CONTENT Course description

basics of ODE §1.1 & §1.2 + Thm about $\exists/!$ §2.4 - §2.8

classification of ODEs §1.3

method of integrating factors §2.1

Syllabus summary: TEXTBOOK: "Elementary Differential Equations and boundary value Problems" 9th ed.

by Boyce & DiPrima. - Ch. 1-7 + 9

PREREQUISITES: Calculus I & II + III + Linear algebra.

GRADING: HW (9 HW - the dates are out on Courseworks) 20%

Midterm (on 24th) 30%

Final (on 11th of August ?) 50%

(the worst 2 grades are automatically dropped)

HW: ⊙ (Usually) due on Monday & Thursday at 11:59pm on Gradescope.

⊙ no late submission accepted (unless there is an emergency) ← Please contact the TA for extensions

⊙ collaboration is encouraged (but you need to write down your own solutions)

⊙ solutions will be posted on courseworks

DISABILITY SERVICES: Contact ODS if you need special exam accomodation.

OFFICE HOURS: Thursday from 3pm to 7pm in Math 407 + on Zoom (link in Courseworks)

Email: mp3947@columbia.edu.

Introduction: what you are going to learn & why

⊛ theoretical point of view: you may have an intrinsic interest in the subject.

E.g. you studied equations $x^2+3x+2=0$ ⇝ find solutions / roots

now: $f''+3f'+2f=0$: which continuous functions $f: \mathbb{R} \to \mathbb{R}$ are solutions?

WHAT

OBJECT of study of this class:

Differential Equations: a DE is any equation involving a function and its derivatives.

Ordinary DE: function is of the type $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$ & the equation is of the form $Eq(t, f(t), f'(t), \dots f^{(n)}(t)) = 0 \quad (t \in I)$.

What do I mean by equation? Ex 1 $f''+3f'+2f$

Ex 2: $\sin(f)+f'$ Ex 3 $f \cdot f' + f'' \cdot t$

Mathematical definition: Eq is a continuous function: $I \times (\mathbb{R}^m)^{n+1} \longrightarrow \mathbb{R}^e$ (*)

$Eq1(t,x,y,z) = z+3y+2x$: $Eq1(t,f,f',f'') = f''+3f'+2f$

$Eq2(t,x,y) = \sin(x)+y$: $Eq(t,f,f') = \sin(f)+f'$

$Eq3(t,x,y,z) = xy+zt$

⇒ So we deal with "two function"

the unknown function ⇝ variable ⇝ f

the equation, which is the "structure of the question / problem"

GOAL of the course: become familiar with these equations and learn how to solve them.

WHY (*) practical point of view: they offer a tool to formulate & understand behavior in the natural world.

Examples: (1) object in free fall: by Newton's second law: $\boxed{m\ddot{x} = -mg}$

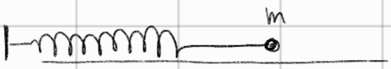
where x is the position.

$$\ddot{x} = x'' = \frac{d^2x}{dt^2} = \text{acceleration.}$$

(2) radioactive decay: $q(t)$ = mass of the radioactive material left at time t.

$\boxed{q' = -k\,q}$ k = decay constant

(3) harmonic oscillator:



you have a mass m attached to a spring on a friction-free surface:

$\boxed{m\ddot{x} = -kx}$

Let's solve them! But let's be smart: equations like these happen all the time. So instead of trying to find solutions ad hoc, let's look for a general strategy.

In order to develop a strategy ⇒ CLASSIFICATION OF ODE'S

Namely we study and classify "Eq".

The HOPE: your own problem → $Eq(t, f', \dots, f^{(n)})$ ⇝

Classification		How to solve it
Form 1	⟶	Strategy 1
Form 2	⟶	Strategy 2
⋮		⋮
Form 4		
⋮		

Pick your specific equation ⟶ Understand the "type / form" ⟶ Apply the right method in order to find solution / answer

Criteria of the Classification

① Explicit dependence on time

if the function Eq has NO explicit dependence on time → it is called **AUTONOMOUS**

otherwise, non-AUTONOMOUS.

Examples: Eq 1 & Eq 2 are autonomous ; Eq 3 is not autonomous.

② Order:
the order of Eq is the "number n" in def (*) : equivalently is the order of the highest derivative appearing.

Examples: Eq 1 & Eq 3 : 2nd Eq 2 : 1st.

