus corresponds to 'flowing down one eigenvalue'.

One should imagine that ξ_3 represents the intersection product with a codimension-3 submanifold of Q, where one "flows one eigenvalue down in $\mathcal{H}$" whenever one crosses this submanifold, and otherwise one does not flow down in eigenvalues.

The proof is given by calculating the coupled Morse complex of $SU(2)$ with its universal family $L : SU(2) \to U(\infty)$, and pulling back this calculation to Q by a simplicial map (where the Morse function and and L all respect this triangulation in an appropriate way).

This is very useful. Because $\overline{HM}$ is computed as the coupled Morse homology of a family of operators on the torus $\mathbb{T}$ — which factors up to homotopy through $SU(2)$ — we may compute it as the twisted homology of $C^*_\Delta(\mathbb{T})$ with respect to this ξ_3.

This theorem already proves that the monopole Floer analogue of Oszváth and Szabó's spectral sequence exists, and that the E^2 and E^3 pages behave as expected. Replacing $SU(2)$ with $U(2)$, their construction also works with little change in the case of local coefficient systems and in the case that $c_1(\mathfrak{s})$ is *non-torsion*.

Working over the reals one may relate this to Atiyah and Segal's *twisted cohomology*, where one twists the differential by a closed 3-form. This only depends on the cohomology class of that 3-form. This uses in an essential way that $\omega^2 = 0$ whenever ω is a 3-form, which is simply false at the level of simplicial cochains: it was very special that $\xi_3^2 = 0$ in the description above.

From this description, Kronheimer and Mrowka were able to verify the monopole Floer analogue of the Oszváth and Szabó conjecture over fields of characteristic zero, but it is not clear that there isn't some unexpected torsion.

Twisted homology

The proof that Atiyah and Segal's twisted cohomology groups only depend on the choice of third cohomology class is very algebraic. If ω is a closed 3-form, their twisted cohomology group is given as the homology of $\Omega^*(Q)$ with the differential $\eta \mapsto d\eta + \omega \wedge \eta$. That this is square-zero follows because $\omega^2 = 0$ and $d\omega = 0$. If $\omega' = \omega + d\zeta$, then

$$\eta \mapsto \eta + \zeta \wedge \eta + \frac{\zeta^2}{2} \wedge \eta + \frac{\zeta^3}{6} \wedge \eta + \cdots ;$$

the crucial property here is that if $b_{2n} = \frac{\zeta^n}{n!}$, then we have

$$db_{2n} + \omega b_{2n-2} - b_{2n-2}\omega' = 0$$

for all n.

When you phrase it in a sufficiently algebraic way like this, it inspires one to hope that a similar analysis might not be hopeless over the integers: one just needs the right perspective. We feel that twisted (co)homology is the right perspective for understanding $\overline{HM}$, thanks to Kronheimer and Mrowka's results above. I'll now exclusively focus on cohomology for ease of discussion (talking about cohomology will amount to talking about algebras; homology will involve talking about algebras and their modules).

Definition 2. *Let A be a dg-algebra (over any ring). A* ***twisting sequence*** *in A is a sequence $(x_3, x_5, \cdots) \in \prod_{n \geqslant 1} A^{2n+1}$ so that*

$$\begin{aligned} dx_3 &= 0 \\ dx_5 + x_3^2 &= 0 \\ dx_7 + x_3x_5 + x_5x_3 &= 0 \\ &\cdots \\ dx_{2n+1} + \sum_{i+j=n} x_{2i+1}x_{2j+1} &= 0. \end{aligned}$$

A ***homotopy*** *between twisting sequences $x_\bullet$ and $y_\bullet$ is a sequence b_{2n} for $n \geqslant 1$ with $db_2 = x_3 - y_3$ and in general*

$$db_{2n} = x_{2n+1} - y_{2n+1} + \sum_{i+j=n} b_{2i}x_{2j+1} - y_{2j+1}b_{2i}.$$

Such sequences are very well-studied in the literature on Lie algebras under the name *twisting cocycle* or *Maurer-Cartan elements*; the first appearance in the literature was in a paper by Ed Brown in the 60s, where he used such cochains to describe a 'twisted' differential on $C_*(B) \otimes C_*(F)$ which recovers the homology of a fiber bundle with base B and fiber F. These general constructions are much too complicated for our desired application, which asks us to understand these twisting sequences as simply and explicitly as possible.