③ Scalar vs system
: if "number m" in def (*) is 1 → scalar ; otherwise → system.

Equivalently, $f: I \longrightarrow \mathbb{R}$ scalar & $f: I \longrightarrow \mathbb{R}^m$, $m>1$ system.

Example: Eq 1, 2, 3 are all scalars.

Example of a system: prey & predator population.

Call $x_1(t)$ = size of the prey population at time t

$x_2(t)$ = " " predator " " " "

⇒ the interaction is given by the system

$$\begin{cases} \dot{x}_1(t) = \dfrac{dx_1(t)}{dt} = (\alpha - \beta x_2)\, x_1 \\ \dot{x}_2(t) = (\gamma x_1 - \delta)\, x_2 \end{cases}$$

[where $\alpha, \beta, \gamma, \delta$ are all given positive constant]

⇒

$$\dot{x}_1 - \alpha x_1 + \beta x_2 x_1 = 0$$

$$\dot{x}_2 - \gamma x_1 x_2 + \delta x_2 = 0$$

$$Eq\left(t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}\right) = 0 \quad \text{when} \quad x : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2 \quad (m=2)$$

$$\underset{\vec{x}}{\uparrow} \qquad \vec{x}\,'$$

④ Linearity: an ODE $Eq(t, f, f', \dots f^{(n)})$ is linear if

$Eq(t, x, y, z \dots)$ is linear in $(x, y, z, \dots)$. Equivalently

$Eq(t, f, f', \dots, f^{(n)})$ has the following form: $\sum_{i=0}^{n} c_i(t) \cdot f^{(i)} + g(t) = 0$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}^m$ is your unknown function

$c_i : I \subseteq \mathbb{R} \longrightarrow M(\ell, m; \mathbb{R})$ ℓ rows, m columns & takes value in $\mathbb{R}$.

$g : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\ell$ is a known function.

Examples: Eq 1 linear ✓

Eq 2 not-linear × due to $\sin(x)$: but $\underbrace{\sin(t)}_{\text{known functions}} + f'(t) = 0$ is linear

Eq 3 " " " due to xy : but $f' + f''t = 0$ is linear

remarks. ▫ the known function needs not to be linear (Eq 2 $g(t) = \sin(t)$)

▫ the coefficients $c_i(t)$ can depend on time (Eq 3 $c_2(t) = t$)

▫ the " " " " need not to be linear: ex: $\underline{f'(t) + \sin(t) f(t) + e^t = 0}$

is linear

▫ "linear ODEs are nice" - (we know how to solve them) &

In practice: we approximate non linear equations by linear ones.

(5a) Homogeneity: if Eq is linear then if $g = 0$ then we say that the ODE is HOMOGENEOUS.

(5b) Constant coeff: if Eq is linear & all the $C_j(t)$ are constant matrices then we say that the ODE has constant coeff.

Examples:

	linear	homogeneous	const. coeff
$f'' + 3f = 0$	✓	✓	✓
$f'' + tf = 0$	✓	✓	X
$f'' + \sin(t) = 0$	✓	X	✓
$(f')^2 + f' = 0$	X	NA	NA

Final Remarks:
- linear easier than non linear
- n-th order easier than (n+1)th order - in particular 1st easier than 2nd
- Usually scalar easier (but sometime we'll transform a scalar into a system)
- auton. / homog. / const. coeff — all better that — non-auton. / non-hom. / non-constant coeff.

It makes sense to try to solve the easier cases first & then the more complicated ones.

DEF. (Solution) a solution of an ODE of order n is a function φ n-times differentiable s.t. $Eq(t, \varphi, \varphi', \dots, \varphi^{(n)}) = 0 \quad \forall t \in I$.

Key fact: an ODE can have more than one solution! $\Rightarrow$ Two questions: $\exists$ (existence), $!$ (uniqueness)

DEFINITION. the family of all the functions that solve a given ODE is called the GENERAL SOLUTION.

$\boxed{\text{First Step:}}$ First Order Scalar Linear Equation

- $c_1(t)\,f' + c_0(t)\,f + g(t) = 0$ $\quad f: I \longrightarrow \mathbb{R}$. Assume that over I,

$c_1(t)$ has no zeros $\Rightarrow$ $f' + \frac{c_0(t)}{c_1(t)} f + \frac{g(t)}{c_1(t)} = 0$ $\Rightarrow$ $\boxed{f' = a(t)\,f + b(t)}$

Rule: it is usually convenient to "isolate" the highest derivative & make it have coeff. $= 1$.

this is called the standard form

CASE 0: a & b are constants : $f' = af + b$

$a = 0$ $\quad f' = b$ $\rightsquigarrow$ we can take the antiderivative on both sides

$\Rightarrow f(t) = bt + k$ (and those are all and only solutions)

$a \neq 0$ $\quad f' = a\left(f + \frac{b}{a}\right)$, suppose $\exists$ a time t_0 : $f(t_0) = -\frac{b}{a}$ then $f'(t_0) = 0$

[we will get back to their argument] $\longrightarrow$ " $\rightsquigarrow$ it would remain constant $\Rightarrow$ $f(t) \equiv -\frac{b}{a}$ is a solution "

suppose that $\nexists\, t \in I$: $f(t) = -\frac{b}{a}$ $\Rightarrow$ $f + \frac{b}{a} \neq 0$ $\forall t \in I$ and we can write

$$\left(\frac{1}{f + \frac{b}{a}}\right) f' = a . \quad \text{Now } f' = \frac{df}{dt} \Rightarrow \left(\frac{1}{f + \frac{b}{a}}\right)\frac{df}{dt} = a \Rightarrow \left(\frac{1}{f + \frac{b}{a}}\right) df = a\,dt$$

$\Rightarrow$ integrating both sides: $\ln\left|f + \frac{b}{a}\right| = at + k$ $\Rightarrow$ taking the exponential

$$\left|f + \frac{b}{a}\right| = \exp(at + k) = C e^{at} \text{ with } C > 0 \longrightarrow f = Ce^{at} - \frac{b}{a} \quad \forall C \in \mathbb{R}\setminus\{0\}$$

Notice: for $C = 0$ we recover the "other solution" $f = -\frac{b}{a}$ (equilibrium point)

Let's try to apply the method to the general form: $f'(t) = a(t)f(t) + b(t)$

even assuming $a(t) \neq 0 \ \forall t$ & $a(t)f(t) + b(t) \neq 0 \ \forall t$, then we may not know how to integrate

$$\frac{1}{f(t) + \frac{b(t)}{a(t)}} df = a(t)dt \,!$$

Leibniz: integrating factor method: what we do is to find a suitable function $\mu(t)$ such that we can find an easy way to integrate:

$f'(t) - a(t)f(t) = b(t)$: $\quad \mu(t)f'(t) - \mu(t)a(t)f(t) = b(t)\mu(t)$

Assume μ is such that & $\mu(t) \neq 0 \ \forall t$ $\quad \frac{d}{dt}[\mu(t)f(t)] = \mu'(t)f(t) + \mu(t)f'(t) = \mu(t)f'(t) - \mu(t)a(t)f(t)$

then $\mu'(t)f(t) = -\mu(t)a(t)f(t) \Rightarrow$ $f(t) \equiv 0$ or $\mu'(t) = -\mu(t)a(t)$ ← we can use a similar method as before

$$\frac{1}{\mu(t)} d\mu(t) = -a(t)dt \Rightarrow \ln|\mu(t)| = -\int a(t)dt + k = -A(t) + k$$

$$\Rightarrow \mu(t) = C \cdot \exp(-A(t)) = C e^{-\int a(t)dt} \quad \text{with } C > 0.$$

$$\Rightarrow \frac{d}{dt}\left(\cancel{C} e^{-\int a(t)dt} f(t)\right) = \cancel{C} e^{-\int a(t)dt} b(t) \Rightarrow \boxed{f(t) = e^{\int a(t)dt}\left[\int e^{-\int a(t)dt} b(t)\,dt + K\right]}$$

Back to the initial examples:

1)	$m\ddot{x} = -mg$	$\ddot{x} + g = 0$	auton., 2nd, scalar, linear, non-homog., const. coeff.
2)	$\dot{q} = -kq$	$\dot{q} + kq = 0$	auton., 1st, scalar, linear, homog., const. coeff.
3)	$m\ddot{x} = -kx$	$\ddot{x} + \frac{k}{m}x = 0$	auton., 2nd, scalar, linear, homog., const. coeff.