For us, the point is: **a twisting sequence gives rise to a twisted homology group; a homotopy between twisting sequences gives rise to an isomorphism between twisted cohomology groups.**

Furthermore, if $f : A \to B$ is a quasi-isomorphism of dg-algebras, the induced map $H_{\text{tw}}(A; x_\bullet) \to H_{\text{tw}}(B; y_\bullet)$ is an isomorphism. (What's more, a quasi-isomorphism induces a bijection on the set of twisting sequences modulo homotopy.)

Goal. Find a zig-zag of quasi-isomorphisms between $C^*_\Delta(\mathbb{T})$ and $H^*(\mathbb{T}) = \Lambda^* H^1(Y)$. Given the twisting sequence $\xi_\bullet = (\xi_3, 0, \cdots)$ on the first arising from monopole Floer theory, transfer it to a sequence $[\xi]_\bullet$ in $\Lambda^* H^1(Y)$. Compute what that sequence is, and hope that it's what we call cup homology.

Problem. If $f : A \to B$ is a quasi-isomorphism of dg-algebras, it's obvious what $f(x_\bullet)$ is; it has n'th term $f(x_{2n+1})$. On the other hand, it is much more difficult to go backwards and lift a twisting sequence in B to a twisting sequence in A (well-defined up to homotopy). The result above guarantees that it's always possible and indeed well-defined up to homotopy, but it's completely inexplicit. It doesn't give any way of computing the result in practice.

Solution. Characteristic classes. If we can cook up computable cohomological gadgets $F_n(x_\bullet)$ which essentially characterize the twisting sequence, we might be able to determine what they are for Kronheimer and Mrowka's twisting sequence $\xi_\bullet$. We then might be able to determine which twisting sequence on $\Lambda^* H^1(Y)$ has the same characteristic classes, and thus corresponds to $\xi_\bullet$.

One such characteristic class is pretty easy to come up with; set $F_1(x_\bullet) = [x_3]$. By definition this is a cocycle, and a homotopy between $x_\bullet$ and $y_\bullet$ amounts in this degree to a chain h_2 with $dh_2 = x_3 - y_3$, so this is indeed a homotopy invariant of the twisting sequence.

Constructing such characteristic classes in higher degrees becomes difficult (indeed, I suspect impossible). One wants to start by taking $[x_5]$, but $dx_5 = -x_3^2 \neq 0$. To construct an appropriate characteristic class, one would need a *natural* chain $E(x_3)$ so that $dE(x_3) = -x_3^2$. If one had such a natural chain, then we could set

$$F_2(x_\bullet) = [x_5 - E(x_3)].$$

If this E is well-behaved this should end up being a homotopy invariant.

Well, why should x_3^2 be null-homotopic to begin with? The element x_3 is odd-degree, so our experience with cochains suggests that x_3^2 should be homotopic to $-x_3^2$ by some cup-1 product. Then (accepting that we have to define our characteristic classes over the rationals, even if we use them to study integral twisting sequences) we may set

$$E(x_3) = 1/2 x_3 \smile_1 x_3.$$

This suggests that if we want to carry this out in general we need some sort of homotopy-commutative structure on our dgas.