1st: is not first order but it is easily integrable $\Rightarrow \ddot{x} = -g \Rightarrow \dot{x} = -gt + A \Rightarrow x = -g\frac{t^2}{2} + At + B$

2nd: exactly of the form $\dot{f} = af + b$ $\therefore$ $\dot{q} = -kq \Rightarrow$ $q = 0$ or $q(t) = Ce^{-kt}$

(3rd: $\ddot{x} + \frac{k}{m}x = 0$ X we still don't know how to solve this)

Example for integrating factors.

(A) $tf' + 2f = 4t^2$ $f: (0, \infty) \longrightarrow \mathbb{R}$ & we want it differentiable. (so that we can take f')

1st: let's isolate f': $t > 0 \Rightarrow t \neq 0 \Rightarrow f' = -\frac{2}{t}f + 4t \Rightarrow a(t) = -\frac{2}{t}$ & $b(t) = 4t$

$$\Rightarrow f(t) = e^{\int[-\frac{2}{t}]dt}\left[\int b(t)\left[e^{\int[\frac{2}{t}]dt}\right]dt + k\right]$$

$$= e^{-2\ln|t|}\left[\int 4t \cdot e^{2\ln|t|} dt + k\right] = (t)^{-2}\left[\int 4t \cdot t^2 dt + k\right] = \frac{1}{t^2}\left[4\int t^3 + k\right] =$$

$t > 0$ so I can remove the $|.|$

$$= \frac{1}{t^2}\left[t^4 + k\right] = t^2 + \frac{k}{t^2}$$

(B) $(4+t^2)y' + 2ty = 4t$

First notice that: $4+t^2$ is never zero $\Rightarrow$ I can divide with no issues:

$\Rightarrow$ put in the standard form: $y' = -\frac{2t}{t^2+4}y + \frac{4t}{t^2+4}$

$a(t) = -\frac{2t}{t^2+4}$ & $b(t) = \frac{4t}{t^2+4}$: $y(t) = e^{-\int \frac{2t}{t^2+4}dt}\left[\int e^{\int \frac{2t}{t^2+4}dt} b(t)\,dt + k\right]$

First solve: $\int \frac{2t}{t^2+4}\,dt = \ln|t^2+4| = \ln(t^2+4)$ indeed $\frac{d}{dt}\ln(t^2+4) = \left(\frac{1}{t^2+4}\right)\frac{d(t^2+4)}{dt} = \frac{2t}{t^2+4}$

$$\Rightarrow y(t) = e^{-\ln(t^2+4)}\left(\int e^{\ln(t^2+4)}\,\frac{4t}{t^2+4}\,dt + k\right)$$

$$= \frac{1}{t^2+4}\cdot\left[\int (t^2+4)\frac{4t}{t^2+4}\,dt + k\right] = \frac{1}{t^2+4}\left(\int 4t\,dt + k\right) = \frac{1}{t^2+4}\left(2t^2+k\right) =$$

$$= \frac{2t^2+k}{t^2+4}$$

SECOND PART : Existence & Uniqueness of solutions.

Now we have a way to find solutions for $f'(t) = a(t)f(t) + b(t)$ $(*)$. But how can we sure that we haven't skipped some? What about the same question but for a general ODE?

QUESTION: given an ODE can we say whether or not it has a solution? Can we say that it has a unique solution? Where is it define / What is the interval of definition I?

We already have seen that the general solution for $(*)$ is $f(t) = e^{\int a(t)dt}\left[\int e^{-\int a(t)dt} b(t)\,dt + K\right]$

when K is an undetermined constant. $\rightsquigarrow$ ∞-many solutions.

$\Rightarrow$ it makes more sense to pose the questions of $\exists$ & $!$ in a context where we are given not only the ODE but also "Extra Data" $\rightsquigarrow$ Initial values

DEFINITION: Consider $f: I \subseteq \mathbb{R} \to \mathbb{R}^m$ & ODE: $Eq(t, f, \dots, f^{(n)}) = 0$ of order n.