Hirsch algebras and characteristic classes

While attempting to define these higher characteristic classes, we spent a lot of time staring at a paper of Kraines on *higher Massey products*; twisting sequences are the same as what he calls defining sequences for Massey powers. In one particularly nice lemma, he shows how you can take any cocycle a and extend it to a twisting sequence

$$(a, 1/2a \smile_1 a, 1/6(a \smile_1 a) \smile_1 a, \cdots);$$

his argument used the *left Hirsch formula*, which for odd-degree elements reads

$$(ab) \smile_1 c = a(b \smile_1 c) + (a \smile_1 c)b.$$

It turns out that these lead to a really well-behaved theory of characteristic classes for algebras which come with a cup-1 product satisfying the left Hirsch formula. **Crucially, this will not work for us.** To construct our minimal model for the torus T^n we will need to work *cubically*; the smallest simplicial model has far too many simplices.

We instead must study *Hirsch algebras* — what A_∞ algebras are to dg-algebras, Hirsch algebras are to dgas with a cup-1 product satisfying the left Hirsch formula. We are given a homotopy-commutator $E_{1,1}$, as well as higher-homotopy operators $E_{p,q}$ which

These were studied in [San16], who noticed a generalization of Kraines' formulas to this setting. His generalizations inspired our construction of the characteristic classes in general.

Theorem 2 (F. Lin–ME)**.** *There are a sequence $F_1(x_\bullet), F_2(x_\bullet), \cdots$ of rational cohomology classes of degree 3, 5, and so on, associated to a twisting sequence in a Hirsch algebra. If two twisting sequences are homotopic, the resulting cohomology classes are equal. These twisting sequences are natural for Hirsch algebra maps. These begin*

$$\begin{aligned} F_1(x_\bullet) &= [x_3] \\ F_2(x_\bullet) &= [x_5 - 1/2x_3 \smile_1 x_3] \\ F_3(x_\bullet) &= [x_7 - x_5 \smile_1 x_3 + 1/3(x_3 \smile_1 x_3) \smile_1 x_3 + 1/3E_{2,1}(x_3, x_3; x_3)] \\ &\cdots \end{aligned}$$

These get progressively more complicated — F_4 is wildly more complicated than the previous, for instance, and includes terms like $(x_3 \smile_1 x_3) \smile_1 x_5$.

The best way to intuit the construction of these classes is as follows. Kraines constructed a natural extension $K(a)$ of a cocycle to a twisting sequence (and Saneblidze taught us how to extend this idea to Hirsch algebras). One can use the operations $E_{p,q}$ to define products of twisting sequences as well; combining these two ideas one gets a **Kraines tower** of approximations $K_{(n)}(x_\bullet)$ of a twisting sequence by these canonical twisting sequences; the characteristic classes above are the obstructions to finding a homotopy from $x_\bullet$ to $K_{(n)}(x_\bullet)$. One should think of $K_{(n)}$ as something like the n'th Taylor polynomial, and the F_{n+1} as the $(n+1)$'th derivative of a function.

When *A has no torsion in its cohomology*, this is enough to completely determine twisting sequences (and are therefore the right tool to use to study twisting sequences on the torus).

Theorem 3 (F. Lin–ME)**.** *Suppose A is a Hirsch algebra with torsion-free cohomology supported in a bounded range of degrees, and let $x_\bullet$ and $y_\bullet$ be twisting sequences.*

Then the characteristic classes $F_n(x_\bullet) = F_n(y_\bullet)$ for all n if and only if $x_\bullet$ and $y_\bullet$ are homotopic.

This is pleasant abstract theory, but to apply this to our specific case we need two things:

(1) We need to construct Hirsch algebra structures on $C^*_\Delta(\mathbb{T})$ and $H^*(\mathbb{T})$ and construct a zigzag of Hirsch algebra structure quasi-isomorphisms between them.
(2) We need to compute the characteristic classes $F_n(\xi_\bullet)$ of Kronheimer and Mrowka's twisting sequence.
(3) We need to determine what cohomology class in $H^*(\mathbb{T})$ has these characteristic classes (and therefore we need a very explicit handle on its Hirsch algebra structure).