To specify initial conditions at time $t_0 \in I$ means to fix the value of

$$f(t_0),\ f'(t_0),\ \dots,\ f^{(n-1)}(t_0) \in \mathbb{R}^m$$

DEFINITION: an INITIAL VALUE PROBLEM (IVP) is the collection of:

$$\begin{array}{ll} \text{an ODE} \to & \left\{ \begin{array}{l} Eq(t, f, \dots, f^{(n)}) = 0 \\ \\ f(t_0) = u_0 \\ \vdots \\ f^{(n-1)}(t_0) = u_{n-1} \end{array} \right. \\ \text{Initial conditions} \to & \end{array}$$

let's go back to $f' = af + b$ $\quad a, b \in \mathbb{R}$ & consider the IVP: $\begin{cases} f' = af + b \\ f(0) = u_0 \end{cases}$

if $u_0 = -\frac{b}{a}$ we know that $f(t) = -\frac{b}{a}$ is a solution.

if $u_0 \neq -\frac{b}{a}$ & f never takes the value $-\frac{b}{a}$ over I $\Rightarrow$ $f(t) = Ce^{at} - \frac{b}{a}$

[double check $f(t) = e^{\int a\,dt}\left[\int e^{-\int a\,dt} b\,dt + K\right]$

$= e^{at}\left[\int e^{-at} b\,dt + K\right] = e^{at}\left[-\frac{1}{a}e^{-at} b + K\right] = -\frac{b}{a} + Ke^{at}$ ✓

we would like to say: if $u_0 = -\frac{b}{a}$ then $f(t) = -\frac{b}{a}$ is the UNIQUE solution!

if $u_0 \neq -\frac{b}{a}$ & $f(t)$ is a solution over I then $f(t) \neq -\frac{b}{a}$ $\forall t \in I$

(& $f(t)$ = above form)

Rmk: notice that if we know $u_0 \neq -\frac{b}{a}$ & $f(t) = -\frac{b}{a}$ never then the above argument for

1. finding $f(t) = Ce^{at} - \frac{b}{a}$ was a "necessary condition" argument.

Namely we assume that f was a solution & we found a form for it. Since if we evaluate the ODE in $f(t) = Ce^{at} - \frac{b}{a}$ we get an identity then $f(t) = Ce^{at} - \frac{b}{a}$ is the general solution.

2. Since we know specify $f(0) = u_0$, we can determine C!

$f(t) = Ce^{at} - \frac{b}{a}$: $f(0) = C - \frac{b}{a} = u_0 \Rightarrow C = u_0 + \frac{b}{a} \Rightarrow \boxed{f(t) = \left(u_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}}$

NOTICE: it is unique now!

THEOREM: Picard – Lindelöf thm : "short-time existence & uniqueness"

Let $Eq(t,f,f') = f' - G(t,f) = 0 \Rightarrow$ write it as $f' = G(t,f)$ be a scalar ODE s.t.

$G(t,x)$ & $\frac{\partial}{\partial x} G(t,x)$ are continuous in

some rectangle $I \times J = (\alpha,\beta) \times (\gamma,\delta)$.

Then $\forall (t_0, u_0) \in I \times J$ $\exists\, h$:

$(t_0 - h, t_0 + h) \subseteq I$ & $\exists!$ solution

$f : (t_0 - h, t_0 + h) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ of the IVP

$$\begin{cases} f' = G(t,f) \\ f(t_0) = u_0 \end{cases}$$

General: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$ &

$G : I \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$

where $G(t,\vec{x})$ &

$D_x G(t,\vec{x}) = \left[\frac{\partial G}{\partial x_1}, \dots, \frac{\partial G}{\partial x_m} \right]$ (Jacobian Matrix)

are continuous on $I \times J$

(Actually: enough Lipschitz continuous in x)

(sketch)

Proof: $f' = G(t,f) \Rightarrow \int_{t_0}^{t} f'(s)\,ds = \int_{t_0}^{t} G(s,f)\,ds \Rightarrow f(t) - f(t_0) = \boxed{f(t) - u_0 = \int_{t_0}^{t} G(s, f(s))\,ds}$ $(*)$

Then by "Banach Fixed-Point thm" there exists a unique solution to $(*)$ and it is

computed by the Picard iteration method: Set: $\varphi_0(t) = u_0$ constant one and define

$\varphi_{k+1}(t) = u_0 + \int_{t_0}^{t} G(s, \varphi_k(s))\,ds$. Then $\{\varphi_k\}_{k \in \mathbb{N}}$ converges to the unique solution. #

THEOREM (Picard – Lindelöf – long-time existence)

Same assumptions as before: the the IVP has a maximal interval of existence and it is of the

form (T_-, T_+). This means that $\exists!$ unique solution $f(t)$ on (T_-, T_+) and

either $T_+ = +\infty$ (resp. $T_- = -\infty$) or $T_+ < \infty$ & $|f(t)| \xrightarrow{t \to T_+} +\infty$ || sloppy!