The second of these is easy: $\xi_\bullet$ is pulled back from the 3-sphere which has $H^{2n+1} = 0$ for $n > 1$, and it is known that

$$[\xi_3] = \cup_Y^3 \in H^*(\mathbb{T}) = \Lambda^*(H^1 Y)^\vee$$

is the triple cup product form of Y.

The construction of a Hirsch algebra structure on simplicial cochains can be found in [Kad04]. Unfortunately, this will not suffice for us. It will turn out that $H^*(\mathbb{T})$ is best described as the **cubical cochains on a minimal cubical model for the torus**. Therefore, we need a formula for the E-operations on cubical sets.

A major part of our work is deriving such formulas from extremely recent diagrammatic/operadic constructions by Ralph Kaufmann and Anibal Medina-Mardones. They constructed an E_∞-algebra structure on cubical and simplicial cochains, which is generated by very simple operations and whose operations can all be represented by certain immersed graphs. We show how to use their machine to produce a Hirsch algebra structure on cubical and simplicial cochains, and what's more we check that their work shows that the natural cubical-to-simplicial comparison map induces a Hirsch algebra map.

However, we still have (3) to think about.

The minimal torus

Let $\mathbb{T}^n_1$ be the cubical model for the torus given by pasting together all opposite sides of the n-cube. Its cubical cochains have zero differential; we have $C^*_\square(\mathbb{T}^n_1) = \Lambda^*(\mathbb{Z}^n)$. What's more, we have an *completely explicit algebraic description of the operations $E_{1,p}$.*

Theorem 4 (F.Lin–ME). *Write a basis element of $\Lambda^*(\mathbb{Z}^n)$ as $e^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$, where $I = \{i_1 < \cdots < i_k\} \subset \{1, \cdots, n\}$.*

Say a family $(e^J; e^{I_1}, \cdots, e^{I_p})$ is $(1;p)$-regular if $I_j \cap I_k = \varnothing$ for distinct j, k, while $|J \cap I_k| = 1$ for all k; writing $J \cap I_k = \{j_k\}$, we also demand that $j_1 < \cdots < j_p$. In this case we write

$$J = J_0 \cup \{j_1\} \cup J_1 \cup \cdots \cup \{j_p\} \cup J_p,$$

where $J_0 < j_1 < J_1$ etc.

Then

$$E_{1,p}(e_J; e^{I_1}, \cdots, e^{I_p}) = \begin{cases} e^{J_0} \wedge e^{I_1} \wedge \cdots \wedge e^{I_p} \wedge e^{J_p} & (J; I_1, \cdots, I_p) \text{ is } (1;p)-\text{regular} \\ 0 & \text{else} \end{cases}$$

Other than the asymmetry, we call this a special case of the *insertion product*; we're feeding the e^I's into one another, so long as the various subsets intersect in a relatively simple way. Iterates of these operations figure into our story in an important way.

Definition 3. *Given a set $(I_1, \cdots, I_p)$ of subsets of $\{1, \cdots, n\}$, we may construct a graph by pasting together p intervals (the j'th with $|I_j|$ vertices) so that the j'th and k'th intervals intersect at precisely $|I_j \cap I_k|$ points (and in fact, at the points corresponding to $I_j \cap I_k$ inside of I_j and I_k).*

We saay $(I_1, \cdots, I_p)$ is ***1-regular*** *if the graph constructed above is a tree. Equivalently, the I's may be reordered so that $I_j \cap (I_1 \cup \cdots \cup I_{j-1})$ is a singleton for all j. Reorder them so that this is the case. In that case, we say the* insertion product

$$j(e^{I_1}, \cdots, e^{I_p}) = (e^{I_1} \smile_1 e^{I_2}) \cdots \smile_1 e^{I_p}$$

is given as the iterated cup-1 product.