(same for T_-).

let's go back to: $\begin{cases} f'(t) = a f(t) + b \\ f(0) = u_0 \end{cases}$ & $\begin{cases} f'(t) = a(t) f(t) + b(t) \\ f(0) = u_0 \end{cases}$

$G(t, f(t)) = a f(t) + b \Rightarrow G(t, x) = ax + b \to C^\infty \Rightarrow$ Theorem applies

given u_0 $\exists!$ solution: ① since $f(t) = -\frac{b}{a}$ for $u_0 = -\frac{b}{a}$ is a solution & $f(t) \not\to +\infty$.

it is the unique one & $\exists$ forever.

② since $f(t) = \left(u_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a} = -\frac{b}{a}$ iff. $\left(u_0 + \frac{b}{a}\right)e^{at} = 0$ then it is never 0.

& it is the unique solution.

Focus on $f'(t) = a(t) f(t) + b(t)$: $G(t, x) = a(t) x + b(t)$

- if $a(t)$ & $b(t)$ are continuous $\Rightarrow a(t) x + b(t)$ is continuous &

$\frac{\partial}{\partial x}(a(t) x + b(t)) = a(t)$ is continuous as well.

$\Rightarrow$ <u>Thm applies!</u> The solut. found is the unique one & $\exists$ always.

<u>Examples</u>: (1) $\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$ First. $G(t, x) = x^2$ which is C^∞. $\Rightarrow$ both thms apply.

- let $y_0 = 0 \Rightarrow y \equiv 0$ is a suitable solution $\Rightarrow$ by $\exists$ & ! thm $\Rightarrow$ this is the unique solution

& $\exists$ forever $(-\infty, +\infty)$.

- let $y_0 > 0$: notice that $y' = y^2 \geq 0 \Rightarrow$ non-decreasing function.

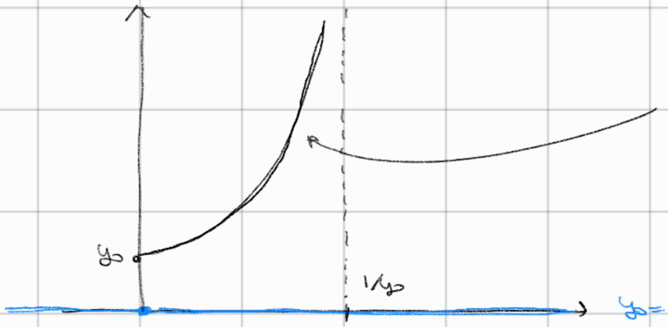
$\Rightarrow y(t) \geq y_0 \ \forall t \geq 0$. in particular $y(t) \neq 0 \ \forall t \geq 0$

$\Rightarrow$ on this interval $[0,\infty)$ $\Rightarrow$ $\frac{1}{y^2}\,dy = dt$ $\Rightarrow$ $-\frac{1}{y}\Big|_{t_0}^{t} = t$ $\Rightarrow$ $-\frac{1}{y(t)} + \frac{1}{y_0} = t$

$\Rightarrow$ $\frac{1}{y(t)} = \frac{1}{y_0} - t = \frac{1-ty_0}{y_0}$ $\Rightarrow$ $y(t)\,(1-y_0t) = y_0$

Notice: at $t = \frac{1}{y_0} > 0$ $\quad 1-y_0t = 0$ $\Rightarrow$ $|y(t)| \xrightarrow{t\to\frac{1}{y_0}} \infty$ $\Rightarrow$ $T_+ < \infty$.

$\Rightarrow$



the graphs of the solutions at different IV (but same time t_0) are called integral curves

Question: can $y(t) = 0$ for any $t<0$ if $y_0 \neq 0$? NO! Why? Uniqueness of solution: indeed if $y(\bar t) = 0$ for $\bar t < 0$ $\Rightarrow$ IVP $\begin{cases} y' = y^2 \\ y(\bar t) = 0 \end{cases}$ would have 2 solutions ↯.

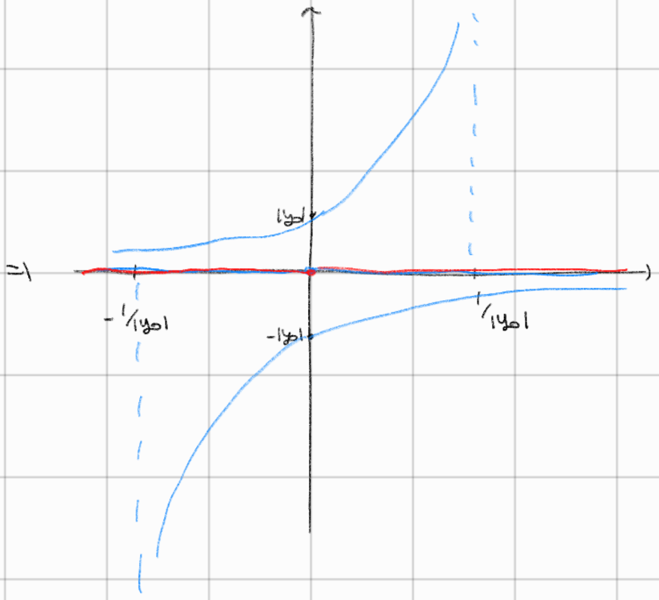
$\Rightarrow$ $y(t) \neq 0$ & actually >0 $\forall t<0$ $\Rightarrow$ $y(t) = \frac{y_0}{1-y_0t}$ well defined (& unique) solution on $(-\infty, \frac{1}{y_0})$

T_-

Corollary (of above thms): integral curves do NOT intersect!

▫ $y_0 < 0$ $\Rightarrow$ $y(t) < 0$ always (same reasoning). $\Rightarrow$ $y(t)\,(1-ty_0) = y_0$

at $t = \frac{1}{y_0} < 0$ you have the asymptote



(2) $\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$: first remark $y=0$ is a solution ⇒ But we cannot use the thm

why not? $G(t,x) = \sqrt{|x|}$ ⇒ $\frac{\partial}{\partial x} = \frac{1}{\sqrt{|x|}}$ at $x=0$ is $+\infty$.

Indeed one can check that $y(t) = \frac{t^2}{4}$ is a solution $\forall t \geq 0$.

However if $y_0 > 0$ ⇒ then no issue!

Hard Exercise:

$\begin{cases} y' - y = 1 + 3\sin(t) \\ y(0) = y_0 \end{cases}$ Find $y_0 \in \mathbb{R}$ s.t. the solution for the IVP problem stays finite $t \to +\infty$

classification: non-auton. + 1st order + scalar + linear + constant coeff. + non-homogeneous.

⇒ by thm & discussion above we have the solution:

$$y(t) = e^{\int a(t)dt} \left[\int e^{-\int a(t)dt} b(t)dt + K \right]$$ $a(t) = 1$ & $b(t) = 1 + 3\sin(t)$

$$\stackrel{!}{=} e^{t}\left[\int e^{-t}(1+3\sin(t))\,dt + K\right]$$

$$= e^{t}\left[-e^{-t} + 3\int e^{-t}\sin(t)\,dt + K\right]$$

$$\int e^{-t}\sin(t)\,dt \overset{\substack{\text{product}\\ \text{rule}}}{\underset{\downarrow}{=}} -e^{-t}\sin(t) - \int\left[-e^{-t}\cos(t)\right]dt$$

$$= -e^{-t}\sin(t) + \int e^{-t}\cos(t)\,dt = -e^{-t}\sin(t) - e^{-t}\cos(t) - \int\left[+e^{-t}\sin(t)\right]dt$$

$$= -e^{-t}(\sin(t)+\cos(t)) - \int e^{-t}\sin(t)\,dt$$

$$\Rightarrow \int e^{-t}\sin(t)\,dt = -\frac{e^{-t}}{2}(\sin(t)+\cos(t))$$

$$\Rightarrow y(t) = e^{t}\left[-e^{-t} - \frac{3e^{-t}}{2}(\sin(t)+\cos(t)) + K\right] = -1 - \frac{3}{2}(\sin(t)+\cos(t)) + Ke^{t}$$

$$y(0) = y_0 = -1 - \frac{3}{2}(\cancel{\sin(0)} + \overset{1}{\cos(0)}) + K = -\frac{5}{2} + K \Rightarrow K = \frac{5}{2} + y_0$$

$\lim_{t\to\infty} y(t) < \infty$. I want $K=0 \Rightarrow y_0 = -5/2$