If $a = \sum_I a_I e^I \in \Lambda^*(\mathbb{Z}^n)$, then we say its ***insertion powers*** *are*

$$a^{\circ m} = \sum_{\substack{\{I_1, \cdots, I_m\} \\ (I_1, \cdots, I_m) \text{ is } 1-\text{regular}}} a_{I_1} \cdots a_{I_m} j(e^{I_1}, \cdots, e^{I_m}).$$

For example,

$$(e^{123} + e^{345})^{\circ 2} = e^{12345}$$

and

$$(e^{123} + e^{345} + e^{146})^{\circ 2} = e^{12345} + e^{12346} + e^{13456},$$

but both of these have insertion cube equal to zero; an expression with nonzero insertion cube is

$$(e^{123}+e^{345}+e^{567})^{\circ 3}=e^{1234567}=(e^{123}+e^{456}+e^{147})^{\circ 3}.$$

After some work with the Kraines construction itself and some combinatorics of subsets of $\{1,\cdots,n\}$, we are able to prove the following.

Theorem 5 (F.Lin–ME)**.** *Let $a\in\Lambda^3(\mathbb{Z}^n)$ be a degree 3 element. Then the Kraines construction returns*

$$K(a)=(a,a^{\circ 2},a^{\circ 3},\cdots).$$

Furthermore,

$$F_nK(a)=\begin{cases}[a] & n=1\\ 0 & n>1\end{cases}$$

Therefore when we transfer $\xi_\bullet$ to $H^*\mathbb{T}$, it is ultimately sent to $K([\xi_3])$. If we write out the triple cup product with respect to some basis of $H^1(Y)$, we obtain an element $a\in\Lambda^3(\mathbb{Z}^n)$, and may then compute its insertion powers as above (which a computer can do nearly instantaneously).

Putting it all together

Kronheimer and Mrowka took us from $\overline{HM}$ to $CMH_*(\mathbb{T}(Y,\mathfrak{s});D)$. They calculate that $K^1(\mathbb{T}(Y))$ injects into $H^{odd}(\mathbb{T}(Y))$, and evaluate its image: it is precisely the class corresponding to the triple-cup-product. Because the higher classes are zero, they are able to show that the map $D:\mathbb{T}(Y)\to U$ is homotopic to a map factoring through $SU(2)$.

Their theorem then takes us to

$$H_*^{\mathrm{tw}}(C_*^{\Delta}(\mathbb{T});\xi_3),$$

the twisted homology of this torus with respect to a certain twisting 3-cycle.

We used a zigzag of quasi-isomorphisms to show that this is isomorphic to

$$H_*^{\mathrm{tw}}(\Lambda^*(\mathbb{Z}^n);K_\bullet)$$

for an appropriate twisting sequence $K_\bullet$ in $\Lambda^*(\mathbb{Z}^n)$.

Lastly, in the previous section, we computed what this $K_\bullet$ was.

Theorem 6 (F.Lin–ME)**.** *Let $(Y,\mathfrak{s})$ be a 3-manifold. Choose a basis $H^1Y\cong\mathbb{Z}^n$, and write $a=\Lambda^3\mathbb{Z}^n$ for the triple cup product written in this basis; write ι_a for contraction against this element. Then we have an isomorphism of $\Lambda^*(H_1Y/Tors)[U,U^{-1}]$-modules*

$$\overline{HM}(Y,\mathfrak{s};\mathbb{Z})\cong H\Big(\Lambda^*(\mathbb{Z}^n)[U,U^{-1}],x\mapsto \iota_a xU+\iota_{a^{\circ 2}}xU^2+\cdots\Big)$$

A version of this with only slight modifications holds for any local coefficient system as well. We call these groups the *extended cup homology groups.* About a thousand random computer computations show that these give the same answer that cup homology does; it probably just takes a clever eye to cook up some isomorphism of abelian groups between the extended cup homology and the usual cup homology. However, it's not clear whether or not one should expect that these are the same as Λ-modules or not; we have not thusfar been checking this (as computing homology over non-integral domains is rather difficult).

